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Ph.D. Thesis

Exact Solutions in General Relativity and Alternative Theories of Gravity: mathematical and physical properties

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The mathematical sciences particularly exhibit
order, symmetry and limitation
and these are the greatest form
of the beautiful.

(Aristotele, Metaphysica)

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Introduction

General Relativity (henceforth GR) is a classical theory of gravitation describing gravitational fields in terms of an elegant mathematical structure, namely the differential geometry of curved spacetime. It was developed by Albert Einstein in 1915 with the introduction of his field's equations which show how the geometrical features of spacetime are related to the matter distribution in the Universe.
An appropriate definition of this fundamental idea is due to John Archibald Wheeler:

> *'Spacetime tells matter how to move; matter tells spacetime how to curve'.*

Furthemore, from a geometrical point of view, the entire content of General Relativity may be summarized as follows: *Spacetime is a differential manifold M endowed with a Lorentz metric g. The metric g is related to the matter distribution in spacetime by Einstein's equations.*

Predictions of GR have been largely confirmed in all observations and experiments so far performed, so it is the simplest theory consistent with experimental data.
This theory deals with several areas of physics like Astrophysics and Cosmology, which also will be discussed in this thesis. In fact, GR allows us a better understanding of astrophysical objects, as white dwarfs and neutron stars, and their structure. Starting from first considerations, carried out by Landau and Chandrasekhar in order to give a relativistic explanation to stellar phenomena, many theoretical predictions have been confirmed by observations. Furthermore, being able to connect experimental data with cosmological models, GR is successful in providing a good description of the large scale structure of spacetime.
Nevertheless, several issues, as the problem of initial singularity, i.e. Big Bang, arise when GR is applied to Cosmology involving a break-up of this theory. In order to solve these open problems, many attempts to generalize

and to quantize GR have been developed with the introduction of alternative theories of gravity, as f(R)-theories and quantum gravity models as Loop Quantum Cosmology and Hořava-Lifshitz.

In this thesis, we discuss several subjects connected with the framework of GR, in order to characterize astrophysical compact objects. The main purpose is to provide simple models describing gravitational fields generated by isolated compact bodies in stationary rotation with extremely simple internal structure. The main tools used for our analysis are exact solutions of Einstein fields equations, which have been approached in different ways.

The work is organized as follows. In the first chapter, after a description of internal structure of neutron stars, we deal with the problem of modeling such rotating systems through a geometrical approach. This task can be carried out making use of the formalism of junction conditions, developed by Darmois and Israel. For this purpose, we give some useful geometrical definitions of stationary and axisymmetric gravitational fields and we describe in detail the used formalism and its applications to known situations as the Oppeneheimer-Snyder collapse and the rotating thin shell. Finally, some considerations concerning our attempts to match rotating solutions of Einstein's equations are drawn.
In Chapter 2, after introducing exact solutions of Einstein equations, we focus on their geometrical properties. We are interested in checking that the Kerr metric belongs to a class of solutions admitting an Abelian bidimensional Lie algebra of Killing fields with an integrable orthogonal distribution. This would provide a new derivation of the Kerr solution based on geometrical requirements. The starting point for this task are the solutions of vacuum Einstein field equations, considered in the geometrical description carried out in [95, 96]. We also give a description of some known derivations of the Kerr metric in order to motivate our main attempt.
With the goal to provide theoretical models for astrophysical objects, Chapter 3 is devoted to the use of a solution generating technique in order to find exact solutions of Einstein's equations. To do this, the Newman-Janis Algorithm is introduced and some results obtained through it, are described, even though relevant ambiguities arising from the application of the algorithm, must be taken into account.
Finally, exact solutions of Einstein's field equations are studied in the framework of Cosmology. In particular, an exact solution, known as Einstein Static Universe describing a closed Friedmann-Robertson-Walker model sourced by a perfect fluid and a cosmological constant, is considered.
Our purpose is to study the stability properties of this solution focusing on

the intriguing possibility of finding static solutions in open cosmological models ($k = -1$). In particular, we will discuss the stability of static solutions in the framework of two alternative theories of Gravity, namely Loop Quantum Cosmology and Hořava-Lifshitz. The original results obtained with this analysis are presented in [83, 17].

Chapter 1

Astrophysical Objects in General Relativity

Even thought a deep and detailed discussion about the main properties of a neutron star and its interior structure, is outside of the main aim of this thesis, it is our interest to give a quite accurate description of these objects, because they can be related with General Relativity. In particular, in these stars lies the starting point of this thesis, so it seems quite natural to pay attention to these aspects. There is a lot of literature regarding neutron stars, their composition and their properties. In our analysis we will mainly refer to the following authors [1],[64], [74], [86], [92], [97] and [108].
The second part of this chapter is devoted to the geometrical description of gravitational fields generated by isolated objects in stationary rotation with a simple internal structure.

1.1 Neutron Stars: an overview

Compact objects as white dwarfs, neutron stars and black holes, represent the final state of stellar evolution. The factor that discriminates whether the star will "die" in a white dwarf, neutron stars or black holes is its initial mass. In fact, white dwarfs are originated from stars with masses $M \leq M_{\odot}$, where $M_{\odot}$ is the solar mass, while larger masses are required to give rise to neutron stars and black holes. The differences with the normal star are, firstly, the fact that these compact objects cannot counteract the gravitational collapse by generating thermal pressure. This is because they do not burn nuclear fuel being born when most of the nuclear fuel of progenitor star has been consumed. But white dwarfs are supported by the pressure of degenerate electrons and neutron stars by the pressure of degenerate neutrons. Black holes are completely collapsed stars. Second, compared to stars of similar mass, compact systems have a smaller radius and accordingly, a stronger surface gravitational field. It is also worth to mention the surface potentials in compact objects because they imply the importance of GR in determining the structure of these objects. All these features are shown in the table 1.1.

Objects	Mass (M)	Radius (R)	Mean Density (gcm^{-3})	Surface Potential (GM/Rc^2)
Sun	$M_\odot$	$R_\odot$	1	10^{-6}
White dwarf	$\leq M_\odot$	$\sim 10^{-2} R_\odot$	$\leq 10^7$	10^{-4}
Neutron Star	$\sim 1 - 3M_\odot$	$\sim 10^{-5} R_\odot$	$\leq 10^{15}$	10^{-1}
Black hole	Arbitrary	$2GM/c^2$	$\sim M/R^3$	1

Table 1.1: Principal characteristics of Compact Objects.

As discussed above, we are interested in dealing with neutron stars for their features that fit so well with the aim of this work.
Neutron stars are really fascinating objects due to the different phenomena occurring in them and this reason makes them complex objects which are of interest for several areas of physics. They can be observed in all ranges of the electromagnetic spectrum, directly as radio sources and pulsars and indirectly as gas accreting and periodic X-ray sources ("X-ray pulsars"). Furthermore these stars can be isolated objects or binary components. Deeply, the neutron stars show intriguing characteristics as:

- a rotation up to several hundred times per second, which is responsible of many important process appearing in these stars;
- a high density $\rho \sim 10^{14}$ g cm^{-3} comparable to the nuclear values, so neutron stars are essentially giant nuclei held together by self-gravity;
- a very strong gravitational field due to their high density and compactness, $\frac{GM}{c^2R} \sim 0.2$, that can be described with General Relativity;
- a magnetic field of the order of 10^{12} G which influences the structure of the crust and the thermal evolution;
- the coexistence of different states of matter: *superconductivity* and *superfluidity*.

For all these aspects, the analysis of neutron stars requires a physical understanding of their structure.

1.1.1 Historical remarks

In our opinion it is worth to devote a paragraph to the historical progress of the idea of neutron stars in order to make their understanding more complete. Neutron stars were firstly predicted from theoretical calculations in 1932 but they were confirmed by observations only in 1968. After the discovery of the neutron by Chadwick in 1932, Rosenfeld, Bohr and Landau discussed possible implications and in that occasion Landau suggested the possibility of cold and dense stars consisting principally of neutrons but his idea was not published until 1937, [63]. In 1934 Baade and Zwicky invented a new class of astrophysical objects called *supernovae* created during the transition from an ordinary star to a very dense neutron star, as can be read in their original paper, [3]:

> *With all reserve we advance the view that supernovae represent the transitions from ordinary stars into neutron stars, which in their final stages consist of extremely closely packed neutrons.*

They also pointed out that neutron stars would be at very high density with small radius and much more gravitationally bound than ordinary stars. The first detailed calculations of neutron stars structure were performed within framework of GR by Oppenheimer and Volkoff in 1939, [80]. In this model, the matter was assumed to be composed of an ideal gas of degenerate neutrons at high density. Subsequently the understanding of these systems was ignored for 30 years since astronomical observations were largely unlikely because of their small area and their too faint residual thermal radiation. However, during the following period, many works were presented on equation of state and neutron star models, (see [50, 51, 53, 52]).
In the early 60's, new discoveries as the cosmic, nonsolar X-ray sources by Giacconi et al. [43] and the identification of the first "quasi-stellar object", (QSO or quasar), by Schimdt at Mt.Palomar [88], renewed interest in the scientific community on neutron stars. When the first pulsars were discovered by Bell and Hewish in 1967, [56], Gold advanced the idea that they were rotating neutron stars formed in supernova events through the collapse of the stellar core to nuclear densities, [46]. Subsequent observations, especially the discovery of the Crab and the Vela pulsar in 1968, confirmed this suggestion and were the starting point of several properties of neutron stars.

1.1.2 The interior structure

The composition of a neutron star deeply depends on the nature of strong interactions which are not well understood in dense matter. Neutron stars can be considered as the most compact of all stars, in fact their mass is around 1.4 $M_\odot$, where $M_\odot$ is the solar mass, with a radius of about 10 km. Accordingly, the mean density of the neutron star matter is a few times ρ_0, where $\rho_0 = 2.8 \times 10^{14}$ g cm^{-3} is the density of the matter in atomic nuclei. According to current views based on recent observations, it is straightforward to consider the structure of a neutron star as a *layer-cake*. They consists of four main layers whose composition changes with the radial distance from the center, because of a strong density gradient from the exterior ($\rho = 7 \times 10^6$ g cm^{-3}) to the interior of the star ($\rho \sim 10^{16}$ g cm^{-3}). We can summarize this structure starting from the surface towards the dense core, as follows, (see also Fig.1.1):

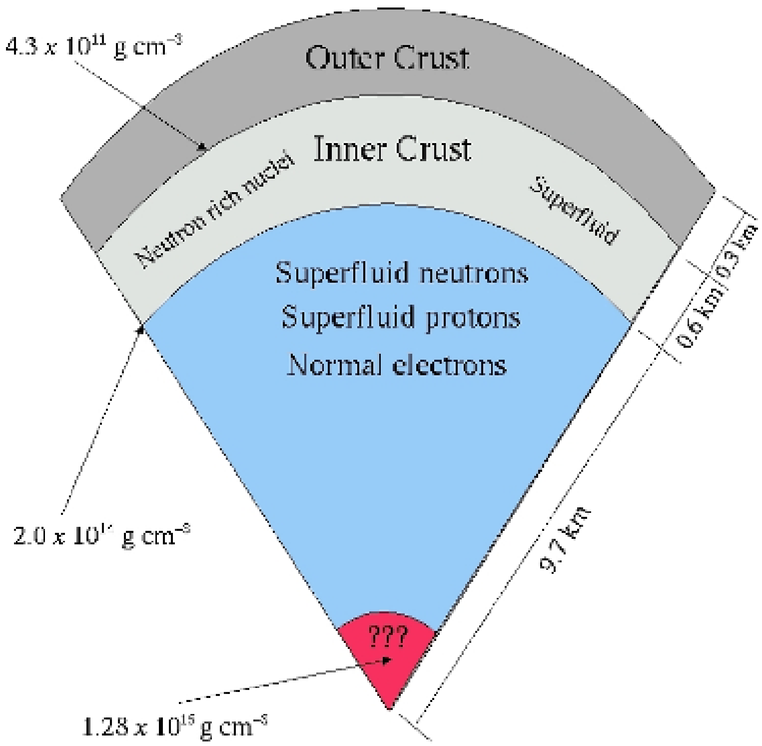

Figure 1.1: Interior structure of a neutron star.

- **Outer crust**: it is the first layer, extending for 1 km, which consists of a lattice of completely ionized nuclei in a β-equilibrium with a gas of highly degenerate relativistic electrons e^-. In this layer the density changes from the initial value of $\rho = 7 \times 10^6$g cm^{-3} to a density of $\rho = 3 \times 10^{11}$ g cm^{-3}. This increase makes the nuclei richer and richer in neutrons and when the density reaches the value of 4.3×10^{11} g cm^{-3}, the neutrons begin to leak the nuclei so the *neutron drip transition* appears. This phenomenon can be considered as the boundary between this layer and the next one. Other characteristics are the high conductivity and the predominant presence of Iron element.

- **inner crust**: in this second layer, whose thickness does not reach 1 km, in addition to increasingly neutron rich nuclei and degenerate relativistic electrons, there is also a degenerate gas of free neutrons. Since the density still increases, the difference between the neutrons in nuclei and the neutrons outside becomes fainter until the nuclei simply dissolve. This happens at the value of $\rho = 2\times10^{14}$ g cm^{-3}. The relevant feature of this layer is the possibility of superfluid neutrons due to the fact that the temperature of neutron stars, typically about T $\simeq 10^6$K, is smaller than the superfluid critical temperature T$_c \sim 10^9 - 10^{10}$K. The superfluidity has important consequences on the rotational dynamics of neutrons stars because it influences many mechanisms such as the *cooling*, the *pulsar emission*, the *glitches* and the *vortex*. We will briefly summarize these aspects in the next section.

- **outer core**: here one finds a gas of free neutrons which coexist with an amount of free protons neutralized by normal electrons in order to maintain the β-equilibrium. The protons are likely superconductors since they can undergo the same as neutrons, i.e. form Cooper pairs.

- **inner core**: the compositions of the final layer, especially in the heavier neutron stars, is still unknown. Many hypothesis have been considered such as the presence of exotic particle as hyperons, the pion or kaon condensation and the quark matter (u,d,s), but it is also taken into account a mixture of different phases, hyperons and quarks. In some models this final layer should not exist.

The nature of matter in the neutron star cores is the main mystery of these objects. Its solution would be of fundamental importance for physics and astrophysics.

1.1.3 Correlated physical phenomena

It is clear that neutron stars provide an interesting laboratory for several branches of physics. In this paragraph we will give a brief description of most relevant phenomena in dealing with neutron stars. A more thorough investigation would be beyond of the interest of this work.
Such astrophysical objects manifest several connection with low-temperature

physics due to the appearance of some superfluid phase when the temperature falls below a critical temperature. In fact, they typically have a temperature of T $\simeq 10^6$ K, which is smaller than the superfluid critical temperature evaluated as T$_c \sim 10^9 - 10^{10}$ K. Studying the nucleon interactions, Migdal [72] suggested the possibility that neutrons near the Fermi surface might be paired according to the Cooper mechanism. This occurs in different ways according to the layers: in the inner crust the neutron Cooper pairs are preferably in a state $^1S_0$, while in the outer core as the density becomes much higher, the neutron pairs form in the triplet state $^3P_2$, because of the strong spin-orbit interaction. However, in the same layer, the protons are free and can undergo the same process as neutrons, i.e. form Cooper pairs. Being fewer than the neutrons, they are expected to condense into a $^1S_0$ state, like electrons in a superconductor. The properties of the superconducting protons were described in [6].
These theoretical predictions have been confirmed by the observations of pulsars, highly magnetized rotating neutron stars emitting short pulses of radiation at very regular intervals. A 'lighthouse beam' effect is produced from motion of charged particles in the intense magnetic field surrounding the star. The pulsar period is identified with the rotational period of neutron star and it is found that the rotational energy lies in the superfluid. For this reason pulsars are considered the strongest evidence of superfluidity in neutron stars. It is possible to observe a steady increase of the pulse period followed by a steady decrease of the angular velocity, typically $\sim 10^{-9}$ s per day. This is interpreted as showing that the rotating neutron star is slowing down and the neutron superfluid is gradually loosing its angular momentum and consequently the rotational energy is transformed into electromagnetic radiation. In addition, a few pulsars show some rare events named *glitches*: within a very short time interval (probably, several minutes) the period suddenly decreases, i.e. the angular velocity suddenly increases, followed by a slow exponential relaxation with a typical time scale of the order of days to months. The two most famous glitching pulsars are the Vela and Crab pulsars with period changes of the order of 10^{-6} and 10^{-8} respectively. These events are thought to be as consequences of angular momentum transfer between the solid crust which rotates at pulsar periodicity and the interior component of neutron star. The latter being weakly coupled is not directly slowed down by the electromagnetic torque and therefore it rotates slightly faster than the solid crust. When a sudden braking of the differential rotation between the two components occurs, a glitch will appear. The observational evidence of this phenomena is the angular velocity of the crust. From this analysis it follows that the principal candidate for the faster component is the superfluid of neutrons.

Furthermore, quite relevant is the presence of vortices pinned in the inner crust, due to the nucleus-vortex interaction. The idea that the rotating superfluid is penetrated by an array of vortex lines was introduced by Packard [81], in order to explain the evidence of the sudden spin-up of the crust. The rotation of a neutron superfluid is so realized in the form of quantized vortices parallel to the neutron star spin axis. The vortices can be considered the responsible of the transfer of angular momentum from the superfluid to the crust.

Superfluidity also influences the thermal evolution of neutron stars. When a neutron star is born, its temperature is about T $\sim 10^{11} - 10^{12}$K, then the star quickly cools and the temperature falls down until T $\sim 10^{10}$. This decrease is governed by neutrino emission. In the absence of superfluidity, we have either the so-called slow cooling, controlled by neutrino reactions of modified Urca process, or fast cooling due to direct Urca process which are more efficient than modified ones but they can occur only if conservation of energy and momentum is satisfied. With the superfluidity, the neutrino emissivity would be reduced by a factor $e^{-\Delta/kT}$, where Δ is the superfluid gap energy. The reason is that, before the beta reaction takes place, one needs to break the Cooper pair. Superfluidity may affect the cooling process in such a way that fast cooling will look like slow cooling, and vice versa thus it becomes a powerful cooling regulator.

1.2 Stationary and Axisymmetric gravitational fields

Gravitational fields, describing equilibrium configurations of rotating bodies in General Relativity have two fundamental properties: *axial symmetry* and *stationarity*. When they are also asymptotically flat, they can describe isolated astrophysical compact objects as neutron stars.

From a geometrical point of view [102], a spacetime is said to be *stationary* if there exists a one-parameter group of isometries σ_t whose orbits are timelike curves. Thus, every stationary spacetime possesses a timelike Killing vector field ξ^a. (Conversely, every spacetime with a timelike Killing vector field whose orbits are complete, is stationary.) Similarly, we call a spacetime axisymmetric if there exists a one-parameter group of isometries, χ_φ whose orbits are closed spacelike curves, which implies the existence of a spacelike Killing field ψ^a, whose integral curves are closed. We call a spacetime stationary and axisymmetric if it possesses both these symmetries and if, in addition, the actions of σ_t, and χ_φ commute:

$$\sigma_t \circ \chi_\varphi = \chi_\varphi \circ \sigma_t$$

i.e., the rotations commute with the time translations. This is easily seen to be equivalent to the condition that the Killing vector fields ξ^a and ψ^a commute:

$$[\xi^a, \psi^a] = 0.$$

This condition ensures that the group of isometries is Abelian. Carter formally proved that such a condition does not involve any loss of generality, [22]. This result can be contained in the following statement:
Let M be both stationary and axisymmetric. Then M is invariant under an action of the form $\pi^S \oplus \pi^A : R(1) \times S0(2) \times M \to M$ of the 2-parameter Abelian cylindrical group $R(1) \times S0(2)$ where π^S is a stationary symmetry action and π^A is an axial symmetry action which commutes with π^S.
Killing vector fields are described in more detail in the Appendix A.

