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SOME TOPICS IN THE THEORY OF GENERALIZED FC-GROUPS

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Introduction

A *finiteness condition* is a group-theoretical property which is possessed by all finite groups: thus it is a generalization of finiteness. This embraces an immensely wide collection of properties like, for example, finiteness, finitely generated, the maximal condition and so on. There are also numerous finiteness conditions which restrict, in some way, a set of conjugates or a set of commutators in a group. Sometimes these restrictions are strong enough to impose a recognizable structure on the group.

R. Baer ([3]) and B.H. Neumann ([37]) were the first authors to discuss groups in which there is a limitation on the number of conjugates which an element may have. An element x of a group G is called *FC-element* of G if x has only a finite number of conjugates in G, that is to say, if $|G : C_G(x)|$ is finite or, equivalently, if the factor group $G/C_G(\langle x\rangle^G)$ is finite. It is a basic fact that the FC-elements always form a characteristic subgroup. An FC-element may be thought as a generalization of an element of the center of the group, because the elements of the latter type have just one conjugate. For this reason the subgroup of all FC-elements is called the *FC-center* and, clearly, always contains the center. A group G is called an *FC-group* if it equals its FC-center, in other words, every conjugacy class of G is finite. Prominent among the FC-groups are groups with center of finite index: in such a group each centralizer must be of finite index, because it contains the center. Of course in particular all abelian groups and all finite groups are FC-groups. Further examples of FC-groups can be obtained by noting that the class of FC-groups is closed with respect to forming subgroups, images and direct products. The theory of FC-groups had a strong development in the second half of the last century and relevant contributions have been given by several important authors including R. Baer ([3]), B.H. Neumann ([37]), Y.M. Gorcakov ([21]), Chernikov ([6]), L.A. Kurdachenko ([27], [28]) and many others. We shall use the monographs [51], [52] and [14] as a general reference for results on FC-groups. The study of FC-groups can be considered as a natural investigation on the properties common to both finite groups and abelian groups.

A particular interest has been devoted to groups having many FC-subgroups or many FC-elements. In this direction, in [4] V.V. Belyaev and N.F. Sesekin described minimal non-FC-groups (i.e., groups that are not FC-groups, but its proper subgroups are FC-groups) in the not perfect case. Moreover, A.V. Izosov e N.F. Sesekin ([30], [31]) started the study of groups which have only a finite number of infinite conjugacy classes, analyzing groups in which all elements that are not FC-elements lie in the same conjugacy class: groups having only two infinite conjugacy classes. More recently, M. Herzog, P. Longobardi e M. Maj ([24]) obtained further information on the structure of groups with restrictions on the number of infinite conjugacy classes.

In the recent years, many authors have investigated wider classes of groups with finiteness restrictions on their conjugacy classes in order to generalize properties of FC-groups. Let $\mathfrak{X}$ be a class of groups. Recall that a group G is called an $\mathfrak{X}C$*-group* (or a group with $\mathfrak{X}$ *conjucacy classes*) if for each element x of G the factor group $G/C_G(\langle x\rangle^G)$ belongs to the class $\mathfrak{X}$. In particular, taking for $\mathfrak{X}$ the class of all finite groups, we obtain in this way the concept of an FC-group defined above. The structure of $\mathfrak{X}C$-groups has been studied for several natural choices of the class $\mathfrak{X}$ in attempt to point out classes of groups generalizing properties of FC-groups. For instance for the class of all Černikov groups ([18], [39]) and for that of polycyclic-by-finite groups ([17], [19]).

An interesting branch of this area is devoted to the analysis of the groups that extend the class FC preserving properties of finiteness and nilpotency. In this direction, the first and most natural definition is that of FC-nilpotent groups (groups having a normal series whose factor are FC-groups, see [22]), nevertheless, for this class of groups, only few properties of finiteness and nilpotency still hold true. To overcome this difficulty, F. de Giovanni, A. Russo and G. Vincenzi in [13] introduced and studied the class of FC^n-groups, a special subclass of FC-nilpotent groups suitable for this purposes.

Denote by FC^0 the class of all finite groups G, and suppose that for some non-negative integer n a group class FC^n has been defined by induction; we define by FC^{n+1} the class of all groups G with conjugacy classes in FC^n, i.e. such that, for every element x, the factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n. Moreover, we put

$$FC^* = \bigcup_{n\geq 0} FC^n.$$

Clearly FC^1-groups are just the FC-groups, and for each non-negative integer n the class FC^n contains all finite groups and all nilpotent groups of class at most n. It is easy to check that, if the n-th term of the upper central series of a group G has finite index, then G is an FC^n-group (see Theorem

1.1.10), in particular every finite-by-nilpotent group is an FC^*-group.

Many investigations on FC^*-groups naturally generalize those for FC. In the first chapter of this dissertation, the reader can find a summary of the most relevant properties of the class of FC^*-groups. In particular, we point out many properties of FC-groups that also hold for FC^n-groups, for every $n \geq 2$. On the other hand, we also present some results concerning FC-groups that do not have an analogue in the class of FC^*-groups.

Recently, D.J.S. Robinson, A. Russo and G. Vincenzi proved in [44] some satisfactory results about Sylow's theory in an FC^*-group (see Theorem 1.4.4). So that arises in a natural way to study the behaviour of the property of pronormality in an in an FC^n-group with $n \geq 1$. Recall that a subgroup H of a group G is said *pronormal in* G if the subgroups H and H^x are conjugate in $\langle H, H^x \rangle$ for every element x of G. Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups. Moreover, Sylow subgroups of finite groups and Hall subgroups of finite soluble groups are always pronormal. The theory of pronormality in FC-groups has been deeply developed by F. de Giovanni and G. Vincenzi in [11] and in [12].

In the second chapter of this dissertation, our aim is to extend some results pertaining to pronormal subgroups of an FC-group to the class of FC^*-groups . In particular, in [46] the problem of the join of pronormal subgroups has been considered and it was shown that the product of two pronormal permuting subgroups of a locally soluble FC^*-group is in turn pronormal (see Theorem 2.3.6), and the join of any chain of pronormal subgroups of a finite-by-nilpotent group is pronormal (see Lemma 2.4.3).

It is easy to check that a subgroup H of a group G is normal if and only if it is both pronormal and subnormal. In particular every group whose subgroups are pronormal is a T-group (i.e. a group in which normality is a transitive relation). On the other hand using some basic properties of T-groups (see [40]) it is shown in [11] that a group whose cyclic subgroups are pronormal is a $\overline{T}$-group (i.e. a group whose subgroups are T-groups). It is well known that for a finite group G the property $\overline{T}$ is equivalent to say that G has all pronormal subgroups (see [38] and Lemma 9 in [10]), but this fails to be true for infinite periodic soluble groups (see [49]). More recently the same characterization has been extended to FC-groups (see Theorem 3.9 in [11]), and furthermore to FC^*-groups (see Theorem 4.6 in [13]). Clearly G is a T-group precisely when $\omega(G) = G$. Here $\omega(G)$ denotes the Wielandt subgroup of G, namely, the intersection of all the normalisers of subnormal subgroups of G. In Theorem 1.3.2, we show that every FC^*-group is either an FC-group or it has a Wielandt subgroup much smaller than G. In the same section we highlight some other relevant properties of the Wielandt subgroup of an FC^*-group (see Theorem 2.4.4 and 2.4.6).

Most of our notation is standard and can be found, for instance, in [43].

CHAPTER 1

Generalized FC-groups

The purpose of this chapter is essentially to present some results concerning the FC^*-groups. Many of the following results can be studied for a wider class of groups, denoted by FC^∞, namely, the class of all groups G such that for any element x of G the group $G/C_G(\langle x\rangle^G)$ has the property FC^n for some non-negative integer n.

1.1 FC*-groups: basic results

In this section, the theory of FC-groups can be taken as a model in order to point out many properties that hold also for FC^n-groups, with $n \geq 2$. On the other hand, we also present some results concerning FC-groups that do not have an analogue in the class of FC^*-groups.

The first lemmas deal with closure properties of the class of FC^n-groups and are easily proved by induction on n. It follows of course that the same properties also hold for the class of FC^∞-groups.

Lemma 1.1.1 *Let G be an FC^n-group and N a normal subgroup of G. Then G/N is likewise an FC^n-group.*

Proof − The statement is obvious for $n = 1$. Suppose $n > 1$ and let G be an FC^{n+1}-group. For every element x of G, the factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n. By induction on n, we have that also $\frac{G/C_G(\langle x\rangle^G)}{C_G(\langle x\rangle^G)N/N}$ is an FC^n-group. It follows that

$$\frac{G}{C_G(\langle x\rangle^G)N} \simeq \frac{G/N}{C_G(\langle x\rangle^G)N/N} \simeq \frac{G/N}{C_{G/N}(\langle xN\rangle^{G/N})}$$

is an FC^n-group. The arbitrary choice of x in G ensures that the statement is true. □

Lemma 1.1.2 *Let G be an FC^n-group and H a subgroup of G. Then H is likewise an FC^n-group.*

Proof − The statement is obvious for $n = 1$. Suppose $n > 1$ and let G be an FC^{n+1}-group. For every element x of H, the factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n and hence, by induction on n, we have that

$$HC_G(\langle x\rangle^G)/C_G(\langle x\rangle^G) \simeq H/H \cap C_G(\langle x\rangle^G) \simeq H/C_H(\langle x\rangle^G)$$

is an FC^n-group. By Lemma 1.1.1, it follows that

$$\frac{H/C_H(\langle x\rangle^G)}{C_H(\langle x\rangle^H)/C_H(\langle x\rangle^G)} \simeq H/C_H(\langle x\rangle^H)$$

is an FC^n-group, as required. □

Lemma 1.1.3 *Let H, K be FC^n-groups. Then $G = H \times K$ is likewise an FC^n-group. Moreover, if for each $i \in I$ H_i is an FC^n-group, then $G = \operatorname*{Dr}_{i\in I} H_i$ is likewise an FC^n-group.*

Proof − The statement is obvious for $n = 1$. Suppose $n > 1$ and let H, K be FC^{n+1}-groups. For each element $x = hk$ of G ($h \in H$ and $k \in K$), we have that

$$H/C_H(\langle h\rangle^H) \in FC^n \quad \text{and} \quad K/C_K(\langle k\rangle^K) \in FC^n$$

and by Lemma 1.1.1 it follows that

$$H/C_G(\langle h\rangle^G) \in FC^n \quad \text{and} \quad K/C_G(\langle k\rangle^G) \in FC^n.$$

Consider now the homomorphism φ defined as follows:

$$\varphi : G \longmapsto \frac{H}{C_G(\langle h\rangle^G)} \times \frac{K}{C_G(\langle k\rangle^G)}$$

$$h_1k_1 \mapsto (h_1C_G(\langle h\rangle^G), k_1C_G(\langle k\rangle^G))$$

Because of Lemma 1.1.2, it follows that $G/Ker(\varphi)$ is an FC^n-group. It's easy to prove that $Ker(\varphi) = C_G(\langle h\rangle^G) \cap C_G(\langle k\rangle^G) \leq C_G(\langle x\rangle^G)$, and hence the lemma follows from Lemma 1.1.1. □

For each positive integer n, let G_n be a finitely generated nilpotent group of class n such that the factor group $G_n/Z_{n-1}(G_n)$ is infinite. Thus G_n is an FC^n-group which does not belong to the class FC^{n-1} (see Proposition 1.1.10), so that

$$FC^0 \subset FC^1 \subset \ldots \subset FC^n \subset FC^{n+1} \subset \ldots$$

Moreover, the direct product

$$G = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, G_n$$

is an FC^∞-group which does not have the property FC^n for any n, and hence $FC^* \subset FC^\infty$.

Let G be a group and $F(G)$ the FC-center of G. The *upper FC-central series* of G is defined as the ascending normal series of G whose terms $F_\alpha(G)$ are defined by the positions

$$F_0(G) = \{1\}, \quad F_{\alpha+1}(G)/F_\alpha(G) = F(G/F_\alpha(G))$$

for every ordinal α, and

$$F_\lambda(G) = \bigcup_{\beta<\lambda} F_\beta(G)$$

if λ is a limit ordinal. The group G is called FC-nilpotent if $G = F_n(G)$ for some non-negative integer n, and this means it has briefly a normal series whose factors are FC-groups. It has been proved (see Theorem 1.1.6) that if G is any FC^n-group, then $F_n(G) = G$, and so G is FC-nilpotent. On the other hand, the following result is useful to shows that the class of FC^*-groups is properly contained in the class of all FC-nilpotent groups.

Lemma 1.1.4 *Let G be an FC^∞-group containing a finite non-empty subset X such that $C_G(\langle X\rangle^G) = \{1\}$. Then G is finite.*

Proof – By hypothesis, for every $x \in X$ the group $G/C_G(\langle x\rangle^G)$ has the property FC^{n_x} for some non-negative integer n_x. Consider the homomorphism between G and the $\mathrm{Dr}_{x\in G}\, G/C_G(\langle x\rangle^G)$ that sends every element y of G in $(yG/C_G(\langle x\rangle^G))_{x\in G}$. Since X is finite and $C_G(\langle X\rangle^G) = \{1\}$, we have that G itself is an FC^n-group for some non-negative integer n. Choose the smallest m such that G has the property FC^m, and assume $m > 0$. Then $G/C_G(\langle x\rangle^G)$ is an FC^{m-1}-group for each element x of X, and hence also G is an FC^{m-1}-group. This contradiction proves the lemma. □

As a consequence of the above result, if A is any infinite abelian group without elements of order 2 and x is the automorphism of A defined by $a^x = a^{-1}$ for all $a \in A$, it is possible to claim that the semidirect product

$G = \langle x \rangle \ltimes A$ does not have the property FC^∞. In fact, if in A there is no element of order 2 it follows that $C_G(\langle X \rangle^G) = \{1\}$ for every finite non-empty subset X of G. Since G is infinite, by Lemma 1.1.4 we can conclude that G is not an FC^∞-group, and hence the class of FC^*-groups is properly contained in the class of all FC-nilpotent groups.

A result of Neumann ([37]) states that the commutator subgroup of any FC-group is locally finite so, in particular, the set of all elements of finite order of an FC-group is a subgroup and torsion-free FC-groups are abelian. The next results prove that similar properties also hold for FC^n-groups when $n > 1$ and they will be very useful to obtain some information about the terms of the lower central series of an FC^n-group.

Lemma 1.1.5 *Let G be an FC^n-group (where n is a positive integer), and let $x_1, \dots, x_t$ be elements of G with $t \leq n$. Then $G/C_G(\langle [x_1, \dots, x_t] \rangle^G)$ is an FC^{n-t}-group.*

Proof – The statement is obvious if $t = 1$. Suppose $t > 1$, and write $y = [x_1, \dots, x_{t-1}]$, so that by induction on t the factor group $G/C_G(\langle y \rangle^G)$ can be assumed to have the property FC^{n-t+1}. Put $C_1 = C_G(\langle y \rangle^G)$ and $C_2 = C_G(\langle x_t \rangle^G)$, and consider the centralizer $H/C_1 = C_{G/C_1}(\langle x_t \rangle^G C_1/C_1)$ and $K/C_2 = C_{G/C_2}(\langle y \rangle^G C_2/C_2)$. Since G/C_1 in an FC^{n-t+1} and G/C_2 is an FC^{n-1}-group, we can consider the homomorphism between G and $G/H \times G/K$ that sends every element y of G in (yH, yK). Since the product $G/H \times G/K$ is an FC^{n-t}-group, it follows that $G/H \cap K$ has also the property FC^{n-t}. Moreover,

$$[H \cap K, \langle y \rangle^G, \langle x_t \rangle^G] = [\langle x_t \rangle^G, H \cap K, \langle y \rangle^G] = \{1\}$$

so that

$$[\langle y \rangle^G, \langle x_t \rangle^G, H \cap K] = \{1\}$$

by the Three Subgroup Lemma. Therefore the subgroup $H \cap K$ is contained in the centralizer $C_G([x_1, \ldots, x_t])$ and hence $G/C_G(\langle [x_1, \ldots, x_t] \rangle^G)$ is an FC^{n-t}-group, and the lemma is proved. □

Theorem 1.1.6 ([13]) *Let G be an FC^n-group (where n is a positive integer). Then the subgroup $\gamma_n(G)$ is contained in the FC-center of G. In particular, G is FC-nilpotent and its upper FC-central series has length at most n.*

Proof – The statement follows directly from Lemma 1.1.5, for $t = n$. □

Corollary 1.1.7 *Let G be an FC^n-group (where n is a positive integer). Then the subgroup $\gamma_{n+1}(G)$ is periodic. In particular, the elements of finite order of G form a subgroup.*

Proof – Clearly we may suppose $n > 0$. By Theorem 1.1.6, the subgroup $\gamma_n(G)$ is an FC-group. It follows that $\gamma_n(G)'$ is periodic and replacing G by $G/\gamma_n(G)'$ it can be assumed, without loss of generality, that $\gamma_n(G)$ is abelian. Consider elements $x \in \gamma_n(G)$ and $g \in G$. Since $G/C_G(\langle g \rangle^G)$ is an FC^{n-1}-group, by induction on n, we obtain that $\gamma_n(G/C_G(\langle g \rangle^G))$ is a periodic group, so $x^k \in C_G(\langle g \rangle^G))$ for some positive integer k. Thus, $[x, g]^k = [x^k, g] = 1$ and the commutator $[x, g]$ has finite order. Therefore the abelian group $\gamma_{n+1}(G)$ is generated by elements of finite order, hence it is periodic. □

Corollary 1.1.8 *Let G be torsion-free FC^n-group. Then G is nilpotent with class at most n.*

Proof – The statement follows directly from Lemma 1.1.7. □

Recall that the group class $\mathfrak{X}$ is called a *Schur class* if for any group G such that the factor group $G/Z(G)$ belongs to $\mathfrak{X}$, also the commutator subgroup G' of G is an $\mathfrak{X}$-group. Thus the famous Schur's Theorem just states that finite groups form a Schur class (see [41, page 287]). It is known that the group classes determined by the most natural finiteness conditions have the Schur property. For the class of FC-groups this has been explicitly proved in [9], but it is possible to see also as a consequence of the following result.

Corollary 1.1.9 *For each non-negative integer n, the class of all FC^n-groups is a Schur class.*

Proof – If $n = 0$ the statement is the well known Schur's Theorem. Suppose $n > 0$, and let G be any group such that $G/Z(G)$ has the property FC^n. Since $Z(G) \leq C_G(\langle g\rangle^G)$ for every $g \in G$, it follows that $G/C_G(\langle g\rangle^G)$ is obviously an FC^n-group for every element g of G, hence G is an FC^{n+1}-group. Let x and y be elements of G. By Lemma 1.1.5, $G/C_G(\langle [x,y]\rangle^G)$ is an FC^{n-1}-group. It follows that $G'/C_{G'}(\langle a\rangle^{G'})$ has the property FC^{n-1} for every $a \in G'$, and hence G' is an FC^n-group. □

Observe that, if G is any group such that $G/Z(G)$ is an FC^n-group for some non-negative integer n, then G obviously has the property FC^{n+1}, and hence the subgroup $\gamma_{n+1}(G)$ is contained in the FC-center of G.
In relation to the above corollary, we mention also that Maier and Rogerio have proved in [34] that FC^n-groups form a Dietzmann class for every non-negative integer n. Here a group class $\mathfrak{X}$ is said to be a *Dietzmann class* if it is closed with respect to subgroups and homomorphic images, and for every element x of a group G such that $\langle x\rangle$ and $G/C_G(\langle x\rangle^G)$ are $\mathfrak{X}$-groups, also the normal closure $\langle x\rangle^G$ belongs to $\mathfrak{X}$. Thus the well known Dietzmann's Lemma states that the class $\mathfrak{F}$ of all finite groups is a Dietzmann class (see [41, page 442]).

It has already been noted that every central-by-finite group is an FC-groups. B.H. Neumann ([37]) proved that, for finitely generated FC-group, the converse is also true. The next result generalizes this property to finitely generated FC^n-groups for any non-negative integer n.

Proposition 1.1.10 *Let G be a finitely generated group. Then G is an FC^n-group for some non-negative integer n if and only if the factor group $G/Z_n(G)$ is finite.*

Proof − Suppose first that G is a finitely generated FC^n-group with $n > 0$, and so $G/C_G(\langle x\rangle^G)$ is an FC^{n-1} for every element x of G. If we consider the homomorphism between G and $\mathop{\mathrm{Dr}}\limits_{x\in G} G/C_G(\langle x\rangle^G)$ that sends every element y of G in $(yG/C_G(\langle x\rangle^G))_{x\in G}$, we have that also $G/Z(G)$ is an FC^{n-1}-group. By induction on n, it follows that $G/Z_n(G)$ is finite.

Conversely, if the factor group $G/Z_n(G)$ is finite for some positive integer n, again by induction on n, we obtain that $G/Z(G)$ has the property FC^{n-1}. In particular, $G/C_G(\langle x\rangle^G)$ is an FC^{n-1}-group for each element x of G, and hence G is an FC^n-group. □

It is important to remark that in the second part of the above proof, the hypothesis that the group is finitely generated is unnecessary.

Lemma 1.1.11 *Let G be an FC^n-group (where n is a positive integer), and let X be a finite subset of G. Then $\langle X\rangle^G/Z_{n-1}(\langle X\rangle^G)$ is finitely generated and $\langle X\rangle^G/Z_n(\langle X\rangle^G)$ is finite.*

Proof − If $n = 1$, G is an FC-group and the statement is obvious. Suppose $n > 1$, and put $C = C_G(\langle X\rangle^G)$. As X is finite, the factor group G/C has the property FC^{n-1}, and hence by induction on n we have that the factors

$$\frac{\langle X\rangle^G C/C}{Z_{n-2}(\langle X\rangle^G C/C)} \quad \text{and} \quad \frac{\langle X\rangle^G C/C}{Z_{n-1}(\langle X\rangle^G C/C)}$$

are finitely generated and finite, respectively. It follows that the group

$$\frac{\langle X\rangle^G C/C}{Z_{n-2}(\langle X\rangle^G C/C)}$$

is finitely generated and central-by-finite. On the other hand, the group $\langle X\rangle^G C/C$ is isomorphic to $\langle X\rangle^G/Z(\langle X\rangle^G)$, and so $\langle X\rangle^G/Z_{n-1}(\langle X\rangle^G)$ is a finitely generated central-by-finite group. The lemma is proved. □

Recall that a group G *satisfies the maximal condition on subgroups*, or briefly *satisfies Max*, if each nonempty subset of subgroups contains at least one maximal element. This property excludes the possibility of infinite well-ordered ascending chains of subgroups. In particular, a group G *satisfies Max locally* if and only if its finitely generated subgroups satisfy Max.

Lemma 1.1.12 *Let G be an FC^*-group. Then G satisfies Max locally.*

Proof – Without loss of generality, we can assume that G is finitely generated FC^n-group, where n is a positive integer. By Proposition 1.1.10, the factor $G/Z_n(G)$ is finite. It follows that $Z_n(G)$ is a finitely generated nilpotent group, and so it is polycyclic. Hence G is polycyclic-by-finite and so G satisfies Max. The lemma is proved. □

Lemma 1.1.13 *Let G be periodic finitely generated FC^*-group. Then G is finite.*

Proof – By Lemma 1.1.12, G satisfies Max and so $Z_n(G)$ is a finitely generated nilpotent group. Hence both $Z_n(G)$ and $G/Z_n(G)$ are finite, and so G is finite. □

It is important to remark the important role of normal closures in the study of FC-groups. It is simple to shown that if G is an FC-group, then $\langle x\rangle^G$ in a finitely generated subgroup of G for every x in G. Moreover, a result of Polovickiĭ states that a group G is an FC-group if and only if $\langle x\rangle^G$ is finite-by-cyclic for every $x \in G$ ([43]). In order to prove a similar result for FC^n-groups, the focus shifts on the terms of the series of successive normal closures.

Recall that, if H is an arbitrary subset of G, *the series of successive normal closures* of H in G is the descending series $\{H^{G,\alpha}\}$ between H and G defined inductively by

$$H^{G,0} = G, \quad H^{G,\alpha+1} = H^{H^{G,\alpha}} \quad \text{and} \quad H^{G,\lambda} = \bigcap_{\beta<\lambda} H^{G,\beta}$$

where α is an ordinal and λ a limit ordinal. It is well known that this is the fastest descending series whose all terms contain H, and it is easy to show by induction that for every positive integer n that $H^{G,n} = H[G,_n H]$.

Lemma 1.1.14 *Let G be an FC^n-group. Then $\langle x\rangle^{G,n}$ is finitely generated for every $x \in G$.*

Proof − Let x be an element of G. By Lemma 1.1.11, the factor group $\langle x\rangle^G/Z_{n-1}(\langle x\rangle^G)$ is finitely generated, and hence it follows that $\langle x\rangle^G \leq YZ_{n-1}(\langle x\rangle^G)$, where Y is a finitely generated subgroup of G containing x. We will show that $\langle x\rangle^{G,m} \leq YZ_{n-m}(\langle x\rangle^G)$ for each $m \in \{1, \dots, n\}$.
By contradiction, let r be the minimum positive integer less then n such that $\langle x\rangle^{G,r}$ is not contained in $YZ_{n-r}(\langle x\rangle^G)$. Clearly $r > 1$, so that

$$\langle x\rangle^{G,r} \leq \langle x\rangle^{\langle x\rangle^{G,r-1}} \leq \langle x\rangle^{YZ_{n-(r-1)}(\langle x\rangle^G)} \leq (\langle x\rangle^Y)^{Z_{n-(r-1)}(\langle x\rangle^G)} \leq$$
$$\leq \langle x\rangle^Y[\langle x\rangle^Y, Z_{n-(r-1)}(\langle x\rangle^G)] \leq \langle x\rangle^Y[\langle x\rangle^G, Z_{n-(r-1)}(\langle x\rangle^G)] \leq$$
$$\leq \langle x\rangle^Y Z_{n-r}(\langle x\rangle^G) \leq YZ_{n-r}(\langle x\rangle^G).$$

From this contradiction, it follows that $\langle x\rangle^{G,n}$ is a subgroup of Y. As Y satisfies Max, it follows that $\langle x\rangle^{G,n}$ is finitely generated. □

Corollary 1.1.15 *Let G be an FC^n-group. Then $\langle x\rangle^{G,n}$ is finite-by-cyclic and $Aut\langle x\rangle^{G,n}$ is finite for every $x \in G$.*

Proof – Since $\langle x\rangle^{G,n} = \langle x\rangle[G,_n \langle x\rangle]$, we have to show that $[G,_n \langle x\rangle]$ is finite. By definition, $[G,_n \langle x\rangle]$ is contained in the periodic subgroup $\gamma_{n+1}(G)$ of G (see 1.1.7). On the other hand, $\langle x\rangle^{G,n}$ is finitely generated by Lemma 1.1.14, so that satisfies Max. It follows that the subgroup $[G,_n \langle x\rangle]$ is periodic and finitely generated, so that it is finite. In addition, it is easy to show that if H is a finite-by-cyclic group, then $AutH$ is finite. □

Recall that a *local system* of normal subgroups of G is a family $\{F_i : i \in I\}$ of normal subgroups of G such that:

i) $G = \bigcup_{i\in I} F_i$,

ii) for each $i, j \in I$, there is a $k \in I$ such that $F_iF_j \leq F_k$.

It follows from Dietzmann's Lemma that a periodic FC-group is generated by its finite normal subgroups. In other words, a periodic group has the property FC if and only if it has a local system of finite subgroups. This result can be considered as a special case of the following property of FC^n-groups.

Theorem 1.1.16 ([13]) *Let G be a periodic group. Then G is an FC^n-group for some positive integer n if and only if it has a local system $\mathfrak{L}$ consisting of normal subgroups such that $L/Z_{n-1}(L)$ is finite for every element L of $\mathfrak{L}$.*

Proof – Suppose first that G is an FC^n-group, and let X be any finite subset of G. Then $\langle X\rangle^G/Z_{n-1}(\langle X\rangle^G)$ is finite by Lemma 1.1.11, and so the local system consisting of the normal closures of all finite subsets of G satisfies the condition of the statement.