1.2.1 The geometrical approach

Referring to the internal structure of neutron stars described in the previous section, we could use a more simple model in which these astrophysical objects can be thought as formed by two different shells. Then, chosen two suitable metrics to describe both shells, we need to check that these metrics can be joined smoothly at their hypersurface of separation Σ, so that their union provides a solution to Einstein's field equations. Such a result can be obtained by solving the junction conditions given by Darmois-Israel, [31, 61].

For this purpose, we will use the mathematical methods of differential geometry, following the approach given by Poisson [85].
Firstly, let we briefly recall some useful geometrical definitions. In a four-dimensional spacetime manifold, a *hypersurface* Σ is a three-dimensional sub-manifold that can be either timelike, spacelike or null. It can be specified either by putting a restriction on the coordinates $\Phi(x^a) = 0$ or by giving parametric equations as:

$$x^\alpha = x^\alpha(y^a) \tag{1.1}$$

where y^a with $a = 1, 2, 3$, are coordinates intrinsic to the hypersurface. We define a unit normal n_α as

$$n^\alpha n_\alpha = \varepsilon = \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike} \\ +1 & \text{if } \Sigma \text{ is timelike} \end{cases} \tag{1.2}$$

The intrinsic metric on the hypersurface Σ is obtained by restricting the line element to displacements confined to Σ. For displacements within Σ we have:

$$ds^2_\Sigma = h_{ab} dy^a dy^b$$

where

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b \tag{1.3}$$

is called *induced metric* or *first fundamental form* of the hypersurface and

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a} \tag{1.4}$$

are the vectors tangent to curves contained in Σ.

We follow the formal definition of extrinsic curvature given in [29].

Let Σ be a p-dimensional submanifold of an n-dimensional manifold X with metric g.
Given u a tangent vector to Σ at point x, the covariant derivative $(\nabla_u v)_x$ of a differentiable vector field v on Σ, is a well defined vector of the tangent space $T_x X$. This covariant derivative has a component along the tangent space $T_x S$, $(\nabla_u v)^{\parallel}_x$ and a normal component $(\nabla_u v)^{\perp}_x$. It can be demonstrated that:

- $(\nabla_u v)_x^{\parallel}$ is the covariant derivative of vector v in the riemannian connection of the induced metric on Σ.

- the normal component is symmetric, $(\nabla_u v)_x^{\perp} = (\nabla_v u)_x^{\perp}$ and $(\nabla_u v)_x^{\perp}$ depends only on the vectors u_x and v_x.

The symmetric mapping

$$k_x : T_x S \times (T_x S)^{\perp} \quad \text{by} \quad (u_x, v_x) \to k_x(u_x, v_x) \equiv (\nabla_u v)_x^{\perp} \tag{1.5}$$

is called *second fundamental form* of the submanifold Σ on X. When the codimension is one, the tangent space $T_x S$ is generated by the unit normal $\boldsymbol{n}$ to Σ and one can set:

$$k_x(u, v) = K_x(u, v)n \tag{1.6}$$

where K_x is called *extrisic curvature* of Σ and is an ordinary symmetric covariant 2-tensor. From (1.6), it follows that

$$K_x(u, v) = (k_x(u, v), n) \tag{1.7}$$

According to our notation and considering the condition (1.5), the previous formula can be expressed as:

$$K(e_a, e_b) = (k(e_a, e_b), n) = (\nabla_{e_a} e_b, n) = -(e_a, \nabla_b, n) \tag{1.8}$$

in which we have used the condition that the covariant derivative of the metric tensor is zero.
Finally, the extrinsic curvature can be expressed as follows:

$$K_{ab} = n_{\alpha;\beta} e_a^{\alpha} e_b^{\beta} \tag{1.9}$$

From these definitions, it ensues that, while h_{ab} is concerned with the purely intrinsic aspects of a hypersurface's geometry, K_{ab} is concerned with the extrinsic aspects. Taken together, these tensors provide a complete characterization of the hypersurface.

A hypersurface Σ divides a spacetime in two regions: M^+ and M^-, in which two different metrics are defined. In facing the problem of joining these two metrics on the hypersurface Σ, we introduce the formalism of junction conditions of Darmois and Israel.
The first condition states that the induced metric (1.3) must be the same on both sides of the hypersurface Σ, (continuity of first fundamental form):

$$[h_{ab}] = 0 \tag{1.10}$$

The second junction condition states that the extrinsic curvature must be the same on both sides of the hypersurface Σ (continuity of second fundamental form):

$$[K_{ab}] = 0 \tag{1.11}$$

Both conditions are expressed independently of coordinates x^α. If the condition (1.11) is violated, then the spacetime is singular at the hypersurface and a thin shell with surface stress tensor

$$S_{ab} = -\frac{\epsilon}{8\pi}([K_{ab}] - [K]h_{ab}) \tag{1.12}$$

is present at hypersurface. But, we can see that when $[K_{ab}] \neq 0$, only the Ricci part of the Riemann tensor acquires a singularity, and this part can be associated with the matter. The remaining part of the Riemann tensor, the Weyl part, is smooth even when the extrinsic curvature is discontinous.

There are many classical examples in which this method of differential geometry is adopted. We shall show the Oppenheimer-Snyder collapse and the case of a slowly rotating shell.

1.2.2 The Oppenheimer-Snyder collapse

In 1939, J.Robert Oppenheimer and his student H. Snyder published the first solution to the Einstein field equations that describes the process of gravitational collapse to a black hole, [79]. In their approach, they modeled the collapsing star as a spherical ball of pressureless matter with a uniform density that is also called *dust* and is described by the energy-momentum tensor of a perfect fluid:

$$T_{\mu\nu} = (\rho + p)g_{\mu\nu}u_\mu u_\nu + pg_{\mu\nu}$$

with $p = 0$. In this case, if the hypersurface Σ divides the spacetime into two regions, called V^+ and V^-, they chose the metric for the inner shell as a Friedmann-Robertson-Walker solution, while the metric for the outer shell, neglecting the gravitational effect of any escaping radiation or matter, is the Schwarzschild solution. The question here considered is whether these metrics can be joined smoothly at their common boundary, the surface of the collapsing star.

The metric for the inner region V^- filled by the collapsing dust, is given by:

$$ds_-^2 = -d\tau^2 + a^2(\tau)(d\chi^2 + \sin^2\chi d\Omega^2) \tag{1.13}$$

where τ is proper time on comoving world lines (along which the others coordinates χ, θ and ϕ are all constant), and $a(\tau)$ is the scale factor. By virtue of the Einstein field equations, this satisfies

$$\dot{a}^2 + 1 = \frac{8\pi}{3}\rho a^2 \tag{1.14}$$

where an overdot denotes differentiation with respect to τ and $G = 1$. By virtue of energy-momentum conservation in the absence of pressure, the dust's mass density ρ satisfies:

$$\rho a^3 = \text{constant} = \frac{3}{8\pi} a_{max}$$

where a_{max} is the maximum value of the scale factor. The solution to the previous equations has the following parametric form

$$\begin{cases} a(\eta) = \frac{1}{2} a_{max}(1 + \cos\eta) \\ \tau(\eta) = \frac{1}{2} a_{max}(1 + \sin\eta) \end{cases}$$

the collapse begins at $\eta = 0$ when $a = a_{max}$ and it ends at $\eta = \pi$ when $a = 0$. The hypersurface Σ coincides with the surface of the collapsing star, which is located at $\chi = \chi_0$ in our comoving coordinates.

The metric outside the dust is given by

$$ds_+^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \tag{1.15}$$

where $f = 1 - 2M/r$ and M is the gravitational mass of the collapsing star. In this case, it is convenient to choose the following spacetime coordinates:

$$x^\alpha = (t, r, \theta, \phi)$$

while for the hypersurface Σ:

$$y^a = (\tau, \theta, \phi)$$

As seen from the outside, this hypersurface is described by the parametric equations:

$$\begin{cases} r = & R(\tau) \\ t = & T(\tau) \\ \theta = & \theta \\ \phi = & \phi \end{cases}$$

where τ is proper time for observers comoving with the surface, so this is the same time that appears in the FRW metric. Now, one has to calculate the induced metric to verify the first junction condition. As seen from inside (V^-), the induced metric on Σ is:

$$ds^2_\Sigma = -d\tau^2 + a^2(\tau)\sin^2\chi_0 d\Omega^2$$

while as seen from outside (V^+):

$$ds^2_\Sigma = -(F\dot{T}^2 + F^{-1}\dot{R}^2)d\tau^2 + R^2(\tau)d\Omega^2$$

where $F = 1 - 2M/R$. Since the induced metric must be the same on both sides of the hypersurface Σ, one has:

$$\begin{cases} R(\tau) = a(\tau)\sin(\chi_0) \\ F\dot{T}^2 - F^{-1}\dot{R}^2 = 1 \end{cases} \tag{1.16}$$

The first equation determines $R(\tau)$ while the second equation can be solved for $\dot{T}$:

$$F\dot{T} = \sqrt{\dot{R}^2 + F} \equiv \beta(R, \dot{R}) \tag{1.17}$$

Last equation can be integrated for $T(\tau)$ and the motion of the boundary in V^+ is completely determined.
Now, to verify if these metrics can be joined on Σ, we need to solve the *second junction condition*, (1.11). To do this, it is necessary to compute the vectors e^α_a given by (1.4) and the component of the unit normal to Σ, n_α, obtainable from the orthogonality and normalization conditions:

$$\begin{cases} n_\alpha e^\alpha_a = 0 \\ n_\alpha n^\alpha = 1 \end{cases} \tag{1.18}$$

This means that for the metric in V^+, one has to calculate the component

$$n_\alpha = (n_t, n_r, n_\theta, n_\phi)$$

and the tangent vectors. Starting from the θ component, one has:

$$\begin{cases} e^\alpha_\theta = 1 & \text{if } \alpha = \theta \\ e^\alpha_\theta = 0 & \text{if } \alpha \neq \theta \end{cases}$$

From the orthogonality condition it follows:

$$n_\theta e^\alpha_\theta = 0$$

which implies that the n_θ component is null. With the same reasoning, for the component n_ϕ it results:

$$n_\phi e^\alpha_\phi = 0$$

For the τ component, one has to consider that the parametric equations for t and r are function of τ, so that:

$$e^\alpha_\tau = \left(\frac{\partial R}{\partial \tau}, \frac{\partial T}{\partial \tau}\right)$$

and from the orthogonality $n_\tau e^\alpha_\tau = 0$, it follows:

$$n_t \dot{T} + n_r \dot{R} = 0$$

Taking into account that $n^\alpha = (-f^{-1}n_t, fn_r, 0, 0)$, the normalization condition $n^\alpha n_\alpha = 1$ turns into:

$$n_t(-f^{-1}n_t) + n_r(fn_r) = 1$$

By solving the resulting system one obtains:

$$\begin{cases} n_t = -\dfrac{f^{1/2}\dot{R}}{\sqrt{f^2\dot{T}^2 - \dot{R}^2}} \\ n_r = \dfrac{\dot{T}f^{1/2}}{\sqrt{f^2\dot{T}^2 - \dot{R}^2}} \end{cases}$$

these are the non vanishing components of normal vector on Σ.
At this point, one has to consider the expression of $n_{\alpha;\beta}$:

$$n_{\alpha;\beta} = \nabla_\beta n_\alpha = \frac{\partial n_\alpha}{\partial x^\beta} - \Gamma^\rho_{\alpha\beta} x_\rho$$

The non vanishing Christofell's symbols for the Schwarzschild metric, are:

$$\Gamma^r_{tt} = \frac{1}{2} f \left(-\frac{\partial f}{\partial r}\right)$$

$$\Gamma^r_{rr} = \frac{1}{2} f^{-1} \left(\frac{\partial f}{\partial r}\right)$$

$$\Gamma^r_{\theta\theta} = -fr$$

$$\Gamma^r_{\phi\phi} = -fr\sin\theta^2$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}$$
$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta$$
$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$
$$\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \frac{1}{2}(-f^{-1})\left(\frac{\partial f}{\partial r}\right)$$

Now, it is possible to compute the respective extrinsic curvatures from the formula:

$$K_{ab} = n_{\alpha;\beta}e^{\alpha}_{a}e^{\beta}_{b} = \left(\frac{\partial n_{\alpha}}{\partial x^{\beta}} - \Gamma^{\rho}_{\alpha\beta}x_{\rho}\right)e^{\alpha}_{a}e^{\beta}_{b}$$

The nonvanishing components are:

$$K_{\theta\theta} = n_{\theta;\theta} = \partial_{\theta}n_{\theta} - \Gamma^{\rho}_{\theta\theta}n_{\rho},$$

since n_θ is vanishing, the only allowed value for the index ρ is r, so that:

$$K_{\theta\theta} = -\Gamma^{r}_{\theta\theta}n_{r} = \frac{fR\dot{T}f^{1/2}}{\sqrt{f^2\dot{T}^2 - \dot{R}^2}}$$

and in the same way:

$$K_{\phi\phi} = n_{\phi;\phi} = -\Gamma^{\rho}_{\phi\phi}n_{\rho} = -\Gamma^{r}_{\phi\phi}n_{r} - \Gamma^{\theta}_{\phi\phi}n_{\theta} = -\Gamma^{\phi\phi}_{r}n_{r} = \frac{Rf\sin^2\theta\dot{T}f^{1/2}}{\sqrt{f^2\dot{T}^2 - \dot{R}^2}}$$

$$K_{\tau\tau} = -a^{\alpha}n_{\alpha}$$

where a^α is the acceleration of an observer comoving with the surface. A straightforward calculation reveals that, as seen from V^+:

$$K^{\theta}_{\theta} = g^{\theta\rho}K_{\rho\theta}$$

and since the only non vanishing component is for $\rho = \theta$:

$$K^{\theta}_{\theta} = g^{\theta\theta}K_{\theta\theta} = \frac{f\dot{T}f^{1/2}}{R\sqrt{f^2\dot{T}^2 - \dot{R}^2}} = \frac{\beta}{R}$$

Similarly one obtains:

$$K^{\phi}_{\phi} = \frac{\beta}{R}$$

where we made use of Eq. (1.17). Furthermore,

$$K^{\tau}_{+\tau} = \frac{\dot{\beta}}{\dot{R}}$$

Performing the same procedure for the side V^-, we get:

$$K^{\tau}_{-\tau} = 0 \quad K^{\theta}_{\theta} = K^{\varphi}_{\varphi} = a^{-1}\cot\chi_0.$$

In order to have a smooth transition at the surface of collapsing star, with geometrical reasonings, it results that β is a constant and this satisfies the second junction condition $[K^{\tau}_{\tau}] = 0$. On the other hand, the condition $[K^{\theta}_{\theta}] = 0$ implies that

$$\beta = \cos\chi_0$$

With the help of Eqs. (1.14), (1.16) and (1.17), the previous result may be turned into

$$M = \frac{4\pi}{3}\rho R^3$$

which relates the gravitational mass of the collapsing star to the product of its density and volume and summarizes the complete solution to the Oppenheimer-Snyder collapse.

1.2.3 Slowly rotating thin shell

Here, we consider the spacetime of a slowly rotating spherical shell. In this case, the exterior metric is assumed to be the slow-rotation limit of the Kerr solution at first order in the parameter a:

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2 - \frac{4Ma\sin^2\theta}{r}dtd\phi \tag{1.19}$$

where $f = 1 - \frac{2M}{r}$, M denoting the shell's gravitational mass and $a = J/M << M$ is a parameter related with the rotation.
Performing a cut off at a radius $r = R$, the induced metric from the exterior is:

$$ds^2_{\Sigma} = -\left(1 - \frac{2M}{R}\right)dt^2 + R^2 d\Omega^2 - \frac{4Ma\sin^2\theta}{R}dtd\phi \tag{1.20}$$

To remove the off-diagonal term we can introduce a new angular coordinate ψ related to ϕ by:

$$\psi = \phi - \Omega t \tag{1.21}$$

where Ω is the angular velocity of the new frame with respect to the inertial frame of (1.19). Since Ω results to be proportional to a

$$\Omega = \frac{2Ma}{R^3},$$

the induced metric becomes:

$$ds^2_\Sigma = -\left(1 - \frac{2M}{R}\right) dt^2 + R^2(d\theta^2 + \sin^2\theta d\psi^2) \tag{1.22}$$

The metric inside the shell is taken to be the Minkowski metric in the form:

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\psi^2) \tag{1.23}$$

where ρ is the radial coordinate. This metric must be cut off at $\rho = R$ and then matched to the metric (1.19). The shell's induced metric agrees with (1.22) and accordingly, the continuity of the induced metric is established. By performing analogous calculations as in the Oppeneheimer-Snyder collapse, we obtain a discontinuity on the hypersurface Σ, which allows us to calculate the stress energy surface tensor S_{ab} given by (1.12). This tensor can be interpreted in terms of a perfect fluid of density σ, pressure p and angular velocity ω. In the classical limit, these quantities reduce respectively:

$$\sigma \simeq \frac{M}{4\pi R^2} \quad p \simeq \frac{M^2}{16\pi R^3} \quad \omega \simeq \frac{3a}{2R^2}$$

1.3 Junction conditions for rotating bodies

Since we were dealing with astrophysical rotating bodies, i.e. neutron stars, we tried to apply the formalism of junction condition of Darmois and Israel with the aim to match rotating solutions, even though a class of interior metrics which, by construction, are matched smoothly to the Kerr solution is still absent.

As a first attempt, we chose the Kerr metric as exterior solution and the so-called *rotating de Sitter metric* as interior solution, both expressed in Boyer-Lindquist coordinates. For the rotating de Sitter metric we have considered several possible choices given by (3.2) and (B.1).

We put these metrics in the following general form:

$$ds^2 = A_\pm dt^2 + B_\pm dt d\varphi + C_\pm dr^2 + D_\pm d\vartheta^2 + E_\pm d\varphi^2 \tag{1.24}$$

where A, B, C, D and E are the elements of tensor metric. A simple choice for the parametric equations describing the hypersurface Σ is:

$$r = R(\tau) \quad t = \tau \tag{1.25}$$

which implies that $dr = \dot{R}dt$ where the overdot denotes differentiation with respect to τ.
The metric induced on the hypersurface defined from (1.25) is:

$$ds^2 = (A + C\dot{R}^2)dt^2 + Bdtd\varphi + Dd\vartheta^2 + Ed\varphi^2 \tag{1.26}$$

In these simple case, the junction conditions were not satisfied.

We also tried to use a more general choice of the hypersurface Σ by using parameterizations different from (1.25) and we also considered different choices of the internal metric (e.q. Kerr de Sitter and further generalizations). Considering more general metrics and hypersurfaces, both involving unknown functions, calculation's complexity increases and new results are still under investigation.

Chapter 2

Exact solutions of Einstein's equations

The aim of this chapter is to check if the Kerr solution may be recovered in a class of metric admitting an Abelian bidimensional Lie algebra of Killing fields as occurs for the Schwarzschild one.
We focus on exact solutions of Einstein equations, which represent the link between the geometrical features of spacetime and the matter distribution in the Universe.
In particular, the solutions of vacuum Einsteins field equations, for the class of Riemannian metrics admitting a non Abelian bidimensional Lie algebra of Killing fields [95, 96], are described. Some of these solutions can also have an interesting physical interpretation. Finally, after a brief introduction of Kerr solution, an attempt to include such a metric in the exact solutions is discussed even though at the time of writing this thesis, a final result is far to be reached.

2.1 Geometrical properties

The investigation of exact solution for Einstein's equations still represents an open issue in General Relativity. The reason lies in the high level of complexity of the field equations that makes really difficult to generate sufficiently general classes of solutions. This problem can be overcome if a high symmetry is present, even though it does not remove the non-linearity of the equations.
Einstein's equations was introduced in order to provide a geometrical description of gravitational field which induces effects of curvature on the spacetime.

They can be written as follows:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R = \kappa T_{\mu\nu} \tag{2.1}$$

where $\kappa = 8\pi G/c^4$ and c is the light velocity. It is worth to recall that $R^{\mu}_{\nu\sigma\rho}$ is the Riemann-Christoffel tensor constructed from the affine connection coefficients, also called Christoffel symbols and from their first derivatives. This tensor is the only one can be made out from at most second order derivatives of the metric tensor g and that is linear in these derivatives. Furthermore, the only $(0,2)$ tensor obtained by contracting the Riemann-Christoffel tensor is the Ricci tensor $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$ while $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature associated to metric tensor g. The energy content is represented by $T_{\mu\nu}$, (the energy-momentum tensor), which also describe the non-gravitational matter fields.
These equations represent a set of non linear second-order differential equations for the metric tensor field $g_{\mu\nu}$.

Nevertheless, it may be pointed out that certain solutions have played very important roles in the discussion of physical problems. Well-known examples are the Schwarzschild and Kerr solutions for black holes, the Friedmann solutions for cosmology which we will analyze in this thesis in dealing with the solution generating techniques.
In the end of 70's several generating techniques have been developed in order to simplify the construction of solutions.
The more suitable method for our aim is the one introduced by Belinskii and Zacharov in order to integrate the equations of gravitational fields when the metric tensor only depends on two coordinates, specifically the time and a spacelike variable [8], which corresponds to cosmological and wave solutions of the equations of gravitation. Subsequently, it was pointed out that this procedure can be also applied to the case in which both the variables on which the metric tensor depends are spacelike, which corresponds to stationary gravitational fields [9]. One possible interpretation of this case is that of a stationary gravitational field with axial symmetry which is the main one we are dealing with.
Performing a suitable generalization of the "Inverse Scattering Method" (ISM), they were able to obtain explicitly a large class of new solutions starting from a known one (thanks to the so-called *dressing ansatz*). Such a method can be adapted to a large class of initial metric to be "dressed" and can be generalized to the electro-vacuum case too.

Now, since we are interested to the analysis carried out in [95], we describe

in more detail the Belinskii-Zacharov approach whose geometrical features have largely inspired the above cited work.
Such approach has allowed to solve Einstein field equations in vacuum for a metric of the form

$$g = f\,(z,t)\left(dt^2 - dz^2\right) + h_{11}\,(z,t)\,dx^2 + h_{22}\,(z,t)\,dy^2 + 2h_{12}\,(z,t)\,dxdy. \quad (2.2)$$

Here the functions f only depend on the variables t and z. For the coordinates is adopted the following the notation $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. The corresponding vacuum Einstein's equations, expressed in more suitable light-cone coordinate, can be written in the form of a single matrix equation:

$$\left(\alpha \mathbf{H}^{-1}\mathbf{H}_\xi\right)_\eta + \left(\alpha\ \mathbf{H}^{-1}\mathbf{H}_\eta\right)_\xi = 0, \quad (2.3)$$

where

$$\mathbf{H} \equiv \|h_{ab}\|\,, \quad \xi = (t+z)\,/\sqrt{2}, \quad \eta = (t-z)\,/\sqrt{2}, \quad \alpha = \sqrt{|\det \mathbf{H}|}$$

The Eq. (2.3) is a non-linear differential equation whose *generalized Lax form* is characteristic for integrable systems. As expected, it was obtained a system of non-linear differential equations whose solution through the ISM gives the so-called *gravitational solitary waves solutions*.
The function f is determined by pure quadrature in terms of given solutions of Eq.(2.3), *via* the relations:

$$\alpha_{,i}\,\partial_i\,(\ln|f|) = \alpha_{,ii} - \alpha_{,i}^{\,2}\,/2\alpha$$

It was the first case in which the powerful methods, introduced in the theory of integrable systems on infinite dimensional manifolds had applied to gravity.

Now let we display some intriguing and useful geometrical properties of such solutions by following the remarks appearing in [95].
From a geometrical point of view, the metric given by (2.2) is invariant under translations along the x, y-axes, *i.e.* it admits two Killing fields, ∂_x and ∂_y, closing on an Abelian two-dimensional Lie algebra $\mathcal{A}_2$. Moreover:

- they have a two-dimensional (Abelian) Lie algebra of Killing fields, so that the distribution generated by the Killing fields is integrable;

- the distribution generated by the vector fields orthogonal to the Killing fields is integrable too.

We briefly recall that a two-dimensional distribution is called *integrable* if the *integral curves* (i.e. Faraday force lines) of two generating vector fields overlap to form a surface. Such surfaces are called *leaves* of the distribution. A non integrable two-dimensional distribution is called *semi-integrable* if it is part (i.e., a suitable restriction) of a three-dimensional integrable distribution.
It is known as for all gravitational fields g admitting a Lie algebra $\mathcal{G}$ of Killing fields the following two properties hold:

I the distribution $\mathcal{D}$, generated by vector fields of $\mathcal{G}$, is $two-$dimensional;

II the distribution $\mathcal{D}^{\perp}$, orthogonal to $\mathcal{D}$ is integrable and transversal to $\mathcal{D}$.

According to whether $dim G$ is 2 or 3, two qualitatively different cases can occur. For the first, a metric satisfying **I** and **II** will be called *G-integrable*, where $\mathcal{G} = \mathcal{A}_2$ or $\mathcal{G}_2$. Many studies of the $\mathcal{A}_{\in}$*-integrable* Einstein metrics exist in literature, starting from the ones of Einstein and Rosen, Rosen, Kompaneyets, Geroch, Belinsky and Khalatnikov, up to the aforementioned works of Belinskii and Zacharov.
Several definitions are pointed out for the case $\mathcal{G}_2$ in [95, 96, 94], for which the Killing fields interact non trivially one another, i.e. $[X, Y] = Y$ and which can be considered useful for our next analysis.
Instead when the case $\mathcal{A}_2$ is investigated [24], these fields may be thought absolutely free as $[X, Y] = 0$. This difference makes the former case more interesting to study and allows a more complete investigation.

When $dim G$ is 3, assumption **II** follows from **I** and the local structure of this class of Einstein metrics can be explicitly described. Some well-known exact solutions, such as the Schwarzschild one, belong to this class.

In [96] the analysis is devoted to construct new global solutions, suitable for all such $\mathcal{G}$*-integrable* metrics.

Let we recall some notational conventional: *Kil(g)* is the Lie-algebra of all Killing fields of a metric g, while *Killing algebra* is a sub-algebra of *Kil(g)*. The geometric approach followed in [24, 95, 96] allows a natural choice of coordinates, i.e., the coordinates adapted to the symmetries of the metrics, even if they do not admit integrable $\mathcal{D}^{\perp}$ distribution. These coordinates can be introduced as follows.

Semi-adapted coordinates

Let g be a metric on the spacetime M admitting $\mathcal{G}_2$ as a Killing algebra whose generators X, Y (i.e. the Killing vector fields), satisfy:

$$[X,Y] = sY$$
$$s = 0,1.$$

The Frobenius distribution $\mathcal{D}$ generated by $\mathcal{G}_2$ is two-dimensional and in the neighborhood of a non singular point of $\mathcal{D}$ a chart (x_α) exists such that:

$$X = \frac{\partial}{\partial x_{n-1}}$$
$$Y = e^{(sx_{n-1})}\frac{\partial}{\partial x_n}.$$

Such a chart will be called *semi-adapted* (with respect to Killing fields).
It can be verified that in this kind of chart a n-metric g admitting the vector fields X and Y has the following form:

$$\begin{aligned} g &= g_{ij}dx^i dx^j + 2\left(l_i + sm_i x_n\right)dx^i dx^{n-1} - 2m_i dx^i dx^n \\ &\quad + \left(s^2 x_n^2 \lambda - 2s\mu x_n + \nu\right)dx^{n-1}dx^{n-1} + \lambda dx^n dx^n \\ &\quad + 2\left(\mu - sx_n\lambda\right)dx^{n-1}dx^n, \\ i,j &= 1,2, \end{aligned} \tag{2.4}$$

with g_{ij}, m_i, l_i, λ, μ, ν, arbitrary functions of x_l with $1 \le l \le n-2$.

Killing leaves

Condition **II** imposed on the metric g allows to construct semi-adapted charts, (x_i), such that the fields $e_i = \frac{\partial}{\partial x_i}$ with $i = 1...,n-2$ belong to $\mathcal{D}^\perp$. In such a chart, that we will call *adapted*, the components of l_i and m_i vanish.

We will call *Killing leaf* an integral two-dimensional submanifold of $\mathcal{D}^\perp$. Since $\mathcal{D}^\perp$ is transversal to $\mathcal{D}$, the restriction of g to any Killing leaf, S, is non-degenerate. Thus, $(S, g|_S)$ is a homogeneous two-dimensional Riemannian manifold. Then, the Gauss curvature $K(S)$ of the Killing leaves is constant (depending on the leave). In the chart $\tilde{x} = x_{n-1}|_S, \tilde{y} = x_n|_S$ one has:

$$\begin{aligned} g|_S &= \left(s^2\tilde{\lambda}\tilde{y}^2 - 2s\tilde{\mu}\tilde{y} + \tilde{\nu}\right)d\tilde{x}^2 + \tilde{\lambda}d\tilde{y}^2 \\ &\quad + 2\left(\tilde{\mu} - s\tilde{\lambda}\right)d\tilde{x}d\tilde{y}, \end{aligned}$$

where $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\nu}$, being the restriction to S of λ, μ, ν, are constant, and

$$K(S) = s^2\tilde{\lambda}\left(\tilde{\mu}^2 - \tilde{\lambda}\tilde{\nu}\right)^{-1}.$$

2.1.1 The Ricci tensor field

Before starting the discussion on four dimensional metrics, we will give some useful conditions on used convention for the indices: Greek letters take values from 1 to 4; the first Latin letters take values from 3 to 4 while the indices i and j from 1 to 2.

Let g a $\mathcal{G}_2$-integrable four-metric at which is possible to associate a blocks matrix $M(g)$:

$$M(g) = \begin{pmatrix} F & 0 \\ 0 & H \end{pmatrix} \tag{2.5}$$

where F and H are 2 x 2 matrices whose elements depend only on x_1 and x_2.
The block F which represents the matrix associated to the metric restricted to $\mathcal{D}^{\perp}$, has negative or positive determinant, det$F < 0$ or det$F > 0$. In both cases the block H assumes the following form:

$$H = \begin{pmatrix} \nu & -\mu \\ -\mu & \lambda \end{pmatrix}$$

If det$F > 0$, the components of the Ricci tensor can be expressed as:

$$(R_{ia}) = s\begin{pmatrix} (\mathbf{H}^{-1}\partial_1(\mathbf{H}))^2_2 - (\mathbf{H}^{-1}\partial_1(\mathbf{H}))^1_1 & -2(\mathbf{H}^{-1}\partial_1(\mathbf{H}))^1_2 \\ (\mathbf{H}^{-1}\partial_2(\mathbf{H}))^2_2 - (\mathbf{H}^{-1}\partial_2(\mathbf{H}))^1_1 & -2(\mathbf{H}^{-1}\partial_2(\mathbf{H}))^1_2 \end{pmatrix} \tag{2.6}$$

and

$$(R_{ab}) = \frac{\mathbf{H}}{2f\alpha}\left[\frac{1}{2}\left[(\alpha\mathbf{H}^{-1}\partial_1(\mathbf{H}))_{,1} + (\alpha\mathbf{H}^{-1}\partial_2(\mathbf{H}))_{,2}\right] + \frac{2s^2}{\alpha}fh_{22}\right] \tag{2.7}$$

which in the Abelian case ($s = 0$) returns the Belinskii-Zacharov Eq.(2.3). The general forms of R_{ij} equations (see [95]), are:

$$\begin{aligned} R_{11} &= \frac{1}{2}\left[\Delta(\ln(\alpha f)) + \frac{1}{2}\mathrm{tr}(\mathbf{H}^{-1}\partial_1\mathbf{H})^2 - \frac{\alpha_{,1}}{\alpha}\partial_1(\ln|f|)\right] \\ &\quad + \frac{1}{2}\left[\frac{\alpha_{,2}}{\alpha}\partial_2(\ln|f|) + \partial_1\left(\frac{\alpha_{,1}}{\alpha}\right) - \partial_2\left(\frac{\alpha_{,2}}{\alpha}\right)\right] \end{aligned} \tag{2.8}$$

$$\begin{aligned} R_{22} &= \frac{1}{2}\left[\Delta(\ln(\alpha f)) + \frac{1}{2}\mathrm{tr}(\mathbf{H}^{-1}\partial_2\mathbf{H})^2 + \frac{\alpha_{,1}}{\alpha}\partial_1(\ln|f|)\right] \\ &\quad -\frac{1}{2}\left[\frac{\alpha_{,2}}{\alpha}\partial_2(\ln|f|) + \partial_1\left(\frac{\alpha_{,1}}{\alpha}\right) - \partial_2\left(\frac{\alpha_{,2}}{\alpha}\right)\right] \end{aligned} \tag{2.9}$$

$$\begin{aligned} R_{12} &= \frac{1}{2}\left[-\frac{\alpha_{,1}}{\alpha}\partial_2(\ln|f|) - \frac{\alpha_{,2}}{\alpha}\partial_2(\ln|f|) + 2\partial_1\partial_2(\ln\alpha)\right] \\ &\quad +\frac{1}{4}\mathrm{tr}[(\mathbf{H}^{-1}\partial_1\mathbf{H})(\mathbf{H}^{-1}\partial_2\mathbf{H})] \end{aligned} \tag{2.10}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

Furthermore as defined in [95], any $\mathcal{G}_2$-integrable four-metric satisfying the vacuum Einstein equations, and such that $\det F > 0$ and $h_{22} \neq 0$, has the following matrix form in the adapted chart (x_μ):

$$M_C(g) = \begin{pmatrix} \begin{matrix} 2f & 0 \\ 0 & 2f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \beta^2 \begin{pmatrix} s^2ky^2 - 2sly + m & -sky + l \\ -sky + l & k \end{pmatrix} \end{pmatrix} \tag{2.11}$$

where k, l, m are arbitrary constants such that $km - l^2 = \pm 1, k \neq 0$;

$$f = -\frac{1}{4s^2k}\Delta\beta^2$$

and β is a solution of the *tortoise equation*:

$$\beta + A\ln|\beta - A| = \Psi,$$

such that $\Delta\beta^2 \equiv \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\beta^2$ is everywhere nonvanishing, A and Ψ being an arbitrary constant and an arbitrary harmonic function.
Moreover in the case $\det F > 0$, by requiring the Ricci flatness, the equations given in Eq. (2.6) have to vanish, i.e. $R_{ia} = 0$. This request is trivially satisfied if $s = 0$ while for $s \neq 0$ it coincides with:

$$\begin{cases} (\mathbf{H}^{-1}\partial_i(\mathbf{H}))^2_2 = (\mathbf{H}^{-1}\partial_i(\mathbf{H}))^1_1, \\ (\mathbf{H}^{-1}\partial_i(\mathbf{H}))^1_2 = 0. \end{cases} \tag{2.12}$$

At this point we have underlined what are the fundamental and mathematical tools which make us able to find a class of solutions of vacuum Einstein's field equations. We have stressed only the cases which will turn out to be useful for our aim as we will see in the next section.

2.2 Some derivations of the Kerr metric

The Kerr solution plays an essential role in relativistic astrophysics to model the exterior gravitational field of rotating masses, which are objects of our study.

As it is well known, Birkhoff's theorem states that the Schwarzschild metric is the only spherically symmetric vacuum solution of Einstein equations. This situation may be compared to that in electromagnetism where the only spherically symmetric field configuration in a region free of charges, will be a Coulomb field.
When dealing with astrophysical objects, it must consider that they rotate and so one would not expect the solution outside them to be exactly spherically symmetric. An appropriate metric was discovered only in 1963 by Roy Kerr.
This metric has a great relevance respect to stationary vacuum solutions, pointed out in the black hole uniqueness theorem which states that, under rather general conditions, the Kerr space-time is the only asymptotically flat, stationary, vacuum black hole. The appearance of the angular momentum of the source makes it more realistic as the rotation is almost an universal property of astrophysical objects.
Due to the complexity of Einstein's equations, several works focusing on the method of derivation of Kerr metric has been developed, as Chandrasekhar [25], who applied the four dimensional Einstein field equations to the general metric for a stationary and axisymmetric field and Ernst [39] who showed how reduce the problem to the solution of Laplace's equations in spheroidal coordinates whose simplest solution gives the Kerr one.
An interesting derivation of Kerr metric was obtained by Newmann and Janis applying a complex coordinates transformation to the Schwarzschild solution, [78]. The same "trick" was applied to the Reissner-Nordtröm metric in order

to get the Kerr-Newmann metric [77]. This method will be largely analyzed in the next chapter.
Here, we recall three interesting derivations of Kerr metric by following in detail the original works.