Conversely, assume that G has a local system $\mathfrak{L}$ such that every element L of $\mathfrak{L}$ is a normal subgroup of G and $L/Z_{n-1}(L)$ is finite. If $n = 1$, G is covered by its finite normal subgroups, and hence it is an FC-group. Suppose $n > 1$. Let x be any element of G, and consider the set $\mathfrak{L}_x$ of all subgroups $L \in \mathfrak{L}$ such that x belongs to L. If $L \in \mathfrak{L}_x$, the center $Z(L)$ is contained in $L \cap C_G(\langle x \rangle^G)$, and so $Z_{n-2}(L/L \cap C_G(\langle x \rangle^G))$ has finite index in $L/L \cap C_G(\langle x \rangle^G)$. It follows that

$$\{LC_G(\langle x \rangle^G)/C_G(\langle x \rangle^G) : L \in L_x\}$$

is a local system of $G/C_G(\langle x \rangle^G)$ consisting of normal subgroups whose $(n-2)$-th term of the upper central series has finite index, and hence by induction on n the group $G/C_G(\langle x \rangle^G)$ has the property FC^{n-1}. Therefore G is an FC^n-group. □

Recall that a *minimal normal subgroup* of a group G is a nontrivial normal subgroup that does not contain a smaller nontrivial normal subgroup of G. The subgroup generated by all the minimal normal subgroups of a group G is called the *socle of* G and it is denoted by $Soc(G)$. If the group fails to have any minimal normal subgroups, as in the case of the infinite cyclic group for example, the socle of G is defined to be 1. If G is a periodic FC-group, then $Soc(G)$ is a direct product of finite normal subgroups of G. The *upper socle series* of a group G is the ascending characteristic series $\{S_\lambda\}$ defined by taking

$$S_0 = 1 \quad , \quad S_{\alpha+1}/S_\alpha = \text{the socle of } \; G/S_\alpha$$

for each ordinal α and $S_\lambda = \bigcup_{\beta<\lambda} S_\beta$, if λ is a limit ordinal This series terminates after a finite or infinite number of steps but, of course, it may not reach G. However if G satisfies Min-n, each non-trivial homomorphic image of G has non-trivial socle, so G coincides with some S_α. Also in the case of a periodic FC-group, the upper socle series terminates in ω steps (see

[51, page 10]). Using this property, in [11] it is shown that locally soluble (respectively, locally nilpotent) FC-groups are hyperabelian (respectively, hypercentral) with length at most ω and have an abelian (respectively, a central) descending normal series of length at most $\omega + 1$. The following two theorems extend these results to the case of FC^n-groups.

Theorem 1.1.17 ([13]) *Let G be an FC^n-group for some positive integer n.*

(a) If G is locally soluble, then it has an ascending characteristic series with abelian factors of length at most $\omega + (n - 1)$.

(b) If G is locally nilpotent, then G is hypercentral and its upper central series has length at most $\omega + (n - 1)$.

PROOF – (a) For each non-negative integer k, let $S_k/Z(\gamma_n(G))$ be the k-th term of the upper socle series of the group $\gamma_n(G)/Z(\gamma_n(G))$. Clearly every S_k is a characteristic subgroup of G, and S_{k+1}/S_k is abelian since G is locally soluble. Moreover, $\gamma_n(G)$ is an FC-group by Theorem 1.1.6, so

$$\gamma_n(G) = \bigcup_{k \in \mathbb{N}_0} S_k$$

and the group G has an ascending characteristic series with abelian factors of length at most $\omega + (n - 1)$.
(b)By Theorem 1.1.6, the subgroup $\gamma_n(G)$ of G lies in the FC-center of G.herefore $\gamma_n(G)$ is also contained in $Z_\omega(G)$ (see [43], Theorem 4.38). It follows that $G/Z_\omega(G)$ is nilpotent with class at most $n-1$ and G is hypercentral group with $Z_{\omega+(n-1)}(G) = G$. □

Recall that a group G is said to be *residually finite* if, given $g \neq 1$ in G, there is $N \lhd G$ such that $g \notin N$ and G/N is finite. It is well known that if G is a group and F is the FC-center of G, then $G/C_G(F)$ is residually finite.

In particular, if G is an FC-group, then $G/Z(G)$ is residually finite (see [51, page 6]). More generally, we have the next result.

Lemma 1.1.18 *Let G be an FC^n-group for some positive integer n. Then the factor $G/Z_n(G)$ is residually finite.*

PROOF – The statement is obvious if $n = 0$. Suppose $n > 0$, and for each element x of G, put

$$Z_x/C_G(\langle x\rangle^G) = Z_{n-1}(G/C_G(\langle x\rangle^G)).$$

As $G/C_G(\langle x\rangle^G)$ is an FC^{n-1}-group, by induction on n we have that G/Z_x is a residually finite group. On the other hand,

$$Z_n(G) = \bigcap_{x\in G} Z_x,$$

and hence $G/Z_n(G)$ is likewise residually finite. □

Theorem 1.1.19 ([13]) *Let G be an FC^n-group for some positive integer n.*

(a) If G is locally soluble, then G is hypoabelian and its derived series has length at most $\omega + n$.

(b) If G is locally nilpotent, then G is hypocentral and its lower central series has length at most $\omega + n$.

PROOF – Lemma 1.1.18 tells us that the group $G/Z_n(G)$ is residually finite. Therefore it is enough to observe that, if X is any residually finite locally soluble (locally nilpotent, respectively) group, then $X^\omega = \{1\}$ ($\gamma_\omega(X) = \{1\}$, respectively). □

The next result deals with the study of chief factors and maximal subgroups of FC^∞-groups. Once again, this theorem generalizes known results concerning FC-groups.

Theorem 1.1.20 ([13]) *Let G be a FC^∞-group. Then every chief factor of G is finite and every maximal subgroup of G has finite index.*

Proof – As the class of FC^∞-groups is closed under homomorphic images, in order to prove the first part of the statement it is enough to show that every minimal normal subgroup of G is finite. By contradiction, assume that G contains an infinite minimal normal subgroup N. Let g be an element of $G/C_G(N)$. Then $G/C_G(\langle g\rangle^G)$ is an FC^n-group for some non-negative integer n and by Theorem 1.1.6 the subgroup $\gamma_n(G/C_G(\langle g\rangle^G))$ lies in the FC-center of $G/C_G(\langle g\rangle^G)$. On the other hand, N is obviously contained in $\gamma_n(G)$, and hence $NC_G(\langle g\rangle^G)/C_G(\langle g\rangle^G)$ is finite. As $N\cap C_G(\langle g\rangle^G)=\{1\}$, it follows that N is finite, a contradiction.

Assume that G contains a maximal subgroup M such that the index $|G:M|$ is infinite. Then M is not normal in G, and replacing G by the factor group G/M_G, we may suppose that M is core-free. Let x be an element of $G\setminus M$, so that $G=\langle x\rangle^G M$ and in particular

$$C_G(\langle x\rangle^G)\cap M=\{1\}.$$

The factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n for some non-negative integer n. Clearly $\gamma_n(G)$ is not contained in M, so $G=\gamma_n(G)M$ and $\gamma_n(G)$ must be infinite. Thus $\gamma_n(G)$ is not a minimal normal subgroup of G, hence $\gamma_n(G)\cap M\neq\{1\}$. Let a be a non-trivial element of $\gamma_n(G)\cap M$. By theorem 1.1.6 the subgroup $\gamma_n(G/C_g(\langle x\rangle^g))$ is contained in the FC-center of $G/C_g(\langle x\rangle^g)$, so that $\langle a\rangle^g C_g(\langle x\rangle^g)/C_g(\langle x\rangle^g)$ satisfies the maximal condition on subgroups. It follows that

$$E=\langle a\rangle^G\cap M\simeq(\langle a\rangle^G\cap M)C_g(\langle x\rangle^g)/C_g(\langle x\rangle^g)$$

is a finitely generated non-trivial subgroup of G. Lemma 1.1.4 tells us that the centralizer $C_G(E^G)$ is a non-trivial subgroup of G, so $G=C_G(E^G)M$

whence $E^G = E^M \leq M$, contradicting the assumption that M is core-free. This contradiction completes the proof of the theorem. □

The *Wielandt subgroups* $\omega(G)$ of a group G is the intersection of all normalisers of the subnormal subgroups of G. Thus a group G is a T-group (i.e. a group in which the normality is a transitive relation) if and only if $\omega(G) = G$. Recall that a subgroup H of a group G is said to be *ascendant* if there is an ascending series between H and G. We denote by $\tau(G)$ the intersection of all the normalisers of ascendant subgroups of G. Clearly every subnormal subgroup is also ascendant, so, for any group G, the subgroup $\tau(G)$ is contained in $\omega(G)$. Moreover, if G is a polycylic-by-finite group, ascendant and subnormal subgroups of G coincide, and hence $\tau(G) = \omega(G)$. We extend these result to the class of FC^*-group.

Lemma 1.1.21 *Let G be an FC^*-group. Then $\omega(G) = \tau(G)$.*

Proof – Let G be an FC^n-group, where n is a positive integer. Let x be an element of $\omega(G)$, and let H be any ascendant subgroup of G. For every element $h \in H$, put $N = \langle h, x\rangle^G$. By Theorem 1.1.11 the factor group $N/Z_n(G)$ is finite, thus $(H \cap N)Z_n(G)$ is subnormal in N and $H \cap N$ is subnormal in G. It follows that x normalises $H \cap N$, and hence h^x belongs to H. The arbitrary choice of h in H yields $x \in N_G(H)$, and so the lemma is proved. □

Here we also mention some aspects of the theory of FC-groups that can not be generalized to FC^n-groups with $n > 2$. For instance, a relevant result by Gorčakov proves that any countable FC-group with trivial center can be embedded in a direct product of finite groups ([21]). The corresponding statement for FC^n-groups with $n \geq 2$ does not hold. Indeed, such a result would, in particular, imply that every FC^2-group with trivial center is an FC-group, while this latter property is in general false. To see this, in the next section we report an example suggested by C. Casolo (see Example 1.3.4).

Moreover, a result of Baer states that if G is any FC-group, then $G/Z(G)$ is periodic (see [51, page 4]). The corresponding result fails for FC^n-groups when $n > 1$. To see this, in the next section we report an example suggested by D.J. Robinson, A.Russo and G. Vincenzi in [44] (see Example 1.3.5).

In [6] S.N. Černikov proved that every FC-group is isomorphic to a subgroup of the direct product of a periodic FC-group and a torsion-free abelian group. The following example shows that a similar result cannot be proved in the case of FC^n-groups with $n \geq 2$.

Example 1.1.22 *There exists an FC^2-group which cannot be embedded in any direct product of a periodic group and a torsion-free group.*

Proof − Consider a periodic FC-group H and a non-periodic nilpotent group K of class 2 such that $Z(K)$ is periodic, and put $G = H \times K$. Let $x = hk$ be any element of G, with $h \in H$ and $k \in K$. Obviously the index $|H : C_H(\langle h\rangle^H)|$ is finite. Since $K/Z(K)$ is abelian and $Z(K) \leq C_K(\langle k\rangle^K)$ for every $k \in K$, it follows that $K/C_K(\langle k\rangle^K)$ is abelian, and hence $K' \leq C_K(\langle k\rangle^K)$. Moreover,

$$C_G(\langle x\rangle^G) = C_H(\langle h\rangle^H) \times C_K(\langle k\rangle^K).$$

It is simple to show that the factor group $G/C_G(\langle x\rangle^G)$ is finite-by-abelian, and hence it is an FC-group and in particular G is an FC^2-group. If N is any torsion-free normal subgroup of G, we have $N \cap Z(K) = \{1\}$. Since K is nilpotent, it follows that $N \cap K = \{1\}$ and hence $N = \{1\}$. Therefore G cannot be embedded into the direct product of a periodic group and a torsion-free group. □

1.2 Subnormal subgroup

A subgroup H of a group G is said to be *serial* in G if there exists a series containing both H and G, i.e. a complete chain Σ of subgroups of G such that if U and V are consecutive terms of Σ, then U is normal in V. If the series between H and G can be chosen to be finite (respectively, well-ordered), we obtain in this way the usual concept of a subnormal (respectively, ascendant) subgroup. A useful result of Hartley (see [23, Lemma 1]) shows that if H is a subgroup of G and there exists a local system $\mathfrak{L}$ of G such that $H \cap L$ is subnormal in L for each $L \in \mathfrak{L}$, then H is a serial subgroup of G.
In [50] Tôgô proved that any serial subgroup of an FC-group is ascendant with length at most ω. What follows is useful to generalize this property to FC^n-groups.

Lemma 1.2.1 *Let G be a finite-by-nilpotent group. Then every serial subgroup of G is subnormal.*

Proof $-$ Let N be a finite normal subgroup of G such that the factor group G/N is nilpotent. If H is any serial subgroup of G, the index $|HN : H|$ is finite, so H is subnormal in HN and hence it is also a subnormal subgroup of G. □

Lemma 1.2.2 *Let G be a locally (finite-by-nilpotent) group. Then the join of any collection of serial subgroups of G is a serial subgroup.*

Proof $-$ Let $(H_i)_{i \in I}$ be a system of serial subgroup of G, and put $H = \langle H_i : i \in I \rangle$. Consider any finitely generated subgroup E of G. Then E satisfies the maximal condition on subgroups, hence $H \cap E$ is contained in $\langle L_{i_1}, \dots, L_{i_t} \rangle$, where $\{i_1, \dots, i_t\}$ is a finite subset of I and L_{i_j} is a suitable finitely generated subgroup of H_{i_j} for each $j \leq t$. Clearly the subgroup $L = \langle E, L_{i_1}, \dots, L_{i_t} \rangle$ is finitely generated, so it is finite-by-nilpotent. By Lemma 1.2.1 every $H_i \cap L$ is subnormal in L. It follows that also $\langle H_{i_1} \cap L, \dots, H_{i_t} \cap L\rangle$ is subnormal in L, hence the subgroup

$$H \cap E = \langle H_{i_1} \cap L, \dots, H_{i_t} \cap L\rangle \cap E$$

is subnormal in E. Therefore H is a serial subgroup of G. □

Lemma 1.2.3 *Let G be a group in which the join of any collection of ascendant subgroups is ascendant and let F be the FC-center of G. If H is an ascendant subgroup of G, then there exists an ascending series from H to HF of length at most ω.*

Proof – We may obviously assume that H is properly contained in HF. Put $H_0 = H$, and suppose that for some non-negative integer n an ascendant subgroup H_n of G has been defined such that $H \leq H_n < HF$. Let $\mathfrak{L}_n$ be the local system of F consisting of the normal closures of all finite subsets of F which are contained in H_n. Let E be any element of $\mathfrak{L}_n$. Since F is the FC-center of G, E satisfies the maximal condition on subgroups, so in particular all ascendant subgroups of E are subnormal and the join of any collection of subnormal subgroups of E is likewise subnormal. As H_n is a proper ascendant subgroup of H_nE, there exists another ascendant subgroup K of H_nE such that H_n is a proper normal subgroup of K. It follows that $E \cap K$ is a subnormal subgroup of E normalizing H_n and $E \cap H_n < E \cap K$. Let $X_n(E)$ be the subgroup generated by all subnormal subgroup of E and

$$E \cap H_n < X_n(E) \leq N_G(H_n).$$

Moreover, if E_1 and E_2 are elements of $\mathfrak{L}_n$ such that $E_1 \leq E_2$, clearly $X_n(E_1)$ is a subnormal subgroup of E_2 normalizing H_n, and hence $X_n(E_1)$ is contained in $X_n(E_2)$. It follows that

$$X_n = \bigcup_{E \in \mathcal{L}_n} X_n(E)$$

is a subgroup of F normalizing H_n, and H_n is a proper normal subgroup of $H_{n+1} = H_n X_n$. As X_n is generated by subnormal subgroups of G, we also have that H_{n+1} is an ascendant subgroup of G. In this way we can define an ascending series