2.2.1 Kerr derivation

In his note [62], Kerr was based on the proof by Goldberg and Sachs, that the algebraically special solutions of Einstein's field equations are characterized by the existence of a geodesic and shear-free ray congruence, called k_μ. He presented a class of solutions for which the congruence is diverging and is not a necessarily hypersurface orthogonal. The first step is to introduce a complex null tetrad, with

$$ds^2 = 2t\bar{t} + 2mk$$

the coordinate system is:

$$\begin{aligned} t &= P(r + i\Delta)d\zeta \\ k &= du + 2Re(\Omega d\zeta) \\ m &= dr - 2Re\{[(r - i\Delta)\dot{\Omega} + iD\Delta]d\zeta\} \\ &\quad + \left\{ r\dot{P}/P + Re[P^{-2}D(D^*\mathrm{P} + \dot{\Omega}^*)] + \frac{m_1 r - m_2\Delta}{r^2 + \Delta^2} k \right\} \end{aligned}$$

where ζ is a complex coordinate, a dot denotes differentiation with respect to u, P is real, Ω and m are complex. The operator D is defined by

$$D = \frac{\partial}{\partial\zeta} - \Omega\frac{\partial}{\partial u},$$

whereas Δ is given by

$$\Delta = Im(P^{-2}D^*\Omega)$$

At this point, Kerr underlined that two natural choices could be made for the coordinate system: P can be take unity, (in this case Ω is complex) or Ω is pure imaginary and P different from unity. The first choice leads to the following field equations:

$$(m - D^*D^*D\Omega) = |\partial_u D\Omega|^2, \tag{2.13}$$

$$\mathrm{Im}(m - m - D^*D^*D\Omega) = 0, \tag{2.14}$$

$$D^*m = 3m\dot{\Omega}. \tag{2.15}$$

The second coordinates system gives more complicated field equations. If $m = 0$ these equations are integrable, while, if $m \neq 0$ the Eqs. (2.13) must satisfy certain integrability conditions. Then, the equations can be solved for m as a function of Ω and its derivatives provided that either $\dot{\Delta}$ or $\ddot{\Omega}$ is nonzero. If both $\dot{\Delta}$ and $\ddot{\Omega}$ are zero, it is possible to lead to a coordinate system with Ω pure imaginary, $P \neq 1$ and $\dot{\Omega} = \dot{P} = 0$. So, the field equations become:

$$m = cu + A + iB \tag{2.16}$$

where c is a real constant, A,B and Ω are determined by

$$\begin{aligned} iB &= (1/2)P^{-2}\nabla(P^{-2}\partial\Omega/\partial\zeta) \\ &\quad -P^{-4}(\partial\Omega/\partial\zeta)\nabla(\ln P), \\ \nabla B &= ic\frac{\partial\Omega}{\partial\zeta}, \\ (\partial/\partial\zeta)(A - iB) &= c\Omega \end{aligned}$$

where $\zeta = \xi + i\eta$. It is worth to note that if c is zero, then $\frac{\partial}{\partial u}$ is a Killing vector. Among the solutions of these equations Kerr was able to find one which is stationary ($c = 0$) and also axially symmetric. After performing the following transformation

$$(r - ia)e^{i\phi}\sin\vartheta = x + iy, \quad r\cos\vartheta = z, \quad u = t + r$$

the resulting metric becomes:

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \left(\frac{2mr^3}{r^4 + a^2z^2}\right)(k)^2, \tag{2.17}$$

with the condition that:

$$(r^2 + a^2)rk = r^2(xdx + ydy) + ar(xdy - ydx) + (r^2 + a^2)(zdz + rdt)$$

This final result is also known as Kerr-Schild solution.

2.2.2 The Chandrasekhar's derivation

Here, we follow the work of Chandrasekhar as it is shown in [26]. The starting point is the metric for stationary axisymmetric spacetimes, written in the following form:

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\varphi - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2 \tag{2.18}$$

where ν, ψ, ω, μ_2 and μ_3 are functions of x^2 and x^3 with the freedom to impose a coordinate condition on μ_2 and μ_3.

Omitting the rather cumbersome calculations, the Einstein's equations for this metric can be reduced to the following Ernst's equation:

$$(1-\tilde{E}\tilde{E}^*)\{[(\eta^2-1)\tilde{E}_{,\eta}]_{,\eta}+[(1-\mu^2)\tilde{E}_{,\mu}]_{,\mu}\} = -2\tilde{E}^*[(\eta^2-1)(\tilde{E}_{,\eta})^2+(1-\mu^2)(\tilde{E}_{,\mu})^2]. \tag{2.19}$$

allows the elementary solution:

$$\tilde{E} = p\eta - iq\mu, \tag{2.20}$$

where

$$p^2 + q^2 = 1, \quad p \quad \text{and} \quad q \quad \text{are real constants.} \tag{2.21}$$

$\tilde{E}$ is given by the transformation:

$$\tilde{Z} = \tilde{\Psi} + i\tilde{\Phi} = -\frac{1+\tilde{E}}{1-\tilde{E}}$$

In this way the solution for $\tilde{Z}$ is:

$$\tilde{Z} = -\frac{1-p\eta-iq\mu}{1+p\eta+iq\mu} \tag{2.22}$$

where $\eta = (r-M)/(M^2-a^2)^{1/2}$ and $\mu = \cos\vartheta$ represent a choice of gauge. Separating the real and the imaginary parts of $\tilde{Z}$, we have

$$\tilde{\Psi} = \frac{p^2\eta^2+q^2\mu^2-1}{1+2p\eta+p^2\eta^2+q^2\mu^2} = \frac{p^2(\eta^2-1)-q^2(1-\mu^2)}{(p\eta+1)^2+q^2\mu^2},$$

$$\tilde{\Phi} = \frac{2q\mu}{(p\eta+1)^2+q^2\mu^2},$$

or, reverting to the variable r,

$$\tilde{\Psi} = \frac{\Delta - [q^2(M^2-a^2)/p^2]\delta}{[(r-M)+(M^2-a^2)^{1/2}/p]^2+[q^2(M^2-a^2)/p^2)]\mu^2}, \tag{2.23}$$

$$\tilde{\Phi} = \frac{2[q(M^2-a^2)/p]\mu}{[(r-M)+(M^2-a^2)^{1/2}/p]^2+[q^2(M^2-a^2)/p^2]\mu^2} \tag{2.24}$$

with the choices

$$p = (M^2-a^2)^{1/2}/M \quad q = \frac{a}{M} \quad \Delta = (M^2-a^2)(\eta^2-1) \tag{2.25}$$

consistent with the condition (2.21), the solutions for $\tilde{\Psi}$ and $\tilde{\Phi}$ simplify considerably to give:

$$\tilde{\Psi} = \frac{1}{\rho^2}(\Delta - a\delta) \quad \text{and} \quad \tilde{\Phi} = \frac{2aM\mu}{\rho^2}, \tag{2.26}$$

where

$$\rho^2 = r^2 + a^2\mu^2 = r^2 + a^2\cos^2\vartheta.$$

Now, by the conjugate Ernst's equations [26], we obtain

$$\tilde{\Psi}_{,2} = -\frac{4aMr\mu}{\rho^4} = \frac{\tilde{\Psi}^2}{\Delta}\tilde{\omega}_{,3} = \frac{(\Delta - a^2\delta)^2}{\rho^4\Delta}\tilde{\omega}_{,3} \tag{2.27}$$

$$\tilde{\Phi}_{,3} = \frac{2aM}{\rho^4}(r^2 - a^2\mu^2) = -\frac{\tilde{\Psi}^2}{\delta}\tilde{\omega}_{,2} = -\frac{(\Delta) - a^2\delta)^2}{\rho^4\delta}\tilde{\omega}_{,2}. \tag{2.28}$$

Accordingly

$$\tilde{\omega}_{,3} = -\frac{4aMr\mu\Delta}{(\Delta - a^2\delta)^2} \quad \tilde{\omega}_{,2} = -\frac{2aM(r^2 - a^2\mu^2)\delta}{(\Delta - a^2\delta)^2} \tag{2.29}$$

and the solution for $\tilde{\omega}$ is

$$\tilde{\omega} = \frac{\omega}{\chi^2 - \omega^2} = \frac{2aMr\delta}{\Delta - a^2\delta} \tag{2.30}$$

We also have, in the case of conjugate equations:

$$\tilde{\Phi} = e^{2\psi}(\chi^2 - \omega^2) = e^{2\nu} - \omega^2 e^{2\psi} = \frac{1}{\rho^2}(\Delta - a^2\delta) \tag{2.31}$$

Making use of the last equation, we obtain:

$$\omega = \frac{2aMr\delta}{\Delta - a^2\delta}(\chi^2 - \omega^2) = \frac{2aMr\delta}{\rho^2}e^{-2\psi} \tag{2.32}$$

Now, combining the equations (2.31) and (2.32), we get

$$\frac{\Delta - a^2\delta}{\rho^2}e^{2\psi} = \frac{\delta}{\rho^4}(\Delta\rho^4 - 4a^2M^2r^2\delta). \tag{2.33}$$

The solutions for ω and $e^{2\psi}$ can be written in a more simple form by using the identities given in [26]. Making use of these identities we find from Eq. (2.33):

$$e^{2\psi} = \frac{\delta\Sigma^2}{\rho^2} \tag{2.34}$$

where $\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\delta$. From Eq.(2.32), it follows that

$$\omega = \frac{2aMr}{\Sigma^2} \tag{2.35}$$

Furthermore, we also have

$$e^{2\nu} = \frac{\rho^2\Delta}{\Sigma^2} \tag{2.36}$$

and

$$\chi = \frac{\rho^2\sqrt{\Delta}}{\Sigma^2\sqrt{\delta}} \tag{2.37}$$

Then, from equations (2.35) and (2.37), it follows

$$X = \chi + \omega = \frac{\sqrt{\Delta} + a\sqrt{\delta}}{[(r^2 + a^2) + a\sqrt{\Delta\delta}]\sqrt{\delta}} \tag{2.38}$$

and

$$Y = \chi - \omega = \frac{\sqrt{\Delta} - a\sqrt{\delta}}{[(r^2 + a^2) - a\sqrt{\Delta\delta}]\sqrt{\delta}} \tag{2.39}$$

Finally, to complete the solution we consider the equations arising respectively from the equation $R_{23} = 0$ and from the difference between the equations $G_{22} = 0$ and $G_{33} = 0$. After some elementary reductions and making use of the derivatives of X and Y (which are shown in [26]), these equations become:

$$-\frac{\mu}{\delta}(\mu_3+\mu_2)_{,2})+\frac{r-M}{\Delta}(\mu_3+\mu_2)_{,3}) = \frac{\mu}{\rho^2\Delta\delta}[(r-M)(\rho^2+2a^2\delta)-2r\Delta] \tag{2.40}$$

and

$$2(r - M)(\mu_3 + \mu_2)_{,2}) + 2\mu(\mu_3 + \mu_2)_{,3}) = 4 - \frac{2(r - M)^2}{\Delta} - \frac{4rM}{\rho^2} \tag{2.41}$$

It is possible to verify that the solution of these equations is given by

$$e^{\mu_3+\mu_2} = \frac{\rho^2}{\sqrt{\Delta}} \tag{2.42}$$

and since $e^{\mu_3-\mu_2} = \sqrt{\Delta}$, we obtain

$$e^{2\mu_2} = \frac{\rho^2}{\Delta} \quad \text{and} \quad e^{2\mu_3} = \rho^2 \tag{2.43}$$

Now the solution for all metric coefficients is complete and the resulting metric is the Kerr metric and can be written as:

$$ds^2 = \rho^2\frac{\Delta}{\Sigma^2}dt^2 - \frac{\Sigma^2}{\rho^2}\left(d\varphi - \frac{2aMr}{\Sigma^2}dt\right)^2 \sin^2\vartheta - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\vartheta^2. \tag{2.44}$$

2.2.3 Straumann's derivation

In the work of Straumann a geometrical approach is presented.
A spacetime (M, g) is axisymmetric if it admits the group $SO(2)$ as an isometry group such that the group orbits are closed spacelike curves. The spacetime (M, g) is stationary and axisymmetric if $R \times SO(2)$ acts isometrically, such that (M, g) is axisymmetric with respect to the subgroup $SO(2)$ and the Killing field belonging to R (time translation) is at least asymptotically timelike.
The two Killing fields belonging to R and $SO(2)$ will be denoted by X and Y. They obviously commute:

$$[X, Y] = 0$$

The orbits of the $R \times SO(2)$ action are bidimensional submanifolds, whose tangent spaces are spanned by k and m. The collection of these tangent space, define an involutive integrable distribution $\mathcal{D}$. Given the orthogonal distribution $\mathcal{D}^{\perp}$, we make the generic assumption that $\mathcal{D} \bigcap \mathcal{D}^{\perp} = \{0\}$.
From the *Frobenius theorem* it follows that $\mathcal{D}^{\perp}$ is involutive if and only if the ideal generated by the 1-forms χ and υ, (which annihilate the distribution $\mathcal{D}^{\perp}$, is differential. The 1-forms χ and υ are defined by

$$< X, Z >= g(X, Z) \quad \forall Z,$$
$$< Y, Z >= g(Y, Z) \quad \forall Z.$$

The aforementioned condition in turn is equivalent to the following *Frobenius conditions*:

$$X \wedge Y \wedge dX = 0, \quad X \wedge Y \wedge dY = 0. \tag{2.45}$$

The vacuum Einstein's equations imply the conditions (2.45), so $\mathcal{D}^{\perp}$ is integrable. When these conditions are satisfied the spacetime (M, g) is called circular and exist adapted coordinates x^a $(a = 0, 1)$, and x^A $(A = 2, 3)$, such that

$$X = \partial_t, \quad Y = \partial_\varphi \quad \text{where} \quad (x^0 = t, x^1 = \varphi),$$

and

$$g = g_{ab}(x^C)dx^a dx^b + g_{AB}(x^C)dx^A dx^B. \tag{2.46}$$

Now, starting from this metric and performing cumbersome calculations, Straumman [] arrives to the Ernst's equation:

$$[(x^2-1)\varepsilon_{,x}]_{,x} + [(1-y^2)\varepsilon_{,y}]_{,y}] = -\frac{2\bar{\varepsilon}}{1-\varepsilon\bar{\varepsilon}}[(x^2-1)\varepsilon_{,x}^2 + (1-y^2)\varepsilon_{,y}^2],$$

where $\varepsilon(x, y)$ is the complex potential.
Then, the Straumann's derivation of the Kerr metric is identical to the Chandrasekhar's one.

2.2.4 The Kerr metric revisited

In both previous derivations, the Kerr metric is obtained in the Boyer-Lindquist coordinates:

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\vartheta d\varphi)^2 + \frac{\sin^2\vartheta}{\Sigma}\left[adt - (r^2+a^2)d\varphi\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\vartheta^2 \tag{2.47}$$

where:

$$\Delta = r^2 - 2mr + a^2 \quad \Sigma = r^2 + a^2\cos^2\vartheta \tag{2.48}$$

or more explicitly:

$$\begin{aligned} ds^2 \;\; = \;\; & -\left[1 - \frac{2mr}{\Sigma}\right]dt^2 - \frac{4mra\sin^2\vartheta}{\Sigma}dtd\varphi + \frac{\Sigma}{\Delta}dr^2 \\ & +\Sigma d\vartheta^2 + \sin^2\vartheta\left[r^2 + a^2 + \frac{2mra^2\sin^2\vartheta}{\Sigma}\right]d\varphi^2. \end{aligned} \tag{2.49}$$

The advantage of these coordinates is that they reduce the number of off diagonal terms from three to one. Furthermore this form of Kerr metric allows a better analysis of the asymptotic behavior and a better understanding the of the difference between an event horizon and an ergosphere.
It is straightforward to verify that Kerr metric in the Boyer-Lindquist form reproduces the Schwarzschild metric when the parameter $a \to 0$ and it is a flat Minkowski space in so-called 'oblate spheroidal coordinates' when $m \to 0$.
Focusing on symmetries arising in the case of the gravitational field for a rotating body, it is clear that this metric form is invariant under simultaneous inversion of t and φ, i.e. under the transformation $t \to -t$ and $\varphi \to -\varphi$, although it is not invariant under inversion of t alone (except when $a = 0$).
This is what one would expect, since time inversion of a rotating object produces an object rotating in the opposite direction.
From a geometrical point of view this means that the Kerr solution, being stationary and axisymmetric, has a two-parameter group of isometries which is necessarily Abelian as shown by Carter in [22]. Therefore the *Frobenius conditions* (2.45) are satisfied and the orthogonal distribution $\mathcal{D}^\perp$ is integrable.

Both characterizations of the Kerr metric, (i.e. the Chandrasekhar and the Straumann derivations), are not still complete from a geometrical point of view.
It would be interesting to derive the Kerr metric from the general approach given by Sparano,Vilasi,Vinogradov ([95]).
A first step is then, to introduce adapted coordinates to the Killing fields and

conformal coordinates for the submanifold orthogonal to the Killing leaves. The Kerr metric in Boyer-Lindquist coordinates (2.49) is not written in these adapted coordinates then we have to make a coordinates transformation by introducing a new coordinate ρ:

$$d\rho = \frac{dr}{\sqrt{r^2 - 2mr + a^2}}$$

which implies:

$$\rho = \int \frac{dr}{\sqrt{|\Delta|}}$$

We obtain the following form of the Kerr metric:

$$\begin{aligned} ds^2 &= f(d\rho^2 + d\vartheta^2) + \sin^2\vartheta \left[\psi(\rho)^2 + a^2 + \frac{2m\psi(\rho)a^2\sin^2\vartheta}{f}\right] d\varphi^2 \\ &\quad - \frac{4ma\psi(\rho)\sin^2\vartheta}{f} dt d\varphi - \left[1 - \frac{2m\psi(\rho)}{f}\right] dt^2 \end{aligned} \tag{2.50}$$

The Kerr metric lies in the case represented by the matrix (2.11), when the parameter s is zero, (Abelian case):

$$M(g) = \begin{pmatrix} \begin{matrix} 2f & 0 \\ 0 & 2f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \beta^2 \begin{pmatrix} m & l \\ l & k \end{pmatrix} \end{pmatrix} \tag{2.51}$$

Furthermore:

$$\det F > 0 \quad h_{22} = g(Y, Y) \neq 0$$

The first condition is necessary for a Lorentzian signature of the metric while the condition on h_{22} element is necessary to avoid that $\det H = 0$. At this point, our purpose is to find dome intrinsic relations which link the parameter α with the properties of matrix H. First of all, it is known that

$$\alpha = \sqrt{|\det H|} \tag{2.52}$$

We observe that:

$$\alpha_{11} = \alpha, \quad \alpha_{22} = -\alpha$$

Such conditions may be sufficient to determine α and also prove that α is harmonic, $\nabla^2\alpha = 0$, as expected from Eq. (2.7).

2.2.5 The $R_{ij} = 0$ equations

Taking into account that the Kerr metric is Ricci flat and that its Killing vector fields commute ($s = 0$), we have to impose the equations (2.7) vanish,

$$R_{ij} == \frac{\mathbf{H}}{2f\alpha}\left[\frac{1}{2}\left[(\alpha\mathbf{H}^{-1}\partial_1(\mathbf{H}))_{,1} + (\alpha\mathbf{H}^{-1}\partial_2(\mathbf{H}))_{,2}\right]\right] = 0. \qquad (2.53)$$

In this way, we obtain from Eqs.(2.8), (2.9), (2.10), respectively:

$$\frac{1}{2}\mathrm{tr}(H^{-1}\partial_1 H)^2 = \frac{\alpha_{,1}}{\alpha}\partial_1(\ln|f|) - \frac{\alpha_{,2}}{\alpha}\partial_2(\ln|f|) - 2\partial_1\left(\frac{\alpha_{,1}}{\alpha}\right) - \Delta(\ln|f|).$$

$$\frac{1}{2}\mathrm{tr}(H^{-1}\partial_2 H)^2 = -\left(\frac{\alpha_{,1}}{\alpha}\right)\partial_1(\ln|f|) + \frac{\alpha_{,2}}{\alpha}\partial_2(\ln|f|) - \Delta(ln|f|) - 2\left(\frac{\alpha_{,2}}{\alpha}\right)$$

and

$$\frac{1}{2}\mathrm{tr}[(H^{-1}\partial_1 H)(H^{-1}\partial_2 H)] = \left(\frac{\alpha_{,1}}{\alpha}\right)\partial_2(\ln|f|) + \left(\frac{\alpha_{,2}}{\alpha}\right)\partial_1(\ln|f|).$$

Furthermore, in this case can be stressed that:

$$\partial_1\partial_2(ln|\alpha|) = 0$$

which implies

$$\alpha_{12} = \frac{\alpha_1\alpha_2}{\alpha}$$

.