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_n \lhd H_{n+1} \lhd \ldots$$

of ascendant subgroups of G contained in HF. Assume that the subgroup

$$\bigcup_{n \in \mathbb{N}_0} H_n$$

is properly contained in HF. Then there exists an element x of F such that $\langle x \rangle^G$ belongs to each $\mathfrak{L}_n$, and hence

$$\langle x \rangle^G \cap H_0 < \langle x \rangle^G \cap H_1 < \langle x \rangle^G \cap H_2 < \ldots < \langle x \rangle^G \cap H_n < \ldots$$

is an infinite ascending series of subgroups of $\langle x \rangle^G$, a contradiction. Therefore

$$HF = \bigcup_{n \in \mathbb{N}_0} H_n,$$

and the lemma is proved. □

Theorem 1.2.4 ([13]) *Let G be a FC^n-group, where n is a positive integer. Then every serial subgroup H of G is ascendant, and there exists an ascending series from H to G of length at most $\omega + (n-1)$.*

Proof − Since G is an FC^n-group, it follows from Theorem 1.1.6 that the subgroup $\gamma_n(G)$ is contained in the FC-center of G. Moreover, by Corollary 1.1.7 $L = \gamma_{n+1}(G)$ is periodic, so it has an ascending G-invariant series

$$\{1\} = L_0 < L_1 < \ldots < L_\tau = L$$

with finite factors. By Proposition 1.1.10, the group G is locally (finite-by-nilpotent), and hence Lemma 1.2.2 yields that for each ordinal $\alpha < \tau$ the product HL_α is a serial subgroup of G, so that HL_α is subnormal in $HL_{\alpha+1}$ as the index $|HL_{\alpha+1} : HL_\alpha|$ is finite. Therefore the subgroup H is ascendant in HL, and so even in G. A second application of Lemma 1.2.2 yields now that the join of any collection of ascendant subgroup of G is likewise ascendant. As $\gamma_n(G)$ is a subgroup of the FC-center of G, it follows from Lemma 1.2.3 that there exists an ascending series from H to $H\gamma_n(G)$ of type at most ω; on the other hand, $H\gamma_n(G)$ is subnormal in G with defect at most $n-1$, and hence there exists an ascending normal series from H to G of type at most $\omega + (n-1)$. □

Corollary 1.2.5 *Let G be an FC^n-group, where n is a non-negative integer. If all subgroups of G are serial, then G is hypercentral.*

Proof – It follows from Theorem 1.2.4 that every subgroups of G is ascendant. Thus G is locally nilpotent, and so even hypercentral by Theorem 1.1.17. □

1.3 FC*-groups that are not FC-groups

The aim of this section is to exhibit some examples of FC^*-groups that are not FC-groups. Of course, one example can be a nilpotent group that is not an FC-group. We first present some cases in which the conditions FC and FC^* are equivalent. In [13], the authors show that this holds for T-groups.

Theorem 1.3.1 ([13]) *Let G be an FC^∞-group in which normality is a transitive relation. Then G is an FC-group.*

The previous result can be considered as a special case of the following theorem. Here we highlight that the proof doesn't depend on Theorem 1.3.1.

Theorem 1.3.2 ([46]) *Let G be an FC^∞-group in which the Wielandt subgroup $\omega(G)$ has finite index. Then G is an FC-group.*

Proof − Suppose first that G is an FC^n-group, where n is a positive integer. Let x be an element of G. Since $\langle x\rangle^{G,n}$ is a subnormal subgroup of G, it follows that $\omega(G)\langle x\rangle^{G,n} \leq N_G(\langle x\rangle^{G,n})$ and so $|G : N_G(\langle x\rangle^{G,n})| < \infty$. By Corollary 1.1.15, $N_G(\langle x\rangle^{G,n})/C_G(\langle x\rangle^{G,n})$ is finite and so $|G : C_G(\langle x\rangle^{G,n})|$ is also finite. This proves that x is an FC-element and hence G is an FC-group. Suppose now that G is an FC^∞-group. If x is any element of G, the factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n for some non-negative integer n. Combining the fact that the index $|G : \omega(G)C_G(\langle x\rangle^G)|$ is finite and $\omega(G)C_G(\langle x\rangle^G)/C_G(\langle x\rangle^G) \leq \omega(G/C_G(\langle x\rangle^G))$, from the first part of the proof it follows that $G/C_G(\langle x\rangle^G)$ is an FC-group for every element x of G. This means that G is an FC^2-group in which the Wielandt subgroup $\omega(G)$ has finite index. Again for the first part of the proof, it follows that G is an FC-group. □

As a consequence of the these results, if G is an FC^*-group and not an FC-group, then the index $|G : \omega(G)|$ is infinite. We also remark that there are easy examples of infinite FC-groups with trivial center in which normality is a transitive relation. So an FC^*-group whose Wielandt subgroup has finite index may not be centre-by-finite. A well known theorem of B.H. Neumann states that a group G is centre-by-finite if and only if every subgroup has finitely many conjugates. It follows that if G is nilpotent, then $Z(G)$ has finite index in G if and only if $\omega(G)$ has finite index in G. Using this characterization and the Schur's Lemma, this property can be easily extended to finite-by-nilpotent groups. This equivalence can be also found as an application of Theorem 1.3.2.

Corollary 1.3.3 *Let G be a finite-by-nilpotent group. If $|G:\omega(G)|$ is finite, then $G/Z(G)$ is finite.*

Proof – By hypothesis, $G/Z_n(G)$ is finite for some positive integer n, in particular G is an FC^*-group, and hence it is an FC-group by Theorem 1.3.2. On the other hand, $Z_n(G)\cap\omega(G)$ is a nilpotent T-group and so is abelian-by-finite. It follows that G is an(abelian-by-finite)-by-finite FC-group. Let A be an abelian-by-finite normal subgroup of G such that the factor group G/A is finite. Let $x_1,\dots,x_t$ be elements of G such that $G=\langle x_1,\dots,x_t,A\rangle$. For each $i\le t$, the factor group $G/C_G(\langle x_i\rangle^G)$ is finite. Considering the homomorphism ϕ between G and $G/A\times G/C_G(\langle x_1\rangle^G))\times\dots\times G/C_G(\langle x_t\rangle^G)$ that sends every element y of G in $(yA, yC_G(\langle x_1,\dots,yC_G(\langle x_t\rangle^G))$. We have that $G/ker\phi$ is finite. It is simple to show that $Ker\phi = A\cap(\bigcap_{i=1}^{t} G/C_G(\langle x_i\rangle^G))\le Z(G)$, and hence $G/Z(G)$ is finite. □

As examples of FC^2-groups that are not FC-groups we present the following:

Example 1.3.4 *There is a periodic FC^2-group with trivial center which is not an FC-group.*

Proof – Let $A=\underset{i\in I}{\mathrm{Cr}}\langle a\rangle_i$ be the cartesian product of infinitely many groups of order 3, and for each index $j\in I$ let b_j denote the automorphism of A defined by the position

$$(a_i^{n_i})_{i\in I}^{b_j}=(u_i)_{i\in I},$$

where $u_i=a_i^{n_i}$ if $i\ne j$ and $u_j=a_j^{-n_j}$. Then $B=\langle b_i|i\in I\rangle$ is a subgroup of exponent 2 of the automorphism group A, and the semidirect product $G=B\ltimes A$ is a group with trivial center which is not an FC-group since the element $(a_i)_{i\in I}$ has infinitely many conjugates. On the other hand, if

$x = ab$ is any element of G (with $a \in A$ and $b \in B$), the centralizer $C_A(b) = C_A(x)$ is normal in G and the index $|A : C_A(b)|$ is finite, so the factor group $G/C_G(\langle x\rangle^G)$ is finite-by-abelian. Therefore G is an FC^2-group. □

Example 1.3.5 *There is a non-periodic FC^2-group G with trivial centre. In particular, this group is not FC.*

PROOF − Let $p_1 < p_2 < \ldots$ be the sequence of odd primes and let $A = \langle a_1\rangle \times \langle a_2\rangle \times \ldots$ where $|a_i| = p_i^2$. Define automorphisms t and x_i of A by the rules: $a_i^t = a_i^{1+p_i}$ and $a_i^{x_i} = a_i^{-1}$, $a_i^{x_j} = a_i$ if $i \neq j$. Then $H = \langle t, x_1, x_2, \ldots\rangle$ is abelian, $|t| = \infty$ and $x_i = 2$ for each i. Consider the semidirect product

$$G = H \ltimes A.$$

We claim that G is an FC^2-group. Clearly all the a_i and x_i are FC-elements. If B is a subgroup generated by the $a_i^{p_i}$, then $t^G = \langle t\rangle B$, and this group is abelian and $G/\langle t\rangle B$ is an FC-group. It follows that G is a non-periodic FC^2-group and evidently $Z(G) = 1$. □

1.4 Sylow subgroups of FC*-groups

Let G be a finite group and p a prime. If $|G| = p^a m$ where $(p, m) = 1$, than a p-subgroup of G cannot have order greater than p^a by Lagrange's Theorem. A p-subgroup of a finite group G which has this maximum order p^a is called a *Sylow p-subgroup* of G. Sylow's Theorem states that Sylow p-subgroups of G always exist and they are pairwise conjugate (see [41, page 39]). In particular, all Sylow p-subgroup of finite groups are isomorphic. If p is a prime, a Sylow p-subgroup of a possibly infinite group G is defined to be a maximal p-subgroup. It is an easy consequence of Zorn's Lemma that every

p-subgroup of G is contained in a Sylow p-subgroup. In particular, Sylow p-subgroup always exist. It follows from Sylow's Theorem that if G is finite, then a maximal p-subgroup of G has order equal to the largest power of p dividing $|G|$. This demonstrates that the foregoing definition is consistent with the notion of a Sylow p-subgroup of a finite group given above.
The main problem of Sylow theory is to determine whether all Sylow p-subgroups of a group are conjugate, possibly in some weak sense. Of course, this is true for finite groups by Sylow's Theorem. However, conjugacy tends to fail in a spectacular manner for infinite groups. Indeed Sylow p-subgroups need not even be isomorphic. For instance, there exist examples of locally finite groups having not equipotent Sylow p-subgroups, hence not isomorphic. For the class of periodic FC-groups the situation is different (see [51, Chapter 5]). It is possible to show that Sylow p-subgroups of a periodic countable FC-group are not always conjugates, yet being isomorphic (without the hypothesis that the group is countable). For each $n \in \mathbb{N}$, let G_n be a group isomorphic to $Sym(3)$, the symmetric group of order 3. Let G be the direct product $\operatorname*{Dr}_{n\in\mathbb{N}} G_n$. Since every G_n contains 3 Sylow 2-subgroups, it follows that the set of the Sylow 2-subgroups of G is not countable. Because G is countable, we can conclude that the Sylow 2-subgroups of G do not lie in the same conjugacy class.

Recall that an automorphism ϕ of a group G is said *locally inner* for if every finite subset $\{x_1, \ldots, x_t\}$ of G there exists an element g of G such that $x_i^\phi = x_i^g$ for every $i \in \{1, \ldots, t\}$. Clearly, the set $LInn\ G$ of all locally inner automorphisms of a group G is a subgroup of $Aut\ G$ and it contains the group $InnG$ of inner automorphisms of G. Moreover, if G is a finite group, it follows that $LInn\ G = InnG$. The subgroups H and K of a group G are said *locally conjugate* if there exists a locally inner automorphism ϕ of G such that $H^\phi = K$. Obviously, conjugate subgroups are also locally conjugate and

locally conjugate subgroups are always isomorphic. On the other hand, there exist locally conjugate subgroups of an infinite group that are not conjugate. In [2], R. Baer proved that Sylow p-subgroups of a periodic FC-group are locally conjugates, and hence they are isomorphic. Moreover, if P is a Sylow p-subgroups of a periodic FC-group having only a finite number of conjugates, the any two Sylow p-subgroups of G are conjugates.
The question of whether FC^*-groups in general have isomorphic Sylow p-subgroups remains open. We present some results of independent interest about Sylow subgroups of FC^*-groups.

Lemma 1.4.1 *Let S a Sylow p-subgroup of a group G. If F is a finite-by-nilpotent normal subgroup of G, then $S \cap F$ is a Sylow p-subgroup of F.*

Proof – If F is finite, the proof is easy since $|F : F \cap S| = |SF : S|$ is a p'-number. Since F is finite-by-nilpotent, there is a positive integer n such that $F/Z_n(F)$ is finite. Put $N = Z_n(F)$ and $N_p = O_p(N)$. As first step we prove that SN/N is a Sylow p-subgroup of SF/N. Let P/N be a p-subgroup of SN/N. Since $N/N_p \leq Z_n((F \cap P)/N_p)$ and $(F \cap P)/N$ ia a finite p-group, $(F \cap P)/N_p$ is nilpotent. Therefore

$$(F \cap P)/N_p = N/N_p \times Q/N_p$$

where Q/N_p is the unique Sylow p-subgroup of $(F \cap P)/N_p$. It follows that Q is characteristic in $F \cap P$ and hence $Q \lhd P$. But S is a Sylow p-subgroup of P, so $Q \leq S$. Therefore $P = P \cap (SF) = S(F \cap P) = S(NQ) = SN$, which implies that SN/N is a Sylow p-subgroup of SF/N.
From the finite case we see that $(SN/N) \cap F/N = (S \cap F)N/N$ is a Sylow p-subgroup of F/N, and in particular $|F : (S \cap N)N|$ is a p'-number. Note that $N_p \leq S \cap F$. Let P_1/N_p be a Sylow p-subgroup of F/N containing $(S \cap N)/N_p$. Now $P_1 \cdot (S \cap F)N = P_1N$ is a subgroup, so $|P_1N : (S \cap F)N|$

divides $|F : (S \cap F)N|$ and hence is a p'-number. Therefore $P_1 N = (S \cap F)N$ and $P_1 = P_1 \cap ((S \cap F)N) = (S \cap F)N_p = S \cap F$; it now follows that $S \cap F$ is a Sylow p-subgroup of F. □

Lemma 1.4.2 ([51]) *Let $\{F_i | i \in I\}$ be a local system of normal subgroups of G and for each $i \in I$ let S_i be a Sylow p-subgroup of F_i. If $S_j \cap F_i = S_i$ whenever $F_j \geq F_i$, then $S = \bigcup_{i \in I} S_i$ is a Sylow p subgroup of G.*

Proof – It is clear that S is a p-subgroup of G. If T is a Sylow p-subgroup of G containing S, then $T \cap F_i$ is a p-subgroup of F_i containing S_i, and so $T \cap F_i = S_i$ and $S = \bigcup_{i \in I} S_i = \bigcup_{i \in I} (T \cap F_i) = T$. □

Lemma 1.4.3 *Let G a periodic locally nilpotent-by-finite group and let N be a normal subgroup of G. If S is a Sylow p-subgroup of G, then SN/N is a Sylow p-subgroup of G/N.*

Proof – Clearly, $O_p(G) \leq S$. Replacing G by $G/O_p(G)$, we may suppose that $O_p(G) = 1$. By hypothesis there exists a normal locally nilpotent subgroup L of G such that G/L is finite. Since $O_p(G) = 1$, it follows that L is a p'-group. Therefore $S \simeq SL/L$ is finite and hence Sylow p-subgroups of G are finite and conjugate. Let Q/N be a finite p-subgroup of G/N. Then $Q = UN$ for some finite subgroup U. Moreover, as $U/U \cap N$ is a p-group, there exists a Sylow p-subgroup S_0 of U such that $U = S_0(U \cap N)$. It follows that $Q = S_0 N$ and $|Q/N| \leq |S|$, and hence Sylow p-subgroups of G/N are finite. Taking Q/N to be one of these, we find by using the argument above that $Q = S^g N$ for some $g \in G$. Thus $SN/N = Q^{g^{-1}}/N$ is a Sylow p-subgroup of G/N. □

Theorem 1.4.4 ([44]) *Let G be an FC^*-group and let S be a Sylow p-subgroup of G. If N is a periodic normal subgroup of G, then:*

(a) $S \cap N$ is a Sylow p-subgroup of N;

(b) SN/N is a Sylow p-subgroup of G/N;

(c) every Sylow p-subgroup of G/N has the form SN/N for some Sylow p-subgroup S of G.

Proof – Since the elements of finite order in an FC^*-group form a subgroup, we may assume that G is periodic.

(a) Let $\mathfrak{L} = \{L_i | i \in I\}$ be a local system consisting of finite-by-nilpotent normal subgroups of G. From Lemma 1.4.1, we know that $S \cap N \cap L_i$ is a Sylow p-subgroup of $N \cap L_i$. If $N \cap L_j \geq N \cap L_i$, then $(S \cap N \cap L_j) \cap (N \cap L_i) = S \cap N \cap L_i$, which implies that $S \cap N = \bigcup_i S \cap N \cap L_i$ is a Sylow p-subgroup of N by Lemma 1.4.2.

(b) Let $\{L_i | i \in I\}$ be a local system consisting of finite-by-nilpotent normal subgroups of G. Clearly the set $\{L_i N/N | i \in I\}$ is a local system of finite-by-nilpotent normal subgroups of G/N. Moreover, Lemma 1.4.1 shows that $S \cap L_i$ is a Sylow p-subgroup of L_i, and hence $(S \cap L_i)(N \cap L_i)/(N \cap L_i)$ is a Sylow p-subgroup of $L_i/(N \cap L_i)$ by Lemma 1.4.3. Put $X_i = (S \cap L_i)N$. Then X_i/N is a Sylow p-subgroup of $L_i N/N$. Assume that $L_i N \geq L_j N$: we claim that $(X_i N) \cap (L_j N/N) = X_j/N$. Since the L_k form a local system, there is a k such that $L_i \cup L_j \subseteq L_k$. Now $(X_k/N) \cap (L_j N)/N$ contains $(X_i/N) \cap (L_j N)/N$ and X_j/N, and by Lemma 1.4.3 all three of these are Sylow p-subgroups of $L_j N/N$. The claim follows at once. We can now deduce from Lemma 1.4.2 that

$$X/N = \bigcup i \in I(X_i/N)$$

is a Sylow p-subgroup of G/N. Then $X = \bigcap_{i\in I}(S \cap L_i)N = SN$ and thus SN/N is a Sylow p-subgroup of G/N.

(c) Let Q/N be a Sylow p-subgroup of G/N and let P be a Sylow p-subgroup of Q. By (b) PN/N is a Sylow p-subgroup of Q/N and hence $Q = PN$. If S is a Sylow p-subgroup of G containing P, then $Q = SN$, as required. □

The Sylow theory in FC^*-groups is more interesting when the ERF-property is also satisfied. Recall that a group G is said to be *extended residually finite* (ERF) if every subgroup is closed in the profinite topology, i.e., every subgroup of G is an intersection of subgroups of finite index. While a complete description of ERF-groups in general is hard to find, the task of identifying them in special cases has seen considerable progresses. For example, the ERF-groups have been determined in the following classes of groups: nilpotent groups ([36]), soluble groups with finite rank ([33]) and FC-groups ([45]). In particular, a relevant result of Smirnov states that the nilpotent ERF-groups are precisely the nilpotent groups with all their quotients residually finite ([48]). The next result gives a complete classification of the ERF-groups belonging to the class of FC^*-groups and it is useful to further investigations on Sylow theory in FC^*-groups. The proof can be found in [44].

Theorem 1.4.5 ([44]) *Let G an FC^n-group where $n \geq 1$. Then G is ERF if and only if the following conditions hold:*

a) Sylow subgroups of G are abelian-by-finite with finite exponent;

b) Sylow subgroups of $\gamma_{n+1}(G)$ are finite;

c) $G/\tau(G)$ is torsion-free nilpotent of finite rank and Prüfer-free.

Here $\tau(G)$ denotes the maximum periodic normal subgroup of a group G. Also a group is said to be *Prüfer-free* if no quotient of a subnormal subgroup is of p^∞-type for any prime p. This theorem extends the result for nilpotent groups previously obtained in [36] by Menth, while the case $n = 1$ is the characterization of FC-groups which are ERF obtained in [45, Theorem 8.1]. Moreover, an application of Theorem 1.4.5 shows that the group of the example 1.3.5 of the previous section is an ERF-group.

The next result is a structural property of FC^n-groups which are ERF: it indicates that these groups are, in a weak sense, close to being nilpotent. The proof of the next result can be found in [44].

Theorem 1.4.6 *If G is an FC^n-group which is in ERF, then $G/Z_n(G)$ is countable.*

The last theorem of this section represents a return to Sylow properties. Here one should keep in mind B.H. Neumanns theorem: Sylow subgroups of FC-groups are isomorphic (and even locally conjugate) (see [51, Chapter 5]). The question of whether FC^*-groups in general have isomorphic Sylow p-subgroups remains open. This is true when the group is ERF, as the following results shows.

Theorem 1.4.7 ([44]) *If G is an FC^*-group which is ERF. Then the Sylow p-subgroups of G are isomorphic for every prime p.*

Proof − Let G be an FC^n-group and let P, Q be Sylow subgroups of G. Since the elements of finite order form a subgroup of G, we can assume that G is periodic. Also P is isomorphic to $PO_{p'}(G)/O_{p'}(G)$, which is a Sylow p-subgroup of $G/O_{p'}(G)$ by Theorem 1.4.4, so we can assume that $O_{p'}(G) = 1$. We will show that P and Q are conjugates. By [44, Lemma 3.5], the group $G/O_p(G)$ is finite. Since $P, Q \geq O_p(G)$, the conjugacy of P and Q follows. $\square$

CHAPTER 2

Pronormality in generalized FC-groups

Our aim here is to extend to FC^*-groups the theory of pronormality in the FC-groups, developed in [11] and [12]. The main theorems deal with the join of pronormal subgroups and the relevant role that the Wielandt subgroup plays in an FC^*-group.

2.1 Some basic properties

Let G be a group and let H be a subgroup of G. Following the notation used in [11], we say that an element x of G *pronormalises* H if the subgroups H and H^x are conjugate in $\langle H, H^x \rangle$. Moreover, H is said a *pronormal subgroup* of G if each element of G pronormalises H or equivalently if any two conjugates of H in G are already conjugate in their join.

The study of pronormality has been subjected to many investigations by several authors (see for instance [5], [12], [20], [25], [29], [32], [42], [53]).

One rather obvious, yet helpful, observation may be made: to prove that a subgroup H is pronormal in a group G, it is enough to show that for any element x of G there exists an element y of $\langle H, H^x \rangle$ such that $H^x = H^y$, so, taking into account that $\langle H, H^x \rangle \leq H[H, x]$, it reduces to find an element y of $[H, x]$ such that $H^x = H^y$. All normal subgroups and all maximal subgroups of a group are clearly included among its pronormal subgroups. Moreover, Sylow subgroups of finite groups as well as Hall and Carter subgroups of finite soluble groups are always pronormal. The concept of a pronormal subgroup was introduced by P. Hall, and the first results about pronormality appeared in a paper by Rose [47]. The next lemma lists some useful properties of pronormal subgroups and their normalisers.

Lemma 2.1.1 *Let H be a subgroup of a group G. Then:*

(i) If H is pronormal in G and $H \leq L \leq G$, then H is pronormal L;

(ii) If N is a normal subgroup of G and $N \leq H \leq G$, then H is pronormal in G if and only if H/N is pronormal in G/N;

(iii) If N is normal in G and H is pronormal in G, then HN/N is pronormal in G/N;

(iv) If N is normal in G and H is pronormal in G, then $N_G(HN) = N_G(H)N$;

(v) If H is a subgroup both pronormal and ascendant in G, then H is normal in G;

(vi) If H is pronormal in G, then $N_G(H) = N_G(N_G(H))$;

(vii) If H is pronormal in G and N is a normal subgroup of G that contains H, then $G = NN_G(H)$;

(viii) No two distinct conjugates of a pronormal subgroup are permutable.

Proof − The properties (i) and (ii) are obvious.

(iii) Since H is a pronormal subgroup of G, for every element x of G there exists $y \in \langle H, H^x\rangle \leq \langle NH, (NH)^x\rangle$ such that $H^x = H^y$. It follows that $(HN)^x = H^xN = H^yN$, and hence HN is pronormal in G.

(iv) Of course, $N_G(H)N \leq N_G(HN)$. Now, let g be an element of $N_G(HN)$. By hypothesis, $H^g = H^x$ where $x \in \langle H, H^g\rangle$. Since $H^gN = (HN)^g = HN$, it follows that $\langle H, H^g\rangle \leq HN$ and $x = hu$ with $h \in H$ and $u \in N$. Obviously, $gx^{-1} \in N_G(H)$, and hence $g = (gx^{-1}h)u$ belongs to $N_G(H)N$.

(v) Let

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_\alpha \lhd H_{\alpha+1} \lhd \ldots H_\tau = G$$

be a ascendant chain of H in G. We want to use transfinite induction on τ. If $\tau = 1$, the statement is obvious. Let τ be greater than 1. If τ is not be limit, then by induction hypothesis we have that H is normal in $H_{\tau-1}$, and hence $G = N_G(HH_{\tau-1}) = N_G(H)H_{\tau-1} = N_G(H)$. Thus H is normal in G. If τ is a limit ordinal, we have that

$$G = H_\tau = \bigcup_{\alpha<\tau} H_\alpha.$$

For every $\alpha < \tau$ $H_\alpha \leq N_G(H)$ hence $H \leq N_G(H)$. It follows that H is a normal subgroup of G.

(vi) Obviously H is a subnormal subgroup of $N_G(N_G(H))$. By the property (v) H is normal in $N_G(N_G(H))$. Therefore $N_G(N_G(H)) = N_G(H)$.

(vii) By the property (iv)

$$G = N_G(N) = N_G(HN) = N_G(H)N.$$

(viii) Let H be a pronormal in G. Suppose that H^x is permutable with H, for some x in G. Then $H^x = H^y$, for some y in $\langle H, H^x\rangle = HH^x$. Now y may be expressed in the form $y = h_1h_2^x$, with h_1, h_2 in H. Therefore $H^x = H^{h_1x^{-1}h_2x} = H^{x^{-1}h_2x}$. It follows that $H = H^{x^{-1}h_2}$ and so $H^x = H$. □

The next lemma provides a local version of some elementary results about pronormal subgroups of a group.

Lemma 2.1.2 *Let G be a group and H and K subgroups of G such that $H^K = H$. If x is an element of G normalizing H and pronormalising K, then x pronormalises HK.*

Proof − Since x pronormalises K, there exists an element $z \in \langle K, K^x\rangle$ such that $K^{xz} = K$. On the other hand,

$$\langle K, K^x\rangle \leq \langle K, x\rangle \leq N_G(H),$$

so that $(HK)^{xz} = H^{xz}K^{xz} = HK$. Therefore x pronormalises HK. □

To localize the property of pronormality we use the concept of *pronormalizer of a subgroup H*, that is the set $P_G(H)$ of all elements of G pronormalizing H. Thus a subgroup H of a group G is pronormal in G if and only if $P_G(H) = G$. The pronormalizer of a subgroup H of a group G is not in general a subgroup; this can be seen from the behaviour of any subgroup of order 2 in the alternating group A_5. Note also that, if G is any group and H is a subgroup of G, then $P_G(H)^\phi = P_G(H^\phi)$ for every automorphism ϕ of G. In the following results we present some cases in which the pronormalizer of a subgroup of a group is a subgroup.

Lemma 2.1.3 *Let G a be a group and H a subgroup of G. If x is any element of G pronormalizing H, then the cosets $N_G(H)x$ and $xN_G(H)$ are contained in $P_G(H)$.*

Proof – The right coset $N_G(H)x$ is obviously contained in the pronormalizer of H. Let a be an element of $N_G(H)$. It follows that x^a pronormalizes $H^a = H$ and so also $xa = ax^a$ is an element of $P_G(H)$. The lemma is proved. □

We shall prove that if H is a subgroup of a group G which is either ascendant or descendant, then the pronormalizer in G of any pronormal subgroup of H is a subgroup. Recall here that a subgroup H of a group G is called *ascendant* (respectively, *descendant*) if there exists an ascending (respectively, descendant) series containing both H and G.