Then we have:

$$\Delta(\ln(\alpha f)) = \Delta(\ln\alpha) + \Delta(\ln|f|) = \partial_1\left(\frac{\alpha_{,1}}{\alpha}\right) + \partial_2\left(\frac{\alpha_{,2}}{\alpha}\right) + \Delta(\ln|f|)$$

By using the general property that:

$$d(\det A) = (\det A)\mathrm{tr}(A^{-1}dA)$$

where A denotes an arbitrary matrix, we obtain:

$$\mathrm{tr}(H^{-1}\partial_1 H) = \frac{\partial_1(\det H)}{\det H} = \frac{\partial_1\alpha^2}{\alpha^2} = \frac{2\alpha\alpha_1}{\alpha^2} = \frac{2\alpha_1}{\alpha}$$

and:

$$\mathrm{tr}(H^{-1}\partial_2 H) = 2\left(\frac{\alpha_{,2}}{\alpha}\right)$$

Now, we consider the Eq. (2.53), where the H block can be written as follows:

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$$

After computing the elements of the matrix $[(\alpha H^{-1}\partial_1(H))_{,1}+(\alpha H^{-1}\partial_2(H))_{,2})]$, we evaluate the difference between the diagonal terms. The result is:

$$\frac{1}{2}\frac{1}{\alpha^3}\left[-h_{22}^2[(\partial_1 h_{11})^2 + (\partial_2 h_{11})^2)] + h_{11}^2[(\partial_1 h_{22})^2 + (\partial_2 h_{22})^2)]\right] +$$

$$\frac{1}{2}\frac{1}{\alpha^3}\left[2h_{22}\alpha^2(\nabla^2 h_{11}) - 2h_{11}\alpha^2(\nabla^2 h_{22})\right] +$$

$$\frac{1}{2}\frac{1}{\alpha^3}\left[2h_{12}(\partial_1 h_{12})[h_{22}(\partial_1 h_{11}) - h_{11}(\partial_1 h_{22})] + 2h_{12}(\partial_2 h_{12})[h_{22}(\partial_2 h_{11}) - h_{11}(\partial_2 h_{22})]\right].$$

This work is still in progress.

Chapter 3

The Newman-Janis Algorithm

This chapter is devoted to the Newman-Janis Algorithm, a solution generating technique used in General Relativity. We mainly consider the problem of finding exact solutions of Einstein equations describing gravitational fields generated by isolated sources, in order to provide theoretical models for astrophysical objects. To this aim, the Newman-Janis Algorithm is described along with the main results obtained through it and underlining some ambiguities which arise in dealing with it. Many issue related with the introduction of a cosmological constant term are also pointed out and some detailed examples are discussed.
We review some interesting results obtained through the Newman-Janis algorithm and we also describe the use of this algorithm in different theories, namely f(R), Einstein-Maxwell-Dilaton gravity, Braneworld, Born Infeld Monopole focusing on the validity of the results.

3.1 The method

We describe the Newman-Janis Algorithm (henceforth NJA) in details considering the two most famous results obtained through it, the vacuum solution (Kerr) and the electro-vacuum solution (Kerr-Newman), (see[78, 77]). This will give us the chance to set up the formalism and stress some ambiguities in the method that will be further discussed in the next section.
Following [78], we show how it is possible to derive the Kerr solution from the Schwarzschild one through the NJA. Let's start by writing the Schwarzschild metric, considered as a static spherically symmetric seed metric, in advanced Eddington-Finkelstein coordinates (i.e. the g_{rr} component is eliminated by

a change of coordinates and a crossterm is introduced):

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

By introducing the formalism of null tetrad, the contravariant metric components can be written as:

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu \tag{3.1}$$

where

$$\begin{aligned} l_\mu l^\mu &= m_\mu m^\mu = n_\mu n^\mu = 0 \\ l_\mu n^\mu &= -m_\mu \bar{m}^\mu = 1 \\ l_\mu m^\mu &= n_\mu m^\mu = 0. \end{aligned}$$

For the Schwarzschild spacetime a possible null tetrad characterized by the vectors $(l^\mu, n^\nu, m^\mu, \bar{m}^\nu)$ is:

$$\begin{aligned} l^\mu &= \delta_1^\mu \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - \frac{2m}{r}\right)\delta_1^\mu \\ m^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right) \end{aligned}$$

This complex null tetrad system is the starting point for the derivation of Kerr space-time. Now, let the coordinate r to take complex values so the complex conjugate of r appears:

$$\begin{aligned} l^\mu &= \delta_1^\mu \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - m\left[\frac{1}{r} + \frac{1}{\bar{r}}\right]\right)\delta_1^\mu \\ m^\mu &= \frac{1}{\sqrt{2}\bar{r}}\left(\delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right) \end{aligned}$$

and then it is possible to perform the following complex coordinate transformation on the null vectors:

$$\begin{aligned} r' &= r + ia\cos\vartheta \\ u' &= u - ia\cos\vartheta \\ \vartheta' &= \vartheta \\ \varphi' &= \varphi, \end{aligned}$$

where a is a real parameter. By requiring that also r' and u' are real (that is considering the transformations as a complex rotation of the ϑ, φ plane), one obtains the following new tetrad:

$$\begin{aligned} l^\mu &= \delta_1^\mu \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - \frac{2mr'}{r'^2 + a^2\cos^2\vartheta}\right)\delta_1^\mu \\ m^\mu &= \frac{1}{\sqrt{2}(r' + ia\cos\vartheta)}\left(ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}(r' - ia\cos\vartheta)}\left(-ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right) \end{aligned}$$

The contravariant components of a new metric can be defined from the above null vectors according to Eq.(3.1). This gives the promised Kerr solution in advanced null coordinates. By performing a transformation on the null coordinate u and the angle coordinate φ, one obtains the usual representation of the Kerr metric in Boyer-Lindquist coordinates, namely the metric (2.49) which we have introduced in the previous chapter.
The same procedure can be used to get the Kerr-Newman metric from the Reissner-Nordström one [77] which, in advanced null coordinates, has the following form:

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{q^2}{r^2}\right)du^2 + 2dudr - r^2(d\vartheta^2 + \sin^2\vartheta\varphi^2)$$

where q represents the charge.
For this space-time the null tetrad is:

$$\begin{aligned} l^\mu &= \delta_1^\mu \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - \frac{2m}{r} - \frac{q^2}{r^2}\right)\delta_1^\mu \\ m^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right) \end{aligned}$$

After the complexification of the radial coordinate r, the tetrad becomes:

$$\begin{aligned} l^\mu &= \delta_1^\mu \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - m\left[\frac{1}{r} + \frac{1}{\bar{r}}\right] - \frac{q^2}{r\bar{r}}\right)\delta_1^\mu \\ m^\mu &= \frac{1}{\sqrt{2}\bar{r}}\left(\delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right) \end{aligned}$$

Performing the complex coordinate transformation as above, one has:

$$\begin{aligned} l^\mu &= \delta_1^\mu \\ n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - \frac{2mr'}{r'^2 + a^2\cos^2\vartheta} - \frac{q^2}{r'^2 + a^2\cos^2\vartheta}\right)\delta_1^\mu \\ m^\mu &= \frac{1}{\sqrt{2}(r' + ia\cos\vartheta)}\left(ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}(r' - ia\cos\vartheta)}\left(-ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right) \end{aligned}$$

Replacing in Eq.(3.1) allows to recover usual the Kerr-Newmann solution in advanced null coordinates.

The ambiguity underlining the NJA can be easily recognized by comparing the two outlined procedures. Indeed, the complex coordinate transformation introduced in [78], has no fundamental explanation or derivation. It should also be noted a certain arbitrariness since the complexification procedure of the terms $1/r$, $2m/r$, q^2/r^2. In [77], the authors just say that if the term q^2/r^2 is replaced by $\frac{1}{2}(1/r^2 + 1/\bar{r}^2)$, as expected, one does not obtain a solution of Einstein-Maxwell equations. This ambiguity in the complexification of the r coordinate will be even more evident in the next section, where some examples and some discording results, previously available in literature, are discussed.

3.2 The NJA in General Relativity

After the introduction of the NJA to generate the Kerr solution from the Schwarzschild metric and the Kerr-Newman solution from the Reissner-Nordström metric, this method has been applied with the aim to generate new solutions. In this section we discuss some interesting results obtained in different cases. Generally the NJA has been treated as an useful procedure for generating

new solutions of Einstein's equations from known static spherically symmetric ones, thus the method turns out to be suitable for studying rotating systems in General Relativity.
Precisely in [32], Demianski and Newman, in order to show that new metrics can be obtained, applied the Newman-Janis technique with a different complex coordinate transformation to the Schwarzschild solution in null polar coordinates, as follows:

$$\begin{aligned} r' &= r + \mathrm{i}(\mathrm{a}\cos\vartheta + \mathrm{b}) \\ u' &= u - \mathrm{i}(\mathrm{a}\cos\vartheta + 2\mathrm{b}\ln(\sin\vartheta)) + 2\mathrm{ib}\ln(\tan\vartheta/2). \end{aligned}$$

The final result is a solution of Einstein's equations in vacuum and appears as a combination of Kerr metric and the Newman-Unti-Tamburino (NUT) space metric. It depends, in fact, on three arbitrary parameters m, a and b. When $a = b = 0$ the Schwarzschild solution is obtained; if $a = 0$ it becomes the NUT space and if $b = 0$ it becomes the Kerr solution. Subsequently, it was demonstrated that by performing a more general complex coordinate transformation, see [33], it is possible to find the most general solution of Einstein field equations obtainable in this way and in which a non vanishing cosmological constant is allowed. With this result, by setting $\Lambda = 0$, one recovers the standard form of the NUT space. However, in this way, it becomes impossible to find the Kerr solution with the cosmological constant and, in particular, this result shows that the Carter's Kerr de Sitter metric cannot be obtained with the NJA, ([23]).
An explanation concerning the success of this "trick" is shown in [49]. Here, Gürses and Gürsey pointed out that, as ensues from [87] where the Kerr-Schild metric is obtained by performing an imaginary displacement (ia) of the coordinates, a complex translation of coordinates is allowed in GR when a coordinates system is found in which the pseudo energy-momentum tensor vanishes or the Einstein equations are linear. This works only in an algebraically special Kerr-Schild geometry.
Several attempts have been made by using the NJA to generate interior Kerr solutions ([35], [55], [60], [100]); however these results were unsuccessful in finding a solution that is both physically reasonable and can be matched smoothly to the Kerr metric.
In particular, in the work of Herrera and Jimenez ([55]), the algorithm was indeed applied to an interior spherically symmetric metric to describe an internal source model for the Kerr exterior solution. The resulting metric was then matched to the vacuum Kerr solution on an oblate spheroid. In [35], Drake and Turolla, in order to obtain new possible sources for Kerr metric, applied the NJA to a generic static spherically symmetric seed metric. Then,

to join any two stationary and axially symmetric metrics, the Darmois-Israel junction conditions were imposed on a suitable separating hypersurface, thus having a vanishing surface stress-energy tensor.
For these reasons, these results were considered as starting point to perform a generalization of the algorithm and to demonstrate why this method is successful. To do this, it is necessary to remove some of the ambiguities appearing in the original derivation, as shown in [34] by Drake and Szekeres, where it was also considered that the only perfect fluid space-time generated by applying the NJA to a static spherically symmetric seed metric, is the Kerr metric and that the Kerr-Newman metric is the most general algebraically special space-time which can be so obtained. The connection with [35] is in the fact that, while the NJA is successful in generating interior space-times, which match smoothly to the Kerr metric and are considered as perfect fluids in the non rotating limit, this is not the case when rotation is included.
The NJA was also used to obtain new metrics describing more general and complicate systems. This is the case of [71, 107], where a rotating radiating charge mass in a de Sitter cosmological background was studied.
Subsequently, Ibohal combined the Newman-Janis method, with the Wang-Wu functions, (see [60]), which are an expansion of the mass in powers of the radius. In this case the seed metric was written in terms of the functions: $M(u, r)$ and $e(u, r)$, where u and r are the coordinates of the space-time geometry, in particular the u-coordinate is related to the retarded time in flat space-time. After the transformation of the metric through the NJA, these two functions depend on the three coordinates (u, r, ϑ). Then the Wang-Wu functions were introduced in the rotating solution to generate new embedded rotating solutions like Kerr-Newman-de Sitter. Furthermore it was shown that all rotating embedded solutions can be written in Kerr-Schild form which seems the most suitable form for the validity of NJA. It is straightforward to underline that the solutions found in [71] are quite different from those found in [60], as noticed by Ibohal himself. This is mainly due to the slightly different approaches followed by the authors and is an example of the ambiguities arising from the NJA already mentioned above. In both cases, the authors provide with a full description of the energy-momentum tensor required by these metrics in order for them to be solutions of the Einstein equations.
In a more recent paper by Viaggiu [100], in order to obtain Kerr interior solutions, starting from the Schwarzschild solution, the NJA is performed. Furthermore, the perturbative expansion at the first order in the parameter a (Slowly Rotating Limit) of the solutions previously obtained, was discussed and some remarks on the energy conditions for these solutions were collected. By starting from these results, a more deep analysis, concerning the ambiguities which arise in dealing with the NJA and the problems appearing when

the cosmological constant is introduced, has been showed in [18].

3.3 Some applications of NJA

In the following paragraphs, we discuss the extension of the NJA to more general seed metrics. When applied to seed metrics other than the two considered above, the NJA does not provide their "standard" rotating generalization, as one might have naively expected. Moreover, due to the arbitrariness in the application of the method, i.e. the complexification of the r coordinate, the algorithm provides discording results.
Firstly, we consider the application of the NJA to a seed de Sitter metric. This is the simplest example allowing to discuss many interesting features of the NJA, when a cosmological constant term is introduced. It turns out that, in order for the new metric to be a solution of the Einstein equations, a suitable matter source is required, which does not allow for a simple interpretation in terms of cosmological constant or perfect fluid. Some comparison with similar results appearing in literature is drawn.
Then, we discuss the application of the NJA to two slightly more general seed metrics: Schwarzschild-de Sitter and Reissner-Nordström-de Sitter. The new metrics so obtained require suitable matter sources in order for them to be solutions of the Einstein equations. Some comparison with similar results appearing in literature is drawn.

3.3.1 Rotating de Sitter metrics

Now we are able to construct a *rotating de Sitter* metric by applying the algorithm to the de Sitter solution. We start by writing the de Sitter metric in terms of its null tetrad vectors:

$$\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - \frac{\Lambda}{3}r^2\right)\delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right)
\end{aligned}$$

The resulting tetrad can be written in the following way:

$$\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \delta_0^\mu - \frac{1}{2}\left(1 - \frac{\Lambda}{3}(r'^2 + a^2\cos^2\vartheta)\right)\delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2}(r' + ia\cos\vartheta)}\left(ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}(r' - ia\cos\vartheta)}\left(-ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right)
\end{aligned}$$

After the complex transformation, the metric becomes:

$$\begin{aligned}
ds^2 &= \left[1 - \frac{1}{3}\Lambda\left(r^2 + a^2\cos^2\vartheta\right)\right]dt^2 + dtdr - (r^2 + a^2\cos^2\vartheta)d\vartheta^2 \\
&+ \left[\frac{1}{3}a\Lambda(r^2 + a^2\cos^2\vartheta)\sin^2\vartheta\right]dtd\varphi - a\sin^2\vartheta drd\varphi \\
&- \left[\frac{1}{3}a^2\Lambda(r^2 + a^2\cos^2\vartheta)\sin^4\vartheta + (r^2 + a^2)\sin^2\vartheta\right]d\varphi^2.
\end{aligned} \tag{3.2}$$

This is not a solution of the Einstein equations with cosmological constant Λ, as one could have expected recalling the fact that the well known Kerr-Newman-de Sitter solution (see Appendix C), after setting $q = m = 0$, is a solution of the Einstein equations with cosmological constant, involving the two remaining parameters a and Λ. This is the first indication that the actual Kerr-Newman-de Sitter solution can't be recovered through the NJA, as further discussed in the following.

The metric in Eq.(3.2) can be recovered as a particular case of metric presented in [71] by setting $Q(u) = M(u) = 0$, while it does not agree with the one dubbed *Rotating de Sitter solution* in [60]. This is due to the fact that in [60] the Wang-Wu functions are introduced in an intermediate step of the NJA and the involved coordinate r is dealt with in a different way in comparison with the complexification scheme discussed above and applied in our example and in [71].

3.3.2 Schwarzschild-de Sitter and Kerr-Newman-de Sitter

Let's consider the Schwarzschild-de Sitter solution as seed metric (i.e. see Appendix B in the case $a = q = 0$). Following the outlined procedure, that is, recasting the metric in advanced null coordinates, considering its representation in terms of null tetrad vectors and applying the NJA, we get the new tetrad:

$$
\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \delta_0^\mu - \frac{1}{2}\left[1 - \frac{\Lambda}{3}(r'^2 + a^2\cos^2\vartheta) - 2m\left(\frac{r'}{r'^2 + a^2\cos^2\vartheta}\right)\right]\delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2}(r' + ia\cos\vartheta)}\left[ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right] \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}(r' - ia\cos\vartheta)}\left[-ia\sin\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right]
\end{aligned}
$$

Now, it is possible to write the new line element using the so obtained tetrad. The result is strongly different from that expected for the analogy with the examples discussed in section (3.1). The metric obtained from this tetrad is not a solution of the Einstein equations with cosmological constant while, as known, the actual Kerr-de Sitter metric (i.e. see Appendix C in the case $q = 0$) is indeed an exact solution.

The same reasoning about not intuitive results obtained through the NJA, can be carried on in the case of Reissner-Nordström-de Sitter seed metric (i.e. see Appendix C in the case $a = 0$). Indeed, it does not provide a solution of the Einstein equations with cosmological constant and a suitable electromagnetic field. In both the aforementioned cases, this can be easily observed by evaluating the corresponding Ricci scalars.

Notice that both the aforementioned metrics can be derived from the metric presented in [71] with ($Q(u) = 0$, $M(u) = const.$) and ($Q = const.$, $M(u) = const.$) respectively. These metrics do not coincide with those presented in [60], as observed by the author himself. This is due to the modification of the NJA introduced by the author since he first applies the algorithm, then he makes use of the Wang-Wu function without applying the complexification procedure of the r involved coordinate. The author also provides a better interpretation of the sources in terms of non-perfect fluids, which, in turn, is quite different from the expected interpretation in terms of an electromagnetic field and a cosmological constant.

3.4 The Newman-Janis Algorithm in other theories

In recent years a new interest in the NJA has risen with the aim to test its validity in other theories of gravity, in order to generate new rotating

solutions.
Now we will show how this algorithm is performed in different theories by focusing on the obtained results.

3.4.1 The NJA in f(R)-gravity

Fourth order theories appear as a quite natural modification of GR theory. They consist in a straightforward generalization of the Lagrangian in the Einstein-Hilbert action by choosing a generic function $f(R)$ of the Ricci scalar. The field equations from this modified Lagrangian are of fourth order, i.e. they contain derivatives up to the fourth order of the component of the metric with respect to the spacetime coordinates. Recently, the interest in $f(R)$ theories, in particular in spherically symmetric solutions of $f(R)$, is increased. This should be the starting point to test the validity of the NJA in $f(R)$ gravity, see [19].