Lemma 2.1.4 *Let G a be a group and X a subgroup of G. If there exists an ascendant subgroup H of G such that $X \leq H \subseteq P_G(X)$, then $P_G(X) = N_G(X)H = HN_G(X)$. In particular the pronormalizer of X in G is a subgroup.*

Proof – Let g be any element of G pronormalizing X. Consider an ascending series

$$H = H_0 < H_1 < \ldots < H_\alpha < H_{\alpha+1} < \ldots < H_\tau = G.$$

Let $\alpha \leq \tau$ be the least ordinal such that g lies into the set $N_G(X)H_\alpha$. Assume by contradiction that $\alpha > 0$. Clearly, α is not a limit ordinal and g does not belong to $N_G(X)H_{\alpha-1}$. Write $g = ah$, where $a \in N_G(X)$ and $h \in H_\alpha$. It follows that $X^g = X^h$ and hence $\langle X, X^g \rangle = \langle X, X^h \rangle \leq H_{\alpha-1}$. Since g pronormalizes X, then $X^g = X^y$ for some element y of $H_{\alpha-1}$. Therefore $g = (gy^{-1})y$ belongs to $N_G(X)H_{\alpha-1}$. This contradiction shows that $P_G(X)$ is contained in the set $N_G(X)H$. Lemma 2.1.3 tells us that the product $HN_G(X)$ is a subset of $P_G(X)$. It follows that $N_G(X)H = HN_G(X) = P_G(X)$, as required. □

Corollary 2.1.5 *Let G be a group and H an ascendant subgroup of G. Then $P_G(H) = N_G(H)$.*

Proof – Since X is an ascendant subgroup of G, Lemma 2.1.4 tells us that $P_G(H) = N_G(H)H = N_G(H)$. □

Theorem 2.1.6 ([12]) *Let G be a locally nilpotent group. If H is any subgroup of G, then $P_G(H) = N_G(H)$. In particular, every pronormal subgroup of a locally nilpotent group is normal.*

Proof – Let g be any element of G pronormalizing H. Then $H^g = H^y$ for some element y of $\langle H, H^g\rangle$. Thus, $gy^{-1} \in N_G(H)$ and $g = (gy^{-1})y \in \langle N_G(H), N_G(H)^g\rangle$. Let E be a finitely generated subgroup of $N_G(H)$ such that g belongs to $\langle E, E^g\rangle$. As E is subnormal in $\langle E, E^g\rangle$, it follows that $g \in E$. It follows that g normalizes H and $P_G(H) = N_G(H)$, as required. □

Lemma 2.1.7 *Let G a be a group and X a subgroup of G. If there exists a descendant subgroup H of G such that $X \leq H \subseteq P_G(X)$, then $P_G(X) = N_G(X)H = HN_G(X)$. In particular the pronormalizer of X in G is a subgroup.*

Proof – Let g be any element of G pronormalizing X and let y be an element of $\langle X, X^g\rangle$ such that $X^g = X^y$. Assume by contradiction that g does not belong to the set $N_G(X)H$. Consider a descending series

$$G = H_o > H_1 > \ldots > H_\alpha > H_{\alpha+1} > \ldots > H_\tau = H.$$

Let $\alpha \leq \tau$ be the least ordinal such that $g \notin N_G(X)H_\alpha$. It follows that g is an element of $N_G(X)H_\beta$ for each ordinal $\beta < \alpha$. Therefore the subgroup $\langle X, X^g\rangle$ is contained in H_β. Suppose first that α is a limit ordinal, so that

$\langle X, X^g\rangle \leq \bigcap_{\beta<\alpha} H_\beta = H_\alpha$. Since gy^{-1} is an element of $N_G(X)$, it follows that g belongs to $N_G(X)H_\alpha$. This contradiction shows that α is not a limit ordinal and $g \in N_G(X)H_{\alpha-1}$. Write $g = uh$, where u normalizes X and $h \in H_{\alpha-1}$. Then $X^g = X^h \leq H_\alpha$, so that again $\langle X, X^g\rangle$ is contained in H_α. We reach a contradiction as in the previous case and therefore $P_G(X)$ in contained in the set $N_G(X)H$. Lemma 2.1.3 tells us that $HN_G(X)$ is a subset of $P_G(X)$ and hence $N_G(X)H = HN_G(X) = P_G(X)$, as required. □

Corollary 2.1.8 *Let G be a group, and let H be a descendant subgroup of G. Then $P_G(H) = N_G(H)$.*

Proof − As H is a descendant subgroup of G, Lemma 2.1.7 tells us that $P_G(H) = N_G(H)H = N_G(H)$. □

Let H be a subgroup of a group G which is either ascendant or descendant. It follows from Corollary 2.1.5 and Corollary 2.1.8 that H is normal in G if and only if it is pronormal. It is an open question whether pronormal subgroups that are also serial must be normal.

2.2 The pronorm of a group

In [1] Baer introduced the *norm N(G)* of a group G as the intersection of all the normalizers of subgroups of G. Clearly, $N(G)$ is a characteristic subgroup of G. It is easy to show that each element of $N(G)$ induces power automorphism on G by conjugation. Therefore $N(G)$ is contained in the second term $Z_2(G)$ of the upper central series of G (see for instance [8]).

In order to define an analogue to the norm for pronormality, we have that if $\mathfrak{X}$ is a property pertaining to subgroups, then the $\mathfrak{X}$-*pronorm* of a group G

is the set $P_{\mathfrak{X}}(G)$ of all elements of G pronormalising every $\mathfrak{X}$ subgroup of G which belongs to $\mathfrak{X}$. If $\mathfrak{A}$ is the class of all groups, $P_{\mathfrak{A}}(G)$ is just the *pronorm* $P(G)$ of G, that is the intersection of all the pronormalizers of subgroups of G. It is easy to see that the pronorm $P(A_5)$ of the alternating group A_5 has order 40. Thus, the pronorm of an arbitrary finite group need not be a subgroup. On the other hand, Theorem 2.1.6 tells us that $P(G) = N(G)$ for every locally nilpotent group G. In order to prove that the pronorm of any finite soluble group is a subgroup, we introduce two relevant series of subgroups.

Let G be a group. The *lower Wielandt series* of G is the descending normal series whose terms $\omega_\alpha(G)$ are defined inductively by positions

$$\omega_0(G) = G, \quad \omega_{\alpha+1} = \bigcap_{K \in \Omega_\alpha(G)} \omega(K),$$

where $\Omega_\alpha(G)$ is the set of all subgroups of G containing $\omega_\alpha(G)$, and

$$\omega_\lambda(G) = \bigcap_{\beta<\lambda} \omega_\beta(K)$$

if λ is a limit ordinal. Clearly $\omega_1(G) = \omega(G)$ and the last term of the lower Wielandt series of G will be denoted by $\bar{\omega}(G)$. The *lower τ-series* of G is the descending normal series obtained by replacing in the above definition the Wielandt subgroup $\omega(X)$ by the subgroup $\tau(X)$ for each group X. The last term of the lower τ-series of G will be denoted by $\bar{\tau}(G)$. Clearly $\tau_1(G) = \tau(G)$. The last term of the lower τ-series of G is a characteristic subgroup of G.

Lemma 2.2.1 *Let G be a group and H an ascendant subgroup of G. If x is an element of G pronormalising H, then $H^x = H$.*

Proof – Assume that $H^x \neq H$. Let

$$H = H_0 < H_1 < \ldots < H_\alpha < H_{\alpha+1} < \ldots H_\tau = G$$

be an ascending series of G. By hypothesis there exists an element y of $\langle H, H^x\rangle$ such that $H^x = H^y$, so that y belongs to $\langle H, H^y\rangle$. Consider now the least ordinal μ such that $y \in H_\mu$. Clearly, $\mu = \alpha + 1$ for some ordinal α and $H \leq H_\alpha \lhd H_\mu$. Therefore $y \in \langle H, H^y\rangle \leq H_\alpha$. This contradiction shows that $H^x = H$, as required. □

Lemma 2.2.1 suggests that the pronorm and the norm of a hypercentral group coincide. In what follows, it is shown that this property also holds for locally nilpotent groups.

Lemma 2.2.2 *Let G be a group and H and K subgroups of G such that $H^K = H$. If N is a normal subgroup of G such that K is pronormal in KN and x is an element of N pronormalising H, then also x pronormalises HK.*

Proof – Since x pronormalises H and $\langle H, H^x\rangle = \langle H, H^x\rangle \cap NH = H(N \cap \langle H, H^x\rangle)$, there exists an element y of $N \cap \langle H, H^x\rangle$ such that $H^{xy} = H$. Since K is pronormal in KN, there exists $z \in \langle K, K^x\rangle$ such that $K^{xyz} = K$. Put $J = HK$. Then $H = H^{xy}$ is a normal subgroup of $\langle J, J^x\rangle$ and hence $H^{xyz} = H^z = H$. Thus $J^{xyz} = J$ where $y \in \langle J, J^x\rangle$ and $z \in \langle K, K^{xy}\rangle \leq \langle J, J^x\rangle$. Therefore yz is an element of $\langle J, J^x\rangle$ and hence x pronormalises J. □

Lemma 2.2.3 *Let G be a finite soluble group and P a p-subgroup of G for some prime number p. If N is a normal subgroup of G such that $N \leq \omega(PN)$, then P is a pronormal subgroup of PN.*

Proof – Without loss of generality it can be assumed that $G = PN$ and so also $G = P\omega(G)$. Let F be the Fitting subgroup of G and put $V = F \cap \omega(G)$. Since every element of $\omega(G)$ acts as (universal) power automorphism on the abelian group F/F', it follows that $[\omega(G), G]$ is contained in $C_G(F/F') = F$, so that $[\omega(G), G] \leq V$ and G/V is a nilpotent group. Let W be the largest

p'-subgroup of the nilpotent group V, then PV/W is a finite p-group. Thus, the subgroup PW is subnormal in G. It follows that PW is normal in $G = P\omega(G)$. Lemma 2.2.2 tells us that P is a pronormal subgroup of G, as required. □

Theorem 2.2.4 *Let G be a finite soluble group, then $P(G) = \overline{\omega}(G)$. In particular, $P(G)$ is a subgroup of G.*

Proof – Suppose that the set $P(G)$ is contained in $\omega_n(G)$ for some non negative integer n. Let H be any subgroup of G containing $\omega_n(G)$. By Corollary 2.1.5, $P(G)$ is contained in the normalizer of every subnormal subgroup. It follows that $P(G) \subseteq \omega(H)$ and $P(G)$ also lies in $\omega_{n+1}(G)$. Therefore $P(G)$ is a subset of the last term $\overline{\omega}(G)$ of the lower Wielandt series of G.

Let X be any subgroup of G. In order to prove that $\overline{\omega}(G)$ is contained in the pronormalizer of X, it can be assumed that (by induction on the order of X) the commutator subgroup X' of X is pronormal in $X'\overline{\omega}(G)$. Put $X/X' = \langle x_1X' \rangle \times \ldots \times \langle x_tX' \rangle$, where every x_i is an element of X with prime-power order. Clearly, $\overline{\omega}(G)$ is contained in $\omega(\langle x_i \rangle \overline{\omega}(G))$. By Lemma 2.2.3, $\langle x_i \rangle$ is a pronormal subgroup of $\langle x_i \rangle \overline{\omega}(G)$. Now, Lemma 2.2.2 tells us that the normal subgroup $\langle x_i, X' \rangle$ of X is pronormalised by $\overline{\omega}(G)$ for each $i = 1, \ldots, t$. Another application of Lemma 2.2.2 proves finally that X is pronormal in $X\overline{\omega}(G)$. This completes the proof. □

It is natural to ask whether this result also holds for soluble FC-groups. In order to studying this situation, we are also interested in the pronorms relative to the class $\mathfrak{C}$ of all cyclic groups and $\mathfrak{F}$ of all finite groups. The subset $P_{\mathfrak{C}}(G)$ of all elements of G pronormalizing all cyclic subgroups is called the *cyclic pronorm* of G.

It is easy to see that all non-cyclic subgroups of the alternating group A_5 are pronormal and hence $P(A_5) = P_{\mathfrak{C}}(A_5)$. On the other hand, a result of Kovacs, Neumann and de Vries shows that in a periodic soluble group G the pronorm $P(G)$ can be a proper subgroup of the cyclic pronorm $P_{\mathfrak{C}}(G)$ ([26]).

What follows is helpful in order to prove that in certain relevant cases the pronorm and the cyclic pronorm of a group are subgroups and sometimes they coincide. In this respect, in [11] there are some interesting results concerning with hyperabelian groups and polycyclic groups. For instance, it has been proved that the cyclic pronorm $P_{\mathfrak{C}}(G)$ of a hyperabelian group G is contained in $\overline{\tau}(G)$. It follows that if G is a periodic hyperabelian group, then $P_{\mathfrak{F}}(G) = P_{\mathfrak{C}}(G) = \overline{\tau}(G)$ (see [11, Theorem 4.6]). Moreover, in the case of polycyclic groups it has been proved that $P(G) = P_{\mathfrak{C}}(G) = \overline{\varpi}(G)$ (see [11, Corollary 4.7]). It follows that the pronorm and the cyclic pronorm of a polycyclic group are subgroups and coincide.

The next results deal with other specific situations in which the pronorm of a group is a subgroup.

Lemma 2.2.5 *Let G be an hyperabelian group such that $\overline{\tau}(G)$ is contained in the FC-center of G. Then the cyclic pronorm $P_{\mathfrak{C}}(G)$ of G is a subgroup of G.*

Proof − Since G is a hyperabelian group, the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a subset of $\overline{\tau}(G)$. By hypothesis, $P_{\mathfrak{C}}(G)$ is contained in the FC-center of G. Let X be a finite subset of $P_{\mathfrak{C}}(G)$ and a any element of G. Since the normal closure $\langle X\rangle^G$ is a polycyclic group, then $L = \langle X, a\rangle$ is polycyclic as well. In particular, the cyclic pronorm of L is a subgroup. It follows that $\langle a\rangle$ is pronormalised by every element of $\langle X\rangle$ and $\langle X\rangle$ is contained in $P_{\mathfrak{C}}(G)$. Therefore $P_{\mathfrak{C}}(G)$ is a subgroup of G, as required. □

Theorem 2.2.6 ([11]) *Let G be a locally soluble FC-group. Then $P(G) = P_{\mathfrak{C}}(G)$. In particular, the pronorm of G is a subgroup.*

Proof − Clearly, G is hyperabelian. By Lemma 2.2.5, the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a normal subgroup of G. If K is any subgroup of G, then X is pronormal in $XP_{\mathfrak{C}}(G)$ for every cyclic subgroup X of K. Lemma 2.2.2 tells us that K is pronormal in $KP_{\mathfrak{C}}(G)$. Therefore $P(G) = P_{\mathfrak{C}}(G)$ and $P(G)$ is a subgroup of G, as required. □

In the following we will show that also in the universe of locally soluble FC^*-groups both the pronorm and the cyclic pronorm of a group G are subgroups.