The standard procedure

Let's consider the spherically symmetric metric as

$$ds^2 = (\alpha + \beta r)dt^2 - \frac{1}{2}\left(\frac{\beta r}{\alpha + \beta r}\right) dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2).$$

By following the standard procedure as shown in Section 2.1, the metric is written in Eddington-Finkelstein coordinates, $(u, r, \vartheta, \varphi)$, and its null tetrad is:

$$\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \left(\sqrt{\frac{2}{\beta r}}\right)\delta_0^\mu - \left(-1 - \frac{2\alpha}{r\beta}\right)\delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(\delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right).
\end{aligned}$$

After the complexification of the radial coordinate r, it is possible to apply the NJA as usual:

$$\begin{aligned}
r' &= r + ia\cos\vartheta \\
u' &= u - ia\cos\vartheta.
\end{aligned}$$

The resulting null tetrad appears in the following way:

$$\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= -\left[1+\frac{\alpha}{\beta}\frac{\Re(r)}{\Sigma^2}\right]\delta_1^\mu - \left(\sqrt{\frac{2}{\beta\Sigma}}\right)\delta_0^\mu \\
m^\mu &= \frac{1}{\sqrt{2}(r+ia\cos\vartheta)}\left[ia(\delta_0^\mu-\delta_1^\mu)\sin\vartheta+\delta_2^\mu+\frac{i}{\sin\vartheta}\delta_3^\mu\right] \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}(r-ia\cos\vartheta)}\left[-ia(\delta_0^\mu-\delta_1^\mu)\sin\vartheta+\delta_2^\mu-\frac{i}{\sin\vartheta}\delta_3^\mu\right],
\end{aligned}$$

where $\Sigma=\sqrt{r^2+a^2\cos^2\vartheta}$. Now, making a *gauge transformation* (see also [35]), in order that the only off-diagonal term is $g_{\varphi t}$, one obtains a new axially symmetric metric as expected:

$$g_{\mu\nu}=\begin{pmatrix}
\frac{r(\alpha+\beta r)+a^2\beta\cos^2\vartheta}{\Sigma} & 0 & 0 & \frac{a(-2\Xi+\Gamma\Sigma^{3/2})\sin^2\vartheta}{2\Sigma} \\
\cdot & \frac{\beta\Sigma^2}{2\alpha r+\Lambda} & 0 & 0 \\
\cdot & \cdot & -\Sigma^2 & 0 \\
\cdot & \cdot & \cdot & -\left[\Sigma^2-\frac{a^2(\Xi-\Gamma\Sigma^{3/2})\sin^2\vartheta}{\Sigma}\right]\sin^2\vartheta
\end{pmatrix},$$

where:

$$\Lambda=\beta(a^2+r^2+\Sigma^2)$$
$$\Xi=\alpha r+\beta\Sigma^2$$
$$\Gamma=\sqrt{2\beta}.$$

The method can be also applied to any spherically solution derived in f(R)-gravity.

3.4.2 The NJA and "rotating dilaton-axion black hole"

In this paragraph, we describe the application of the Newman-Janis method in Einstein-Maxwell-dilaton-axion gravity, which is an interesting generalization of Einstein-Maxwell theory obtained in the low energy limit of the heterotic string theory. Precisely, in [109] is shown how the NJA can be used to derive the rotating dilaton-axion black hole solution from the static spherically symmetric charged dilaton black hole solution, found by Gibbons and independently by Garfinkle, Horowitz and Strominger. Since Sen (see [91]) was able to generate the rotating charged black hole solution by starting from the Kerr solution, it seems natural to verify if Sen's solutions can be

generated via NJA from the GGHS solutions.
The first step is to write the metric describing the dilaton black hole solution, namely GGHS, in the suitable form directly generated from the Schwarzschild solution ([110]):

$$ds^2 = \left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right) dt^2 - \left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right)^{-1} dr^2 - r^2\left(1+\frac{r_2}{r}\right)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$

where

$$r_1 + r_2 = 2M \quad r_2 = \frac{Q^2}{M},$$

and M and Q are the mass and the charge of the dilaton black hole. After expressing the metric in advanced coordinates with:

$$dt = du + \left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right)^{-1} dr,$$

is it possible to introduce the null tetrad:

$$\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \delta_0^\mu - \frac{1}{2}\left(\frac{1-\frac{r_1}{r}}{1+\frac{r_2}{r}}\right) \\
m^\mu &= \frac{1}{\sqrt{2}r\sqrt{1+\frac{r_2}{r}}}\left(\delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}r\sqrt{1+\frac{r_2}{r}}}\left(\delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right).
\end{aligned}$$

Following the standard procedure, one obtains the new null tetrad:

$$\begin{aligned}
l^\mu &= \delta_1^\mu \\
n^\mu &= \delta_0^\mu - \frac{1}{2}\left(\frac{1-\frac{r_1 r}{\Sigma}}{1+\frac{r_2 r}{\Sigma}}\right)\delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2}(r+ia\cos\vartheta)}\frac{1}{\sqrt{1+\frac{r_2 r}{\Sigma}}}\left(ia\cos\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\vartheta}\delta_3^\mu\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}(r-ia\cos\vartheta)}\frac{1}{\sqrt{1+\frac{r_2 r}{\Sigma}}}\left(-ia\cos\vartheta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu - \frac{i}{\sin\vartheta}\delta_3^\mu\right),
\end{aligned}$$

where $\Sigma = r^2 + \cos^2\vartheta$. After further simplifications and a suitable choice of coordinates, the rotating dilaton-axion black hole metric reads:

$$\begin{aligned} ds^2 &= \left(1-\frac{2Mr}{\tilde{\Sigma}}\right)dt^2 - \tilde{\Sigma}\left(\frac{dr^2}{\Delta}+d\vartheta^2\right) + \frac{4Mra\sin^2\vartheta}{\tilde{\Sigma}}dtd\varphi \\ &\quad - \left[r(r+r_2)+a^2+\frac{2Mra^2\sin^2\vartheta}{\tilde{\Sigma}}\right]\sin^2\vartheta d\varphi^2, \end{aligned}$$

where:

$$\tilde{\Sigma} = r(r+r_2)+a^2\cos^2\vartheta$$
$$\Delta = r(r-r_1)+a^2.$$

The final result coincides with that expected. However it is known that the static spherically symmetric charged dilaton black hole is also a solution to the truncated theory without axion field (i.e. Einstein-Maxwell-dilaton gravity) but, in this case, the Newman-Janis method does not work. The reason should be that the full theory has a larger symmetry group than the truncated one, [109].

3.4.3 The NJA in Braneworld

In the framework of Braneworld, the NJA is applied to three static, spherically symmetric seed metrics in the following form [105]:

$$ds^2 = -e^{2\varphi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2,$$

with three different choices of the functions $e^{2\varphi(r)}$ and $e^{2\lambda(r)}$, which are respectively:

$$\begin{aligned} e^{2\varphi(r)} &= \left[(1+\epsilon)\sqrt{1-\frac{2M}{r}}-\epsilon\right]^2, \quad & e^{2\lambda(r)} &= \left(1-\frac{2M}{r}\right)^{-1} \\ e^{2\varphi(r)} &= \left(1-\frac{2M}{r}\right), \quad & e^{2\lambda(r)} &= \frac{\left(1-\frac{3M}{2r}\right)}{\left(1-\frac{2M}{r}\right)\left(1-\frac{r_o}{r}\right)} \\ e^{2\varphi(r)} &= \left(1-\frac{2M}{r}-\frac{4}{3}\frac{Ml^2}{r^3}\right), \quad & e^{2\lambda(r)} &= \left(1-\frac{2M}{r}-\frac{2Ml^2}{r^3}\right)^{-1}. \end{aligned}$$

It is noticed that all these three metrics reduce to the Schwarzschild solution in the appropriate limits which are respectively:

$$\begin{aligned} \epsilon &\to 0 \\ r_0 &\to 3m/2 \\ l &\to 0, \end{aligned}$$

where l is the curvature lenght. After the usual complexification and the introduction of Boyer-Lindquist coordinates, the resulting metric is:

$$
\begin{aligned}
ds^2 &= -e^2\varphi dt^2 - 2a\sin^2\vartheta e^{\varphi}(e^{\lambda}-e^{\varphi})dt d\psi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\vartheta^2 \\
&\quad + \sin^2\vartheta[\Sigma + a^2\sin^2\vartheta e^{\varphi}(2e^{\lambda}-e^{\varphi})]d\psi^2,
\end{aligned}
$$

where the exponential functions become:

$$
\begin{aligned}
e^{\varphi(r)} &= \left[(1+\epsilon)\sqrt{1-\frac{2Mr}{\Sigma}}-\epsilon\right], \qquad e^{\lambda(r)} = \left(1-\frac{2Mr}{\Sigma}\right)^{-1/2} \\
e^{\varphi(r)} &= \left(1-\frac{2Mr}{\Sigma}\right)^{1/2}, \qquad e^{\lambda(r)} = \frac{\left(1-\frac{3Mr}{2\Sigma}\right)^{1/2}}{\left(1-\frac{2Mr}{\Sigma}\right)^{1/2}\left(1-\frac{r_o r}{\Sigma}\right)^{1/2}} \\
e^{\varphi(r)} &= \left(1-\frac{2Mr}{\Sigma}-\frac{4}{3}\frac{Ml^2r}{\Sigma^2}\right)^{1/2}, \qquad e^{\lambda(r)} = \left(1-\frac{2Mr}{\Sigma}-\frac{2Ml^2r}{\Sigma^2}\right)^{-1/2}.
\end{aligned}
$$

and:

$$
\begin{aligned}
\Sigma &= r^2 + a^2\cos^2\vartheta \\
\Delta &= \Sigma e^{-2\lambda} + a^2\sin^2\vartheta.
\end{aligned}
$$

It is straightforward to see that all three obtained metrics do not satisfy the condition to be a valid braneworld solution, i.e. $R = 0$. Then this form of NJA does not appear to be useful to get more general rotating Braneworld Black Holes solutions from the static, spherically symmetric ones, even though it partially works in order to generate the metric for a rotating source on the brane and for the tidal Kerr-Newman black hole.

3.4.4 The NJA in Born Infeld Monopole

The Born-Infeld theory is a non linear generalization of Maxwell electrodynamics, considered as the only *completely exceptional* regular non linear electrodynamics. With the advent of new developments of the string and brane theories, the BI electrodynamics has undergone a revival interest.
In [69] the application of the NJA to the static spherically symmetric metric of a Born-Infeld monopole, firstly investigated by Hoffmann [57], is discussed. The aim is to determine if the metric obtained via NJA coincides with the

metric obtained from the Born-Infeld monopole with rotation. By following the original steps given by Newman and Janis, the starting metric in Eddington-Finkelstein coordinates is:

$$ds^2 = -\left(\frac{\Delta}{r^2}\right) du^2 - 2dudr + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

where:

$$\begin{aligned} u &= t - r - f(r) \\ \Delta &= r^2 - 2GMr + Q^2(r), \end{aligned}$$

and $Q^2(r)$ is a complicated function of the Born-Infeld radius. Performing the usual complex transformation, the final result, in the suitable Boyer and Lindquist coordinates, is:

$$g_{\mu\nu} = \begin{pmatrix} \frac{a\sin^2\vartheta - \Delta}{\rho^2} & 0 & 0 & \frac{a\sin^2\vartheta[\Delta-(r^2+a^2)]}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ \frac{a\sin^2\vartheta[\Delta-(r^2+a^2)]}{\rho^2} & 0 & 0 & -\left[\frac{sin^2\vartheta[(r^2+a^2)^2-\Delta a^2\sin^2\vartheta]}{\rho^2}\right] \end{pmatrix},$$

where $\Delta \simeq r^2 - 2GMr + Q^2(r) + a^2$.
This metric corresponds to the Kerr metric in Boyer-Lindquist coordinates when $Q(r) = 0$ and to the static Born-Infeld monopole when $a = 0$. It results that, even though this new metric reproduces the behavior of the metric for a rotating spherical charged source, it cannot be associated with the source of the rotating Born-Infeld monopole. This aspect is deeply analyzed in [69], where this problem is pointed out comparing the structure of the energy-momentum tensors (considered on the same basis vectors) for both metrics. From this study it comes out that the interpretation given by Newmann and Janis to the complex coordinates transformations works only for a linear theory.

3.5 Discussions

From the previous analysis about the NJA, it is pointed out that, even in the standard examples, some ambiguities arise in the complexification procedure for the radial coordinate r. This ambiguity is even more evident when the NJA is carried out using the Wang-Wu functions, rather then applying the algorithm on a specific explicit metric. We have considered various examples in which the NJA does not provide a solution belonging to the family of

Kerr-Newman-de Sitter solutions, as one may have expected.
We have also taken into account two class of theories in which the method is successful: the f(R)-theories and the Einstein-Maxwell-Dilaton gravity. Is is worth noticing some interesting aspects which arise from the application of the method in these theories. As pointed out in Section (3.4.4), in the truncated theory, without axion field, the NJA does not generate the expected rotating solution.
Then, we have considered two classes of theories in which the NJA does not return the expected results: the Braneworld scenario and the Born-Infeld theory. In both cases it is still unclear why this method fails and it seems useful to perform a modification of the algorithm in order to extend its application to other theories.

Chapter 4

Einstein's solutions in Cosmology

In this chapter the stability properties of the Einstein Static solution of General Relativity are studied. These properties are altered when corrective terms arising from modifications of the underlying gravitational theory appear in the cosmological equations.
Firstly, we will give a brief introduction on Cosmology, as described in the framework of GR and on Einstein Static Universe. Then, using dynamical system techniques and numerical integrations, we will discuss the stability of static cosmological solutions in the framework of two recently proposed quantum gravity models, namely Loop Quantum Cosmology and Horava-Lifshitz gravity. This work is based on the original results presented in [17, 83].

4.1 Cosmology: a brief introduction

The understanding of Universe has always attracted attention from physicists and curiosity from people but an exhaustive knowledge is far to be reached. Our Universe is a great mixture of structures which recover a wide range of scales: stars collected into galaxies ($\sim$ Kpc where 1Kpc $\simeq 3.1 \times 10^{19} m$), galaxies gravitationally bounded into clusters ($\sim Mpc$),and clusters compacted into superclusters ($\sim 150h^{-1}$Mpc, where $h \simeq 0.7$ is a parameter related to the expansion rate of the Universe).
Many astronomical observations such as the distribution of galaxies on the sky and the distribution of their apparent magnitudes and redshifts as well the distribution of radio sources on the sky, also reveal that Universe is homogeneous and isotropic on large scales ($> 150h^{-1}Mpc$) while it is visibly anisotropic on small ones. Homogeneity means that there are no preferred

locations in the Universe; isotropy means that there are no preferred directions in the Universe. It is worth noting that homogeneity does not imply isotropy, for example a Universe with a uniform magnetic field is homogeneous, as all points are the same, but it fails to be isotropic because directions along the field lines can be distinguished from those perpendicular to them. Both properties are included in what is called **Cosmological Principle**,[66]:

'the Universe looks the same whoever and wherever you are'

which can be considered valid when we are concerned with the Universe as a whole, assuming the large-scale invariance. This principle is strictly related to the discovery of Hubble about the expansion of the Universe.

From a relativistic point of view, Cosmology can be considered as a task of finding solutions to Einstein's field equations that are consistent with the large-scale matter distribution in the Universe.
Friedmann, Lemaître and other theorists showed how the expansion of the Universe could be described by a spatially homogeneous and isotropic model obeying the field equations of General Relativity, but the geometrical approach is due to Robertson and Walker whose resulting metric is:

$$ds^2 = -dt^2 + a(t)^2[d\chi^2 + \Phi_k^2(\chi)(d\vartheta^2 + \sin^2\varphi)]$$

where:

$$\Phi_k^2(\chi) = \begin{cases} \sinh^2(\chi) & \text{for } k = -1, \\ \chi^2 & \text{for } k = 0, \\ \sin^2(\chi) & \text{for } k = 1. \end{cases}$$

The parameter k describes the curvature of the spatial sections (slices at constant cosmic time): $k = +1$ corresponds to positively curved spatial sections (locally isometric to 3- spheres); $k = 0$ corresponds to local flatness, and $k = -1$ corresponds to negatively curved (locally hyperbolic) spatial sections.
The dynamics of the Universe as characterized by the evolution of the scale factor $a(t)$ is determined by the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

with $T_{\mu\nu}$ the stress-energy tensor describing the matter content of the model which is forced by the symmetry of the model to have the algebraic form of a perfect fluid:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$$

where U^μ is the fluid four-velocity; ρ and p are energy density and pressure in the rest frame of the fluid respectively.
The Einstein equations, lead to two non linear differential equations called Raychaudhuri equation:

$$\dot{H} = -\frac{\kappa}{2}\rho\,(1+w) + \frac{k}{a^2} \tag{4.1}$$

and Friedmann equation:

$$H^2 = \frac{\kappa}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{4.2}$$

where $H = \dot{a}/a$ is the *Hubble parameter*, $\kappa = 8\pi G/3$ and the dot denotes derivative with respect to cosmic time t.
According to General Relativity which encodes energy conservation, it is straightforward to obtain a single energy-conservation equation by performing the covariant derivative of the stress-energy tensor, $\nabla_\mu T^{\mu\nu} = 0$:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{4.3}$$

These three equations are not independent, indeed the Friedmann equation is a first integral of Eq. (4.1) and Eq. (4.3) whenever $\dot{a} \neq 0$; it is a constraint which relates the expansion rate of the Universe to the energy density the spatial curvature of the Universe.
Adding a cosmological constant to the Einstein's equations is equivalent to including a new component of the energy density in the Universe described by an energy-momentum tensor of the form $T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$ with pressure and density $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ and $p_\Lambda = -\rho_\Lambda$.

4.2 Einstein Static Universe

The Einstein Static (ES) Universe is an exact solution of Einstein's equations describing a closed Friedmann-Robertson-Walker model sourced by a perfect fluid and a cosmological constant (see, for example, [54]). This solution is unstable to homogeneous perturbations as shown by Eddington [36], furthermore it is always neutrally stable against small inhomogeneous vector and tensor perturbations and neutrally stable against adiabatic scalar density inhomogeneities with high enough sound speed [4].

In recent years there has been renewed interest in the ES Universe because of its relevance for the Emergent Universe scenario [37, 38, 75] in which the ES solution plays a crucial role, being an initial state for a past-eternal inflationary cosmological model. In the Emergent Universe scenario the horizon problem is solved before inflation begins, there is no singularity, no exotic physics is involved, and the quantum gravity regime can even be avoided. This model, relying on the choice of a particular initial state, suffers from a fine-tuning problem which is ameliorated when modifications to the cosmological equations arise but then a mechanism is needed to trigger the expanding phase of the Universe (see [67, 68]).