Theorem 2.2.7 ([46]) *Let G be a locally soluble FC^*-group. Then the cyclic pronorm $P_{\mathfrak{C}}(G)$ of G is a subgroup of G.*

Proof − Assume G to be an FC^n-group, where n is a positive integer. Let X be a finite subset of $P_{\mathfrak{C}}(G)$ and let a be an element of G. By Theorem 1.1.11, $\langle X, a\rangle^G/Z_n(\langle X, a\rangle^G)$ is finite, it follows that $\gamma_{n+1}(\langle X, a\rangle^G)$ is finite. Therefore $\gamma_{n+1}(\langle X, a\rangle)$ is a finite soluble group and so it is polycyclic. On the other hand, the nilpotent factor $\langle X, a\rangle/\gamma_{n+1}(\langle X, a\rangle)$ is also polycyclic so that $\langle X, a\rangle$ is polycyclic and $P_{\mathfrak{C}}(\langle X, a\rangle)$ is a subgroup of $\langle X, a\rangle$. Clearly X is contained in $P_{\mathfrak{C}}(\langle X, a\rangle)$, thus every element of $\langle X\rangle$ pronormalises the subgroup $\langle a\rangle$. The arbitrary choice of the element a in G shows that $P_{\mathfrak{C}}(G)$ is a subgroup of G. □

Though the hypotheses of the next two lemmas may look bizarre, these results are sufficient to prove that also in a locally soluble FC^*-group the pronorm is a subgroup .

Lemma 2.2.8 *Let G be a group, and let K be a subgroup of G. Let Ω be a chain of normal subgroups of K such that $K = \bigcup_{H\in\Omega} H$. Let x be an element of G such that $x = ky$, where $k \in K$ and $y \in G$ pronormalises every element of Ω. If the subgroup $[K, x]$ is finite, then both x and y pronormalise K.*

Proof – Let H be an element of Ω, then $H^y = H^z$ where $z \in [H, y]$. Put $[K, x] = \{y_1, \dots, y_t\}$ and observe that $[H, y]$ is a subgroup of $K[K, x]$, so $z = k_H y_i$ with $k_H \in K$ and $y_i \in [K, x]$. For each $i \le t$, let Ω_i be the subset of Ω consisting of all subgroups $H \in \Omega$ such that $H^y = H^{k_H y_i}$ where k_H is a suitable element of K, and so

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_t.$$

For each $i \le t$, put

$$K_i = \bigcup_{H\in\Omega_i} H$$

so that $K_i^y = K_i^{k_H y_i}$. If h, j are indices such that K_h is not contained in K_j, there exists an element $\overline{H}$ of Ω_h which is not contained in K_j. By hypothesis Ω is a chain, so that every element of Ω_j is contained in $\overline{H}$, and hence also in K_h. Thus K_j is contained in K_h, and the finite set $\{K_1, \dots, K_t\}$ is a chain. On the other hand,

$$K = \bigcup_{H\in\Omega} H = \langle K_1, \dots, K_t\rangle,$$

so that $K = K_n$ for some $n \le t$. Since every element of Ω is a normal subgroup of K, we have that:

$$K^x = K^y = \bigcup_{H\in\Omega_n} H^{k_H y_n} = \bigcup_{H\in\Omega_n} H^{y_n} = K^{y_n},$$

where y_n is an element of $[K, x]$ and so x pronormalises K. On the other hand, $y_n \in \langle K, K^x\rangle = \langle K, K^y\rangle$ and so also y pronormalises K. □

Lemma 2.2.9 *Let G be a locally soluble FC^n-group. Then every subgroup K of G is a pronormal subgroup of $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$.*

Proof − Let K be a subgroup of G. By Theorem 1.1.17, it is possible to consider an ascending characteristic series

$$\{1\} = K_0 \leq K_1 \leq \ldots \leq K_n \leq K_{n+1} \leq \ldots \leq K_\gamma = K$$

with abelian factors of the locally soluble FC^n-group K.

Assume that the lemma is false, so that some K_α is not a pronormal subgroup of $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$, and α can be chosen to be smallest with respect to this condition. Let $x = ky$, where $k \in K$ and $y \in P_{\mathfrak{C}}(G)$, be an element of $\gamma_{n+1}(KP_{\mathfrak{C}}(G))$ which does not pronormalise K_α. As G is an FC^n-group, by Theorem 1.1.6, the subgroup $[K, x]$ is finite. If $\alpha = 1$, then K_α is abelian and so there exists an ascending normal series of K_α say

$$\{1\} = H_0 \leq H_1 \leq \ldots \leq H_\mu = K_\alpha,$$

whose factors are cyclic. Since K_α is a normal subgroup of K, if x does not pronormalise K_α, then y does not even pronormalise K_α. Let δ be the first ordinal such that y does not pronormalise K_δ. It's obvious that $\delta > 1$. If δ were a limit ordinal, then $H_\delta = \bigcup_{\beta<\delta} H_\beta$ and H_β is pronormalised by y for each $\beta < \delta$. As the subgroup $[H_\delta, x]$ is finite, it follows by Lemma 2.2.8 that y pronormalises H_δ, a contradiction. Hence δ cannot be a limit ordinal and y pronormalises $H_{\delta-1}$. Since the factor $H_\delta/H_{\delta-1}$ is cyclic, there exists an element h_δ of H_δ such that $H_\delta = H_{\delta-1}\langle h_\delta\rangle$. We can observe that $H_{\delta-1}$ and $\langle h_\delta\rangle$ are subgroups of G such that $H_{\delta-1}^{\langle h_\delta\rangle} = H_{\delta-1}$, moreover $P_{\mathfrak{C}}(G)$ is a normal subgroup of G such that $\langle h_\delta\rangle$ is pronormal in $\langle h_\delta\rangle P_{\mathfrak{C}}(G)$ and the element y of $P_{\mathfrak{C}}(G)$ pronormalises $H_{\delta-1}$, so that y pronormalises H_δ by Lemma 2.2.2. This contradiction shows that $\alpha > 1$. If α is not a limit ordinal, we can assume that x pronormalises $K_{\alpha-1}$. Since the factor $K_\alpha/K_{\alpha-1}$ is abelian, there exists a chain of ascending normal subgroup of K_α whose factors are cyclic that is

$$H_0 = K_{\alpha-1} \leq H_1 \leq \ldots \leq H_\gamma = K_\alpha.$$

Let λ be the smallest ordinal such that y does not pronormalise H_λ. By assumption $\lambda > 0$. Hereafter, we can argue as above to get contradiction both in the case that λ is limit or not. Finally assume that α is limit, then $K_\alpha = \bigcup_{\delta<\alpha} K_\delta$ and K_δ is pronormalised by x for each $\delta < \alpha$. Because of K_δ is a normal subgroup of K for every $\delta < \alpha$, it follows that K_δ is pronormalised by y for each $\delta < \alpha$. The subgroup $[K_\alpha, x]$ is finite, and a new application of Lemma 2.2.8 yields that x pronormalises K_α. This contradiction proves the lemma. □

Theorem 2.2.10 ([11]) *Let G be a locally soluble FC^*-group. Then the pronorm $P(G)$ of G is a subgroup of G.*

Proof – Assume G to be an FC^n-group, where n is a positive integer. Let K be a subgroup of G. By Lemma 2.2.9, $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$ is contained in $P_{KP_{\mathfrak{C}}(G)}(K)$. Since $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$ is a subnormal subgroup of $KP_{\mathfrak{C}}(G)$, by Lemma 2.1.4 it follows that $P_{KP_{\mathfrak{C}}(G)}(K)$ is a subgroup of $KP_{\mathfrak{C}}(G)$. We can observe that $P_{KP_{\mathfrak{C}}(G)}(K) = P_G(K) \cap KP_{\mathfrak{C}}(G)$.
On the other hand,

$$P(G) = \bigcap_{K\leq G} P_G(K) = (\bigcap_{K\leq G} P_G(K)) \cap P_{\mathfrak{C}}(G) = \bigcap_{K\leq G} (P_G(K) \cap P_{\mathfrak{C}}(G))$$

and

$$P_G(K) \cap P_{\mathfrak{C}}(G) = P_G(K) \cap KP_{\mathfrak{C}}(G) \cap P_{\mathfrak{C}}(G) = P_{KP_{\mathfrak{C}}(G)}(K) \cap P_{\mathfrak{C}}(G).$$

It follows that $P(G)$ is a subgroup of G because it is an intersection of subgroups. □

2.3 Product of pronormal subgroup

Many authors are interested in studying in which case the product of two pronormal subgroups of a group is still a pronormal subgroup.

Recall that two subgroups H and K of a group G are said to *permute* if $HK = KH$, and this is precisely the condition for HK to be a subgroup. In particular, H and K permute if $K \leq N_G(H)$.

Theorem 2.3.1 ([47]) *Let H and K be subgroups of a group G such that $K \leq N_G(H)$. If H and K are both pronormal in G, then HK is pronormal in G.*

Proof – Put $J = HK$. For any x in G, $H = H^{xy}$ for some y in $\langle H, H^x \rangle \leq \langle J, J^x \rangle$ because H is pronormal in G. Since $H \lhd J$, we have $H \lhd \langle J, J^{xy} \rangle$. By hypothesis K is pronormal in G, so that $K = K^{xyz}$ for some z in $\langle K, K^{xy} \rangle \leq \langle J, J^{xy} \rangle \leq \langle J, J^x \rangle$. It follows from $H \lhd \langle J, J^{xy} \rangle$ that $H = H^{xyz}$, and so $J = HK = H^{xyz}K^{xyz} = J^{xyz}$ with yz in $\langle J, J^x \rangle$. Hence J is pronormal in G. □

Corollary 2.3.2 *Let H and K be subgroups of a group G such that $\langle H, K \rangle$ is abelian. If H and K are both pronormal in G, then $\langle H, K \rangle$ is pronormal in G.*

Proof – The proof follows from Theorem 2.3.1. □

An example of Chambers in [7] shows the existence of non soluble finite groups having primary subgroups H, K of coprime order such that their product is a non pronormal subgroup. To see this, consider the simple group S of order 168, and let H be a subgroup of S of order 24, so that $H = XY$, where $|X| = 8$ and $|Y| = 3$. Clearly, S can be embedded into its automorphism group G. It is easy to prove that X and Y are pronormal subgroups of G but H is not pronormal. On the other hand, in the same paper Chambers proved that in the finite soluble case a subgroup V of a group G such that its Sylow subgroups are pronormal in G is also pronormal in G. Legovini in [32] generalized this result (and Rose's result in the soluble case) using the concept of Sylow system of a finite group.

Suppose that G is a finite group and $p_1, \ldots, p_k$ denote the distinct prime divisors of $|G|$. If Q_i is a Hall p_i'-subgroup of G, then the set $\{Q_1, \ldots, Q_k\}$ is called a *Sylow system of* G . It is easy to check that a finite group has a Sylow system if and only if it is soluble.

Now, let G be a finite soluble group and X a subgroup of G. A Sylow system Σ of G is said to *reduce into* X if the set $\Sigma_X = \{X \cap S : S \in \Sigma\}$ is a Sylow system of X. It follows from some results of P. Hall that for each subgroup X of a finite soluble group G there exists at least one Sylow system reducing into X. As Sylow systems of finite soluble groups are pairwise conjugate, we obtain that every Sylow system of G reduces into some conjugate of X. Using this concept, pronormal subgroups of finite soluble groups have been characterized completely by Mann. The proof of the following result can be found in [35].

Theorem 2.3.3 ([35]) *Let G be finite soluble group and X a subgroup of G. Then X is pronormal in G if and only if every Sylow System of G reduces into exactly one conjugate of X.*

The next result is due to Legovini and is concerned with the product of two pronormal subgroups of a finite soluble group. Theorem 2.3.3 is crucial in the following proof.

Theorem 2.3.4 ([32]) *Let G be finite soluble group and let X and Y be pronormal subgroups of G that permute. Then the product XY is a pronormal subgroup of G*

Proof – Put $H = XY$. Let Σ be a Sylow system of G reducing into H and H^g for some element g of G. Since X is pronormal in G, it follows from Theorem 2.3.3 that Σ reduces into a unique conjugate of X. Moreover the set $\Sigma_H = \{H \cap S : S \in \Sigma\}$ is a Sylow system of H, so that it reduces into a conjugate X^y of X with $y \in Y$. As Σ also reduces into H^g, we have that Σ also reduces into a conjugate of X^g in H^g, so that X^y is contained in H^g. A similar argument shows that Y^x is a subgroup of H^g for some element x of X. Let x_1 and y_1 be element of X and Y, respectively, such that $yx^{-1} = x_1y_1$, and put $z = x_1^{-1}y$. Then $y = x_1z$ and $x = y_1^{-1}z$, so that $X^y = X^z$ and $Y^x = Y^z$. Thus $H = H^x = X^zY^z = X^yY^x$, and hence $H^g = H$. Therefore Σ reduces into exactly one conjugate of H. Theorem 2.3.3 tells us that H is a pronormal subgroup of G. □

It was proved in [11, Theorem 2.8] that a subgroup X of a polycyclic-by-finite group G is pronormal if and only if X^σ is pronormal in G^σ for every finite homomorphic image G^σ of G. Therefore Theorem 2.3.4 can also be stated in the case of polycyclic group. The next result provides a further extension of Theorem 2.3.4 to the class of locally soluble FC-groups.

Theorem 2.3.5 ([12]) *Let G be a locally soluble FC-group and let X and Y be pronormal subgroups of G that permute. Then the product XY is a pronormal subgroup of G.*

Proof – Put $H = XY$ and let g be any element of G. In order to prove that H and H^g are conjugate in $\langle H, H^g\rangle$ it can obviously be assumed that $G = \langle H, g\rangle$. Moreover, replacing G by the factor group G/H_G, we may also suppose without loss of generality that the subgroup H is core-free, so that in particular $H \cap Z(G) = \{1\}$ and H is periodic. If the element g has finite order, its normal closure $\langle g\rangle^G$ is finite by Dietzmann's Lemma, and so H has finite index in G; then the group G is finite and Theorem 2.3.4 applies.

Suppose now that g has infinite order, and let n a positive integer such that g^n belongs to $Z(G)$. As the factor group $G/\langle g^n\rangle$ is periodic, the above argument shows that $H\langle g^n\rangle$ is a pronormal subgroup of G, and hence $H\langle g^n\rangle = H^u\langle g^n\rangle$ for some element u of $\langle H, H^g\rangle$. Let T be the subgroup consisting of all elements of finite order of G. Then $H^g = H^g\langle g^n\rangle \cap T = H^u\langle g^n\rangle \cap T = H^u$. Therefore the subgroup H is pronormal in G. □

The above proof relies heavily on the classical theorem of Baer stating that if G is an FC-group, then $G/Z(G)$ is locally finite and, in particular, the elements of finite order of G form a subgroup (see [51, page 4]). In order to prove the corresponding result for FC^*-groups, the main obstacle we had to face is that Baer's result fails for FC^2-groups, even if they are ERF (see Chapter 1). The proof of next theorem doesn't depend on theorem 2.3.5.