The existence of ES solutions along with their stability properties has been widely investigated in the framework of General Relativity for several kinds of matter fields sources ([5] and references therein). ES solutions also exist in several modified gravity models [15] ranging from the Randall-Sundrum braneworld scenario [40, 48, 90, 27, 111] to Gauss-Bonnet modified gravity and $f(R)$ theories [11, 28, 12, 47, 45, 89, 13]. The issue of the existence and stability of ES solutions has also been considered in the semiclassical regime of Loop Quantum Cosmology (LQC), in either the case of correction to the matter sector [76] or the case of correction to the gravitational sector [83]. Recently the same issue has been also considered in the framework of Hořava-Lifshitz (HL) gravity [106] and IR modified Hořava gravity [14, 84]. When dealing with higher order modified cosmological equations, the existence of many new ES solutions is possible, whose stability properties, depending on the details of the single theory or family of theories taken into account, are substantially modified with respect to the classical ES solution of General Relativity (GR).

Often in such analysis the case of closed ($k = 1$) cosmological models is the only one considered, neglecting the intriguing possibility of static solutions in open ($k = -1$) cosmological models. Here we point out that, due to the aforementioned corrections to the cosmological equations, open ES models may be found even in the case of a vanishing cosmological constant or when the perfect fluid has vanishing energy density.

4.2.1 Einstein Static Universe in Loop Quantum Cosmology

The singularities arising in dealing with curvatures represent a breakdown of General Relativity (GR) and require an extended theory for a meaningful description. Among the theories leading to modifications of GR, Loop Quantum Cosmology (henceforth LQC), which can be considered an application of Loop Quantum Gravity (or quantum Riemannian geometry) to cosmological models, allows us to resolve the cosmological singularity. In fact, initial quantizations of LQC lead to a regularization of the big bang singularity [10] resulting from the fact that the quantum Einstein equation is non-singular

as well as from modifications to the scalar field energy density and dynamics. The modifications to the scalar field dynamics were based on effects arising from quantum inverse scale factor operators.
Mulryne et al. [76] used the scalar-field modification approach to investigate the stability of the Einstein static model to homogeneous perturbations. They found that the new LQC Einstein static model is a center fixed point in phase space, i.e. a neutrally stable point, for a massless scalar field with $w \equiv p_\phi/\rho_\phi = 1$. This modification of stability behavior has important consequences for the Emergent Universe scenario, since it ameliorates the fine-tuning that arises from the fact that the Einstein static is an unstable saddle in GR.

Our aim is to investigate how the LQC corrections affect the stability properties in the Einstein static Universe.

For the sake of simplicity, in this paragraph we consider the modified Friedmann equations arising in the semiclassical regime of LQC [2, 99]. We consider gravitational modifications only, neglecting the inverse volume correction to the matter sector. The motivation is twofold: the analysis of this system allows a more transparent comparison with the case of GR; moreover it allows us to follow the notations introduced in [83] which will also be easily used in the analysis of the HL gravity presented in the next section.

The model considered is sourced by a perfect fluid with linear equation of state $p = w\rho$ plus a cosmological constant Λ. The classical energy conservation equation still holds,

$$\dot{\rho} = -3\rho H(1+w), \tag{4.4}$$

while the loop quantum effects lead to a modification to the classical Friedmann equation,

$$H^2 = \left(\frac{\kappa}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa\rho_c} + \frac{3k}{\kappa\rho_c a^2}\right) \tag{4.5}$$

and to the Raychaudhuri equation,

$$\begin{aligned} \dot{H} &= -\frac{\kappa}{2}\rho\,(1+w)\left(1 - \frac{2\rho}{\rho_c} - \frac{2\Lambda}{\kappa\rho_c}\right) \\ &\quad + \left[1 - \frac{2\rho}{\rho_c} - \frac{2\Lambda}{\kappa\rho_c} - \frac{3\rho(1+w)}{\rho_c}\right]\frac{k}{a^2} + \frac{6k^2}{\kappa\rho_c a^4}. \end{aligned} \tag{4.6}$$

Notice that we are considering at once the $k = 0$ case and the $k = \pm 1$ cases [2, 99]. Here $\kappa = 8\pi G = 8\pi/M_P^2$, and the critical LQC energy density is $\rho_c \approx 0.82 M_P^4$.

4.2.2 Static solutions

The system of Eqs.(4.4)-(4.6) admits two static solutions, i.e. solutions characterized by $\dot{a} = \dot{H} = \dot{\rho} = 0$. The first solution corresponds to the standard ES Universe in GR; the second solution arises from the LQC corrective terms:

$$\rho_{GR} = \frac{2\Lambda}{\kappa(1+3w)}\,, \quad a^2_{GR} = \frac{2k}{\kappa\rho_{GR}(1+w)}\,, \tag{4.7}$$

$$\rho_{LQ} = \frac{2(\Lambda-\kappa\rho_c)}{\kappa(1+3w)}\,, \quad a^2_{LQ} = \frac{2k}{\kappa\rho_{LQ}(1+w)}\,. \tag{4.8}$$

The conditions under which these static solutions exist are summarized in Table 4.1; they follow from $a^2 > 0$ and $\rho > 0$. The presence of the curvature index k is worth stressing, indeed the previous analysis [83] can be enlarged to enclose the $k = -1$ case where the two solutions still exist.

4.2.3 Stability analysis

The stability of the solutions Eqs.(4.7) and (4.8) can be characterized using dynamical system theory and performing a linearized stability analysis. To this aim, we first have to rewrite the system of Eqs.(4.4)-(4.6) in the form of a genuine dynamical system. Indeed, in these equations the three variables a, H and ρ appear but the actual dynamics is constrained on a two-dimensional surface described by the modified Friedmann equation,(see fig. LQCSurface). Thus, following [83], we solve Eq.(4.5) for $1/a^2$.
Two solutions are found:

$$\frac{1}{a^2} = g_\pm(\rho, H) \tag{4.9}$$

where

$$g_\pm = \frac{2(\kappa\rho+\Lambda)+\kappa\rho_c\left(1\pm\sqrt{1-12H^2/\kappa\rho_c}\right)}{6k}. \tag{4.10}$$

Substituting Eq.(4.9) into Eq.(4.6), we find two branches for the time derivative of the Hubble parameter, thus the original system splits in a pair of two-dimensional nonlinear dynamical systems in the variables ρ and H:

$$\mathrm{GR} \;:\; \dot{\rho} = -3H\rho\,(1+w) \quad \text{and} \quad \dot{H} = F_-(\rho, H), \tag{4.11}$$

$$\mathrm{LQ} \;:\; \dot{\rho} = -3H\rho\,(1+w) \quad \text{and} \quad \dot{H} = F_+(\rho, H), \tag{4.12}$$

where

$$\begin{aligned} F_\pm \;&=\; -\frac{\kappa}{2}(1+w)\rho\left(1-\frac{2\rho}{\rho_c}-\frac{2\Lambda}{\kappa\rho_c}\right)+\frac{6k^2g_\pm^2}{\kappa\rho_c} \\ &\quad +g_\pm k\left[1-\frac{2\rho}{\rho_c}-\frac{2\Lambda}{\kappa\rho_c}-3(1+w)\frac{\rho}{\rho_c}\right]. \end{aligned} \tag{4.13}$$

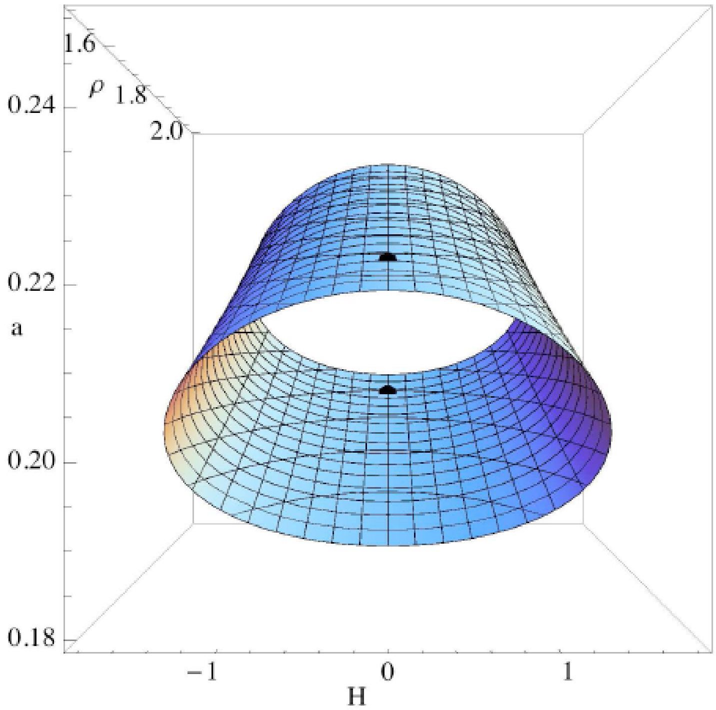

Figure 4.1: Friedmann constraint as hypersurface in the a, H, ρ space the for the case $k = -1$, $\Lambda < 0$, $w < -1$ with $\Lambda = -100$, $w = -2$, $\kappa = 25.13274123$. The ES and LQ solutions are depicted as black dots on top and underneath the surface respectively.

Each one of the systems (4.11) and (4.12) admits a fixed point representing a static solution, that is,

$$\text{GR} \quad : \quad H = 0 \quad \text{and} \quad \rho_o = \frac{2\Lambda}{\kappa(1+3w)}, \tag{4.14}$$

$$\text{LQ} \quad : \quad H = 0 \quad \text{and} \quad \rho_o = \frac{2(\Lambda - \kappa\rho_c)}{\kappa(1+3w)}, \tag{4.15}$$

respectively. Substituting these values of ρ_o in Eq.(4.5) one gets exactly the values of the constant scale factor in terms of the parameters as in Eqs.(4.7) and (4.8). Finally, to characterize the stability of the solutions Eqs.(4.7) and (4.8) we evaluate the eigenvalues of the Jacobian matrix for the two systems Eqs.(4.11) and (4.12) at the fixed points Eqs.(4.14) and (4.15) respectively. For the system in Eq. (4.11), we recover the usual properties of the ES solution in GR. The eigenvalues of the linearized system at the fixed point are

$$\lambda_{GR} = \pm\sqrt{\Lambda(1+w)}. \tag{4.16}$$

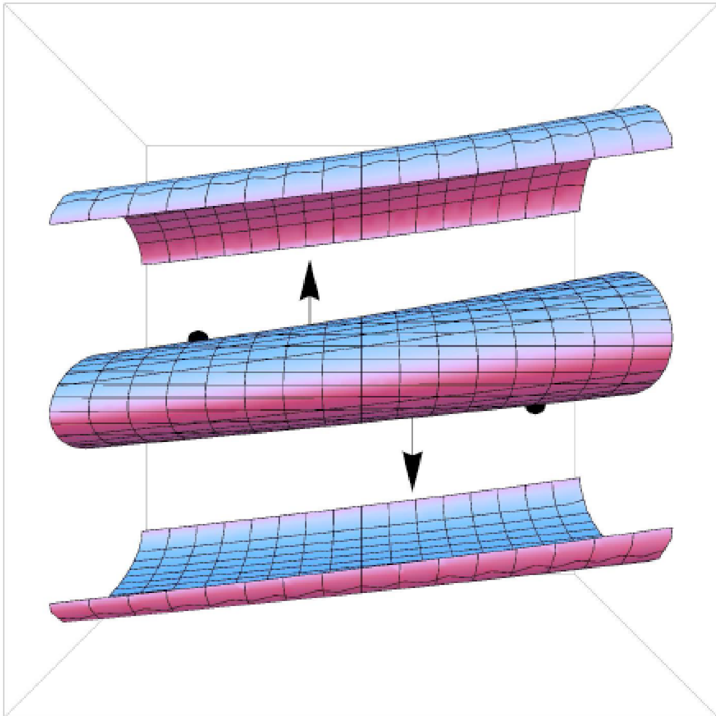

Figure 4.2: Splitting of the Friedmann constraint in two local charts around the fixed points.

In the case of positive curvature index $k = 1$, these are either real with opposite signs for $\Lambda > 0$ and $w > -1/3$ - thus the fixed point is unstable (of the saddle type) - or purely imaginary for $\Lambda < 0$ and $-1 < w < -1/3$, so the fixed point is a center. In the case of negative spatial curvature index $k = -1$, these are again real with opposite signs for $\Lambda < 0$ and $w < -1$, so the fixed point is unstable (of the saddle type). In Fig. 4.3 an example of the latter case is depicted.

For the system Eq. (4.12) the eigenvalues at the fixed point are

$$\lambda_{LQ} = \pm\sqrt{(\kappa\rho_c - \Lambda)(1 + w)}\,. \tag{4.17}$$

In the case of positive curvature index $k = 1$, the LQ fixed point is either unstable (of the saddle kind), when $\kappa\rho_c > \Lambda$ and $-1 < w < -1/3$, or a center for the linearized system, i.e. a neutrally stable fixed point, when $\kappa\rho_c < \Lambda$ and $w > -1/3$. In the case of negative spatial curvature index $k = -1$, the eigenvalues are purely imaginary for $\kappa\rho_c > \Lambda$ and $w < -1$, so we have a center for the linearized system again. In the latter case, the fixed point is nonhyperbolic thus the linearization theorem does not apply. Nevertheless,

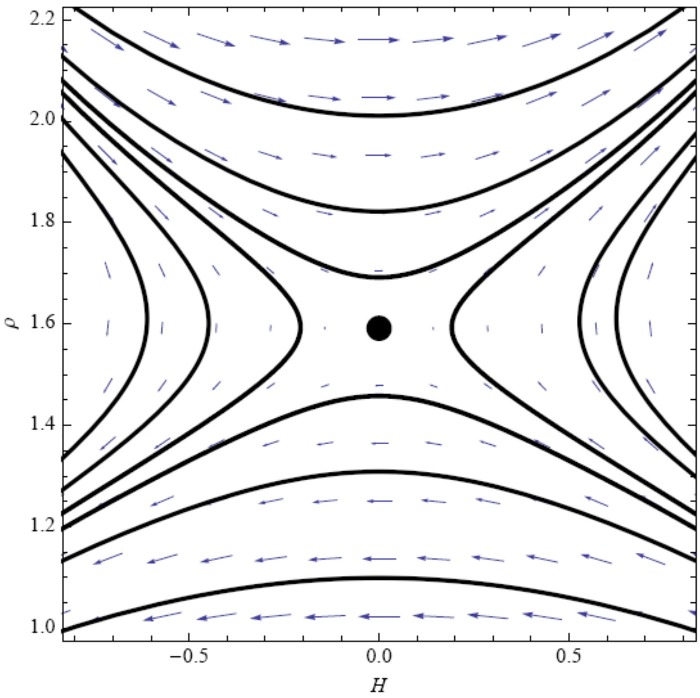

Figure 4.3: Dynamical behavior of the system around the GR fixed point for the case $k = -1$, $\Lambda < 0$, $w < -1$ with $\Lambda = -100$, $w = -2$, $\kappa = 25.13274123$.

a numerical integration of the fully nonlinear system Eq.(4.12) for initial conditions near the fixed point, confirms the result of the linearized stability analysis (see Fig. 4.4). It is worth stressing that in open LQC models a stable ES solution exists in the case of positive values of the cosmological constant as long as $\Lambda < \kappa\rho_c$.

The results of the linearized stability analysis are summarized in Table 4.1.

4.3 Einstein Static Universe in Hořava-Lifshitz

The Hořava-Lifshitz gravity (HL), [58, 59], is a power-counting renormalizable theory of (3+1)-dimensional quantum gravity. In the ultraviolet limit, the theory has a Lifshitz-like anisotropic scaling between space and time characterized by the dynamical critical exponent $z = 3$. In the IR limit the theory flows to the relativistic value $z = 1$.

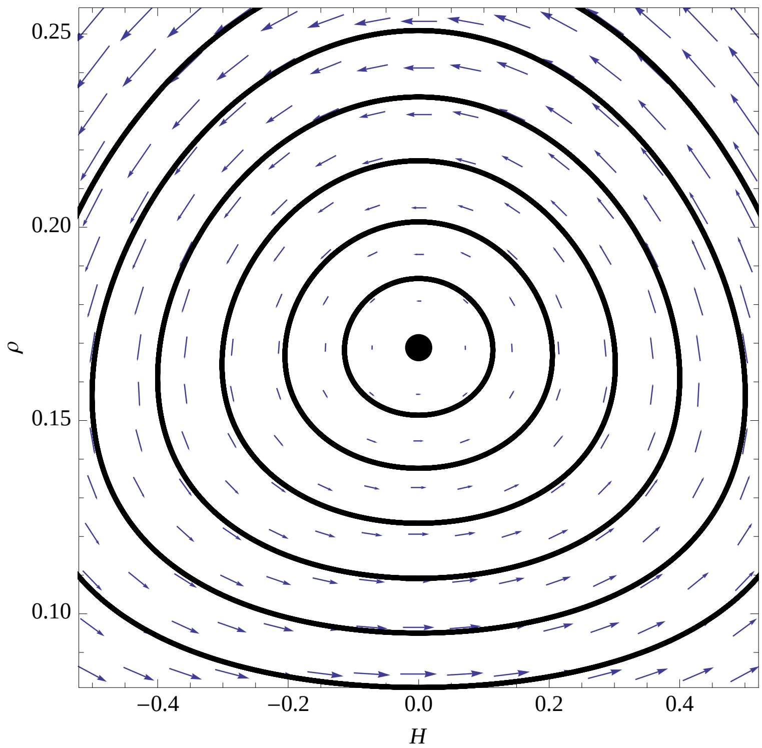

Figure 4.4: Dynamical behavior of the system around the LQ fixed point for the case $k = -1$, $\Lambda < \kappa\rho_c$, $w < -1$ with $\Lambda = 10$, $w = -2$, $\kappa = 25.13274123$.

	k	Λ	w	Stability
GR	1	> 0	$w > -1/3$	saddle
		< 0	$-1 < w < -1/3$	center
	-1	< 0	$w < -1$	saddle
LQ	1	$< \kappa\rho_c$	$-1 < w < -1/3$	center
		$> \kappa\rho_c$	$w > -1/3$	saddle
	-1	$< \kappa\rho_c$	$w < -1$	center

Table 4.1: Existence conditions and stability conditions for the static solutions in Eqs.(4.7) and (4.8).

The effective speed of light c, the effective Newton constant G and the effective cosmological constant Λ of the low-energy theory, emerge from the relevant deformations of the deeply nonrelativistic $z = 3$ theory which dominates at short distances [58]:

$$c = \frac{\kappa^2 \mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}}, \qquad G = \frac{\kappa^2}{32\pi c}, \qquad \Lambda = \frac{3}{2}\Lambda_W. \tag{4.18}$$

The first of the equations in (4.18) imposes a relation among the parameters c, Λ_W and λ; thus, in order to have a real emergent speed of light c, for $\lambda > 1/3$ the cosmological constant has to be negative Λ_W. However, after an analytic continuation of the parameters (see [70]), a real speed of light for $\lambda > 1/3$ implies a positive cosmological constant Λ_W. Thus, mimicking the notation introduced in [73], we introduce a two-valued parameter $\epsilon = \pm 1$, in order to examine both the aforementioned cases at once.
The HL cosmology has been systematically studied using dynamical systems theory in [20, 30, 65, 93], it has also been investigated in [103] using conservation laws of mechanics. Here we consider static solutions of the cosmological equations for the HL gravity when both the detailed balance condition and projectability condition hold.

First we recast the modified Friedmann equations of [70] in a form which allows an easy comparison with the formerly considered case of LQC [1].