Theorem 2.3.6 ([46]) *Let G be a locally soluble FC^*-group. Let X and Y be pronormal subgroups of G that permute. Then the product XY is a pronormal subgroup of G.*

Proof − Assume G to be an FC^n-group, where n is a positive integer. Put $H = XY$, and let g be an element of G. In order to prove that H and H^g are conjugate in $\langle H, H^g\rangle$, it can be obviously assumed that $G = H^{\langle g\rangle}\langle g\rangle$. Since g pronormalizes X, there exists $u \in [g, X]$ such that $X^g = X^u$. The element u pronormalises X, then there exists $v \in [u, X] \leq [[g, X], X]$ such that $X^u = X^v$. Iterating this process, we find an element a in $\gamma_{n+1}(G)$ such that $X^g = X^a$, and similarly $X^{g^{-1}} = X^b$ with b in $\gamma_{n+1}(G)$. With the same argument for Y, we have that $Y^g = Y^c$ and $Y^{g^{-1}} = Y^d$, where c and d are elements of $\gamma_{n+1}(G)$. All the elements a, b, c and d are periodic and lie in the FC-center of G by Theorem 1.1.6, so that $N = \langle a, b, c, d\rangle^G$ is finite by Dietzmann's Lemma. Remark that for each $z \in \mathbb{Z}$, $X^{g^z} = X^{a_1}$, where $a_1 \in N$ and $Y^{g^z} = Y^{b_1}$ where $b_1 \in N$, thus $H^{\langle g\rangle} \leq NH$. This means that H has finite index in $H^{\langle g\rangle}$. If there exists a positive power of g lying in $H^{\langle g\rangle}$, then $|H^{\langle g\rangle}\langle g\rangle : H^{\langle g\rangle}|$ is finite and H has finite index in $G = H^{\langle g\rangle}\langle g\rangle$. Arguing in the finite soluble factor G/H_G, we have that $H/H_G = (XH_G/H_G)(YH_G/H_G)$ is pronormal in G/H_G by the quoted result of Legovini. Therefore we may assume that g has infinite order and $G = \langle g\rangle \ltimes H^{\langle g\rangle}$. As N is finite, there exist two distinct integer i and j such that $X^{g^i} = X^{g^j}$ and so g^s lies in $N_G(X)$ for some $s \in \mathbb{N}$. Similarly, there exists $t \in \mathbb{N}$ such that g^t lies in $N_G(Y)$. Put $m = st$, it follows that g^m normalizes H. If x is an element of $G = \langle g\rangle \ltimes H^{\langle g\rangle}$, then $x = g^i g^{mj} yh$, where $i = 0, \dots, m-1$, $j \in \mathbb{Z}$, $y \in N$ and $h \in H$. It follows that $x = g^i y' g^{mj} h$, where y' is a suitable element of the finite normal subgroup N. It turns out that the finite set $T = \{g^i y : i = 0, \dots, m-1 \text{ and } y \in N\}$ is a left transversal of $\langle g^m\rangle H$ in G, and for each $t \in \mathbb{N}$ we may claim that $\langle g^{mt}\rangle H$ has finite index in G. Put $J_t = (\langle g^{mt}\rangle H)_G$ and consider the finite factor G/J_t. By hypothesis, XJ_t/J_t and YJ_t/J_t are pronormal subgroup of G/J_t and so $HJ_t/J_t = (XJ_t/J_t)(YJ_t/J_t)$ is pronormal in G/J_t so that HJ_t is a pronormal subgroup of G, again by Legovini's result. It follows that $(HJ_t)^g = (HJ_t)^y$, where $y = jz$ with $j \in J_t$ and $z \in \langle H, H^g\rangle$. Since $\langle H, H^g\rangle$ is a subgroup of HN, then $z \in HN \cap \langle H, H^g\rangle = H(N \cap \langle H, H^g\rangle)$. Put $N \cap \langle H, H^g\rangle = \{u_1, \dots u_s\}$ so that $(HJ_t)^g = (HJ_t)^{u_i}$ for some u_i. For each $r \leq s$, let Ω_r be the subset of $\mathbb{N}$ consisting of all integers n such that $(HJ_n)^g = (HJ_n)^{u_r}$. Clearly, there exists $i \leq s$ such that Ω_i is infinite. The following relation holds:

$$H \subseteq \bigcap_{n\in\Omega_i} HJ_n \subseteq \bigcap_{n\in\Omega_i} \langle g^{mn}\rangle \ltimes H \subseteq H$$

and hence all these terms coincide. Then:

$$H^g = (\bigcap_{n\in\Omega_i} HJ_n)^g = \bigcap_{n\in\Omega_i} (HJ_n)^g = \bigcap_{n\in\Omega_i} (HJ_n)^{u_i} = (\bigcap_{n\in\Omega_i} \langle g^{mn}\rangle \ltimes H)^{u_i} = H^{u_i},$$

and hence H is a pronormal subgroup of G. □

2.4 Joins of pronormal subgroups

In this section, we analyze some interesting situations involving the join of pronormal subgroup of a group.

Lemma 2.4.1 *Let G be a group, and let K be a subgroup of G and Ω a chain of subgroup of G such that $K = \bigcup_{H\in\Omega} H$. If x is an element of G pronormalising every element of Ω and the subgroup $[K,x]$ is finite, then x also pronormalises K.*

Proof – Put $[K,x] \cap \langle K, K^x\rangle = \{y_1, \ldots, y_t\}$. For each $i = 1, \ldots, t$ let Ω_i be the subset of Ω consisting of all subgroups $H \in \Omega$ such that $H^x = H^{y_i}$. For every $H \in \Omega$ the subgroup $\langle H, H^x\rangle$ is contained in the product $H([H,x] \cap \langle K, K^x\rangle)$, and so $\Omega = \Omega_1 \cup \ldots \cup \Omega_t$. For each $i \leq t$ put $K_i = \bigcup_{H\in\Omega_i} H$, so that $K_i^x = K_i^{y_i}$. If i, j are indices such that K_i is not contained in K_j, there exists an element H of Ω_i which is not contained in K_j, so that every element of Ω_j is contained in H, and hence also in K_i. Thus K_j is contained in K_i, and the finite set $\{K_1, \ldots, K_t\}$ is a chain. On the other hand, $K = \bigcup_{H\in\Omega} H = \langle K_1, \ldots, K_t\rangle$. Therefore $K = K_i$ for some $i \leq t$ and x pronormalises K, as required. □

Corollary 2.4.2 *Let G be an FC-group and Ω a chain of pronormal subgroups of G. Then also $\bigcup_{H\in\Omega} H$ is a pronormal subgroup of G.*

Proof – Put $K = \bigcup_{H\in\Omega} H$. Since G is an FC-group, the subgroup $[K,x]$ is finite for each element x of G. Lemma 2.4.1 tells us that K is pronormalised by every element of G and hence it is a pronormal subgroup of G. □

The question whether the same result also holds for FC^n-groups when $n \geq 2$ is still open. The following result shows that at least for finite-by-nilpotent groups, the join of a chain of pronormal subgroups is likewise pronormal.

Theorem 2.4.3 ([46]) *Let G be a finite-by-nilpotent group, and let Ω be a chain of pronormal subgroups of G. Then $\bigcup_{H\in\Omega} H$ is a pronormal subgroup of G.*

Proof – Since G is a finite-by-nilpotent group, there exists a finite normal subgroup N of G such that the factor G/N is nilpotent of class n (n positive integer). In particular $\gamma_{n+1}(G) = \{z_1, \dots, z_t\}$ is finite. Let x be an element of G and let H be an element of Ω. Since x pronormalises H, there exists $y \in [x, H]$ such that $H^x = H^y$. The element y also pronormalises H, so there exists $z \in [y, H] \leq [[x, H], H]$ such that $H^x = H^z$. Iterating this process, it follows that $H^x = H^{z_i}$ where z_i is a suitable element of $\gamma_{n+1}(G) \cap \langle H, H^x\rangle$. For each $i = 1, \dots, t$, let Ω_i be the subset of Ω consisting of all subgroups $H \in \Omega$ such that $H^x = H^{z_i}$ and so $\Omega = \Omega_1 \cup \ldots \cup \Omega_t$. Put $K = \bigcup_{H\in\Omega} H$ and for each $i = 1, \dots, t$ put $K_i = \bigcup_{H\in\Omega_i} H$, so that $K_i^x = K_i^{z_i}$. If i, j are indices such that K_i is not contained in K_j, there exists an element H of Ω_i which is not contained in K_j, so that every element of Ω_j is contained in H, and hence also in K_i. Thus K_j is contained in K_i, and the finite set $\{K_1, \dots, K_t\}$ is a chain. On the other hand, $K = \bigcup_{H\in\Omega} H = \langle K_1, \dots, K_t\rangle$, so that $K = K_i$ for some $i \leq t$, and $K^x = K^{z_i}$. □

The last part of this section is devoted to the study of the Wielandt subgroup of a finite-by-nilpotent group.

Theorem 2.4.4 ([46]) *Let G be a periodic FC^*-group. If G contains a locally nilpotent normal subgroup N such that the factor G/N is locally nilpotent, then $\tau(G) = \bar{\tau}(G) = \omega(G)$.*

Proof – Let K be a subgroup of G. In order to prove that the theorem is true, we show that $K \cap \tau(G)$ is contained in $\tau(K)$. Put $L = K \cap \tau(G)$ and let X be any ascendant subgroup of K. Since N is a locally nilpotent FC^*-group, it is hypercentral by Theorem 1.1.17. It follows that $Y = X \cap N$ is an ascendant subgroup of G and L is contained in the normaliser of Y. The factor group X/Y is isomorphic with a subgroup of G/N, so that it is hypercentral and has a unique Sylow p-subgroup X_p/Y, for every prime p. In particular X_p is ascendant in K. In order to prove that L normalises X, it is enough to show that L is contained in $N_G(X_p)$ for all p, so we may assume that X/Y is a p-group for some prime p. Let $N = U \times V$, where U is a p-group and V has no elements of order p, and write $\bar{G} = G/V$. Since $\bar{N}$ is a p-subgroup of $\bar{G}$ such that $\bar{G}/\bar{N}$ is locally nilpotent, it follows that $\bar{G}$ has a unique Sylow p-subgroup $\bar{P}$. Moreover, $\bar{X}$ is a p-subgroup of $\bar{G}$, so that $\bar{X} \leq \bar{P}$ and so $\bar{X}$ is ascendant in $\bar{G}$. It follows that $\bar{X}$ is normalised by $\bar{L}$, and hence $L \leq N_G(XV)$. Therefore LY/Y lies in the normaliser of $(XV \cap K)/Y = X(V \cap K)/Y$. On the other hand, the ascendant subgroup X/Y is a Sylow p-subgroup of $X(V \cap K)/Y$, so that X/Y is characteristic in $X(V \cap K)/Y$ and L is contained in $N_G(X)$. This proves that $K \cap \tau(G)$ is contained in $\tau(K)$, for all subgroups K of G. Now, let K be any subgroup of G containing $\tau(G)$, then $\tau(G)$ is contained in $\tau(K)$, so that $\tau_2(G) = \tau(G)$, and hence $\bar{\tau}(G) = \tau(G)$. □

Lemma 2.4.5 *Let G be a finite-by-nilpotent group, and let K be a locally soluble subgroup of G. If N is a normal subgroup of G such that X is pronormal in XN for every cyclic subgroup X of K, then K is a pronormal subgroup of KN.*

Proof – Let G be an FC^n-group, where n is a positive integer. Theorem 1.1.17 tells us that it is possible to consider an ascending characteristic series with abelian factors of K of length at most $\omega + (n-1)$:

$$\{1\} = K_0 \leq K_1 \leq \ldots \leq K_{\omega+(n+1)} = K.$$

Assume that the lemma is false, so that it follows that K_δ is not pronormal in $K_\delta N$ for some ordinal δ, and δ can be chosen to be the smallest with respect to this condition. Let x be an element of N which does not pronormalise K_δ. If $\delta = 1$, then K_δ is abelian. The ordered set $\mathfrak{L}$ consisting of all subgroups of K_δ pronormalised by x is inductive by Theorem 2.4.3, thus by Zorn's Lemma $\mathfrak{L}$ contains a maximal element M. If $K_\delta \neq M$, we may consider $y \in K_\delta \backslash M$, and the subgroup $\langle y \rangle$ is pronormal in $\langle y \rangle N$ by hypothesis, so that x also pronormalises $\langle y \rangle M$ by Lemma 2.2.2. This contradiction shows that $\delta > 1$. If δ were a limit ordinal, then $K_\delta = \bigcup\limits_{\beta<\delta} K_\beta$ and K_β is pronormalised by x for each $\beta < \delta$, and a new application of Theorem 2.4.3 yields that x pronormalises K_δ. Hence δ cannot be a limit ordinal and x pronormalises $K_{\delta-1}$. We can use again Theorem 2.4.3 to show that the ordered set $\mathfrak{L}$, consisting of all subgroups of K_δ containing $K_{\delta-1}$ and pronormalised by x, is inductive. Let M be a maximal element of $\mathfrak{L}$. Clearly M is a normal subgroup of K_δ, and for every element y of K_δ we have that $\langle y \rangle M$ is pronormalised by x from Lemma 2.2.2. Therefore $M = K_\delta$, a contradiction. □

It has already been remarked that the Wielandt subgroup and the pronorm coincide for polycyclic groups with nilpotent commutator subgroup and for periodic groups with nilpotent and finite commutator subgroup (see [13]). In what follows these results are extended to the class of soluble finite-by-nilpotent group.

Theorem 2.4.6 ([46]) *Let G be a soluble finite-by-nilpotent group. Then $P(G) = P_{\mathfrak{C}}(G)$. Moreover, if G is periodic and metanilpotent, then $P(G) = \omega(G)$.*

PROOF – Let G be an FC^n-group where n is a positive integer. By Theorem 2.2.7, the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a normal subgroup of G. Let K be any subgroup of G. Then X is pronormal in $XP_{\mathfrak{C}}(G)$ for every cyclic subgroup X of K, and K is pronormal in $KP_{\mathfrak{C}}(G)$ by Lemma 2.4.5. Therefore $P(G) = P_{\mathfrak{C}}(G)$. Suppose now that G is periodic so that $P_{\mathfrak{C}}(G) = \bar{\tau}(G)$ (see [11, Theorem 4.6]), and moreover if G is metanilpotent then $\bar{\tau}(G) = \omega(G)$ by Theorem 2.4.4. The proof is complete. □

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