The modified Friedmann equation reads

$$H^2 = \frac{2}{3\lambda - 1}\left[\frac{\kappa}{3}\rho + \epsilon\left(\frac{\Lambda}{3} - \frac{k}{a^2} + \frac{3k^2}{4\Lambda a^4}\right)\right] \tag{4.19}$$

and the modified Raychaudhuri equation reads

$$\dot{H} \;=\; \frac{2}{3\lambda - 1}\left[-\frac{\kappa}{2}\rho(1+w) + \epsilon\left(\frac{k}{a^2} - \frac{3k^2}{2\Lambda a^4}\right)\right]. \tag{4.20}$$

The conservation equation for the energy density of the perfect fluid still holds unchanged:

$$\dot{\rho} = -3\rho H(1+w). \tag{4.21}$$

Besides the overall factor $\frac{2}{3\lambda-1}$ on the right hand side of Eqs.(4.19) and (4.20), the modifications to the cosmological equations of GR consist of the higher order terms $\propto k^2/\Lambda a^4$ which become dominant at short distance scales and do not affect the classical cosmological equations in the case of flat models.

[1]According to the definitions given in Sec. 4.2, $c = 1$ and $\kappa = 8\pi G$; Eq.(4.19) and Eq.(4.20) have been written accordingly.

4.3.1 Static solutions

It can be readily found, imposing the conditions $\dot{a} = \dot{H} = \dot{\rho} = 0$, that the system of Eqs.(4.21)-(4.20) admits the following two static solutions:

$$\rho_{HL1} = 0, \qquad a^2_{HL1} = \frac{3k}{2\Lambda}, \tag{4.22}$$

$$\rho_{HL2} = \frac{-16\epsilon\Lambda}{(3w-1)^2\kappa}, \qquad a^2_{HL2} = \frac{(3w-1)k}{2\Lambda(1+w)}. \tag{4.23}$$

The conditions under which these static solutions exist are summarized in Tables 4.2 and 4.3.

The presence of the curvature index k and the parameter ϵ in Eqs.(4.22) and (4.23) is worth being stressed; indeed the analysis presented in [106] can be enlarged to enclose the $k = -1$ case where new interesting possibilities arise. For instance a physically meaningful ES solution is present even in the case of vanishing energy density of the perfect fluid, i.e. Eq.(4.22).

4.3.2 Stability analysis

The stability analysis can be easily performed reducing the original system to an actual two-dimensional autonomous dynamical system by making use of the Friedmann constraint, (see Fig.4.5).

In this case, the simplest and most straightforward choice is to eliminate the dependence on ρ from the other equations, being Eq.(4.19) linear in ρ, that is to consider the projection on the (H,a)-plane, (see Fig. 4.5). This allows us to describe the dynamics with just one set of equations. Indeed, solving Eq.(4.19) for ρ,

$$\rho = \frac{3}{2\kappa}\left(3\lambda - 1\right)H^2 - \frac{\epsilon}{\kappa}\left(\Lambda - \frac{3k}{a^2} + -\frac{3k^2}{4\Lambda a^4}\right), \tag{4.24}$$

and substituting into Eq.(4.20) one gets a first order nonlinear differential equation,

$$\begin{aligned} \dot{H} &= \frac{\epsilon}{3\lambda - 1}\left[(1+w)\Lambda - \frac{(3w+1)k}{a^2} + \frac{3k^2(3w-1)}{4\Lambda a^4}\right] + \\ &\quad -\frac{3}{2}\left(1+w\right)H^2, \end{aligned} \tag{4.25}$$

which, together with the definition of the Hubble parameter,

$$\dot{a} = aH \tag{4.26}$$

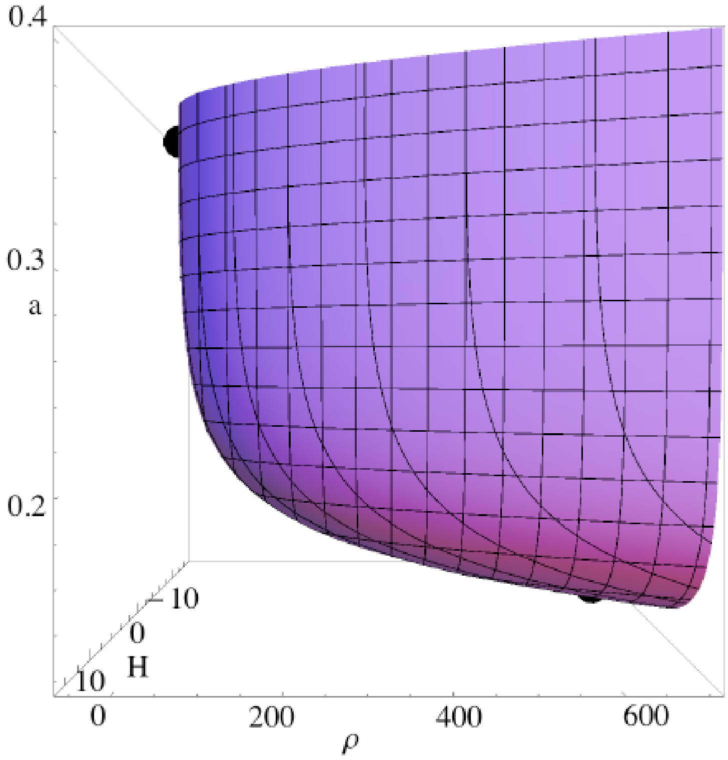

Figure 4.5: Friedmann constraint as hypersurface in the a, H, ρ space the for the case $k = -1$ with $\epsilon = 1$, $\lambda > 1/3$, $\Lambda < 0$, $w > 1/3$. The two black dots represent the HL1 (upper) and HL2 (lower) static solutions.

provides a genuine two-dimensional autonomous dynamical system in the variables a and H. The system admits two fixed points with energy densities as in Eqs.(4.22) and (4.23); thus, to characterize the stability of these solutions, we evaluate the eigenvalues of the Jacobian matrix for the system Eqs.(4.25) and (4.26) at the fixed points corresponding to Eqs.(4.22) and (4.23) respectively.

The eigenvalues at the fixed point $HL1$ read

$$\lambda_{HL1} = \pm \frac{2\sqrt{6(3\lambda - 1)\epsilon\Lambda}}{3(3\lambda - 1)}. \tag{4.27}$$

For all the admitted values of the parameters this is a pair of purely imaginary eigenvalues thus the fixed point is a center for the linearized system. The point is nonhyperbolic, so the linearized analysis may fail to be predictive at nonlinear order, nevertheless a numerical integration proves that this fixed point is actually a center (see Fig. 4.7).

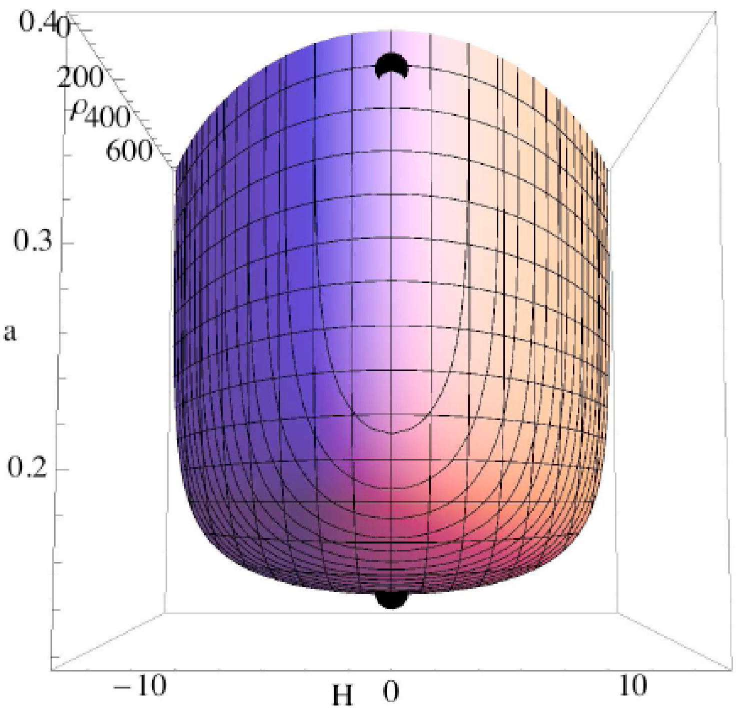

Figure 4.6: Friedmann constraint as seen from the (H, a)-plane.

The results of the stability analysis for the fixed point $HL1$ are summarized in Table 4.2.

ϵ	λ	k	Λ	Stability
-1	$< 1/3$	-1	< 0	center
	$> 1/3$	1	> 0	
1	$< 1/3$	1	> 0	center
	$> 1/3$	-1	< 0	

Table 4.2: Existence conditions and stability conditions for the static solution $HL1$.

The eigenvalues at the fixed point $HL2$ read

$$\lambda_{HL2} = \pm \frac{2\sqrt{-2(3w-1)(3\lambda-1)(1+w)\epsilon\Lambda}}{(3\lambda-1)(3w-1)}. \tag{4.28}$$

According to the admitted values of the parameters, this is either a pair of purely imaginary eigenvalues, so the fixed point is a center for the linearized

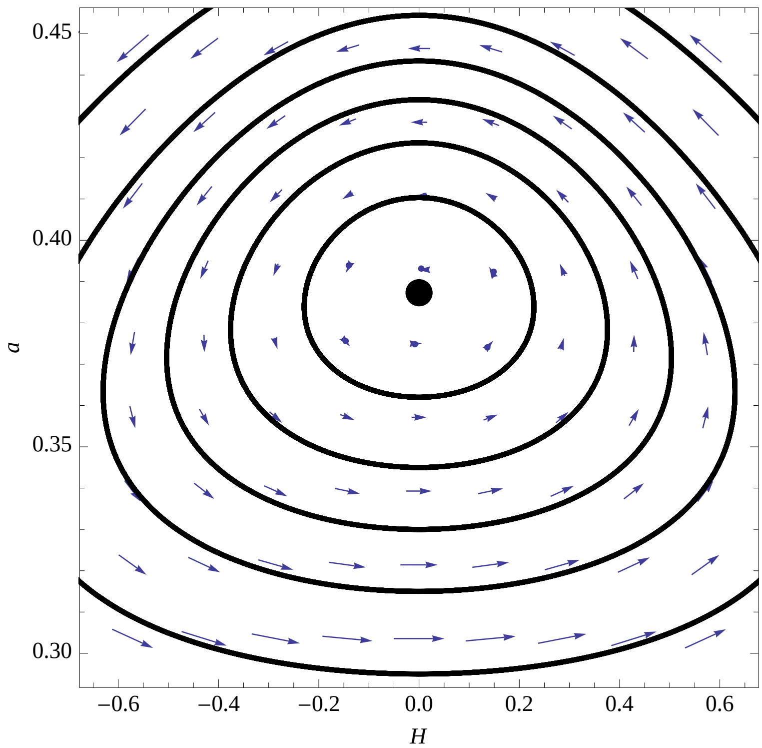

Figure 4.7: Dynamical behavior of the system around the $HL1$ fixed point for the case $k = -1$ with $\epsilon = 1$, $\lambda > 1/3$, $\Lambda < 0$, $w > 1/3$.

system, or a pair of real eigenvalues with opposite signs, so the fixed point is unstable (of the saddle type). In particular, the solution is a center for $-1 < w < 1/3$ and is a saddle for $w < -1$ or $w > 1/3$ (for an example of the latter case see Fig. 4.8).

The results of the stability analysis for the fixed point $HL2$ are summarized in Table 4.3.

4.4 Remarks

We have considered the existence of static solutions in the framework of two recently proposed quantum gravity models, namely, LQC and HL gravity. We have shown that the inclusion of a negative curvature index $k = -1$ enlarges the ranges of existence of the solutions affecting their stability properties thus providing new interesting results. The solutions found display stability conditions rather different from those of the corresponding solutions in closed models and from the stability properties of the standard ES solution of GR.

In the case of LQC, gravitational modifications to the Friedmann equa-

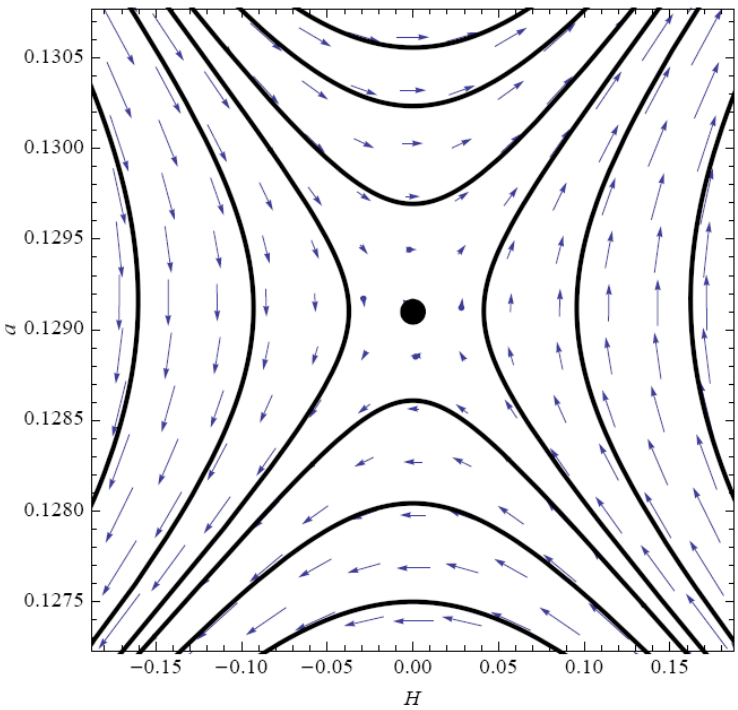

Figure 4.8: Dynamical behavior of the system around the $HL2$ fixed point for the case $k = -1$ with $\epsilon = 1$, $\lambda > 0$, $\Lambda < 0$, $w > 1/3$.

tions, a negative curvature index allows a neutrally stable static solution with $\Lambda < \kappa\rho_c$ and $w < -1$, in contrast to the GR case. In particular the LQC static solution exists and is stable in the case of positive values of the cosmological constant as long as $\Lambda < \kappa\rho_c$.

In the case of HL gravity, two static solutions are found. The inclusion of the negative curvature index leads to a static solution ($HL1$) with negative cosmological constant and vanishing energy density which is neutrally stable against homogeneous perturbations. Furthermore, a negative curvature index allows a static solution ($HL2$) which can be either a saddle, for $w < -1$ and $w > 1/3$, or a center for $-1 < w < 1/3$.

As already observed in the frameworks of different modified models [67, 76, 83], the regime of infinite cycles, about the center fixed points, must be eventually broken in order to enter the current expanding universe phase. To this aim a further mechanism is needed, whose analysis is beyond the scope of this appendix.

<table>
<tr><th>ϵ</th><th>λ</th><th>k</th><th>Λ</th><th>w</th><th>Stability</th></tr>
<tr><td rowspan="3">-1</td><td rowspan="3">$> 1/3$</td><td>-1</td><td>> 0</td><td>$-1 < w < 1/3$</td><td>center</td></tr>
<tr><td rowspan="2">1</td><td rowspan="2">> 0</td><td>$w < -1$</td><td rowspan="2">saddle</td></tr>
<tr><td>$w > 1/3$</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">$> 1/3$</td><td rowspan="2">-1</td><td rowspan="2">< 0</td><td>$w < -1$</td><td rowspan="2">saddle</td></tr>
<tr><td>$w > 1/3$</td></tr>
<tr><td>1</td><td>< 0</td><td>$-1 < w < 1/3$</td><td>center</td></tr>
</table>

Table 4.3: Existence conditions and stability conditions for the static solution $HL2$.

Conclusions

In Chapter 1, we are focused on the fascinating problem of providing a more realistic description of astrophysical rotating bodies. We have considered gravitational fields which are stationary and axisymmetric and have used the formalism of junction conditions for finding new solutions of Einstein equations in presence of matter by matching metrics representing two shells of a compact body. We considered the matching of metrics describing internal gravitational field of rotating bodies, such as neutron stars, with the Kerr solution which describes the external gravitational field generated by these bodies. This is a hard task indeed a class of interior metrics which, by construction, are smoothly matched to the Kerr solution is still absent. In the simplest case (exterior Kerr metric and interior *rotating de Sitter metric*), the junction conditions are not satisfied. Then we have generalized the problem, considering several metrics and several hypersurfaces. Equations expressing junction conditions are rather complicated and new possible results are still under investigation.
In Chapter 2, exact solutions of Einstein equations are analyzed following a geometric approach. We have verified that the Kerr metric belongs to a class of exact solutions described in [95]. Then, we started the problem of a full geometric derivation of the Kerr solution. The work is still in progress.
In Chapter 3, the application of the NJA in several cases is described, with the aim to find sensible internal solutions for the gravitational field of isolated sources to be joined with the known external one. Some ambiguities arising in dealing with the NJA are pointed out; they are mainly related with the complexification used for the radial coordinate r, the introduction of a cosmological constant term and the interpretation of matter sources. We have considered various examples in which the NJA does not provide a solution belonging to the family of Kerr-Newman-de Sitter solutions, as one may have expected.
We have also taken into account two classes of theories in which the method is successful: the f(R)-theories and the Einstein-Maxwell-Dilaton gravity. In the latter case, considering the truncated theory, without axion field, the

NJA does not generate the expected rotating solution.
Then, we have considered two classes of theories for which this algorithmic method fails: the Braneworld scenario and the Born-Infeld theory. To remove the problems which make the NJA an incomplete solutions generating technique, it seems useful to perform a modification of the algorithm in order to extend its application to other theories.
In Chapter 4, we have investigated the existence of static solutions of Einstein equations, in the framework of two recently proposed quantum gravity models, namely, LQC and HL gravity, performing a complete characterization of the solutions along with their stability properties. It follows that the inclusion of a negative curvature index $k = -1$ enlarges the ranges of existence of the solutions affecting their stability properties thus providing new interesting results. The solutions found display stability conditions rather different from those of the corresponding solutions in closed models and from the stability properties of the standard ES solution of GR. In the case of LQC, gravitational modifications to the Friedmann equations, a negative curvature index allows a neutrally stable static solution with $\Lambda < \kappa\rho_c$ and $w < -1$, in contrast to the GR case. In particular the LQC static solution exists and is stable in the case of positive values of the cosmological constant as long as $\Lambda < \kappa\rho_c$.

In the case of HL gravity, two static solutions are found. The inclusion of the negative curvature index leads to a static solution $(HL1)$ with negative cosmological constant and vanishing energy density which is neutrally stable against homogeneous perturbations. Furthermore, a negative curvature index allows a static solution $(HL2)$ which can be either a saddle, for $w < -1$ and $w > 1/3$, or a center for $-1 < w < 1/3$.
These results are relevant for the so called Emergent Universe scenario in which the ES solution plays a key role, being the initial state for a past-eternal inflationary cosmological model. The new stability properties of the ES solution, due to the higher order terms appearing in the modified Friedmann equations, provide a wider range of possible initial states for the Emergent Universe scenario thus ameliorating its fine-tuning problem.

In this thesis we have underlined the importance of exact solutions of Einstein equations not only for providing interesting mathematical results but also for giving important inputs in solving various problems of theoretical physics and astrophysics.

Appendix A

Killing fields

The Einstein equations are very hard to solve and there is no hope to find the general solution due to the nonlinearities. For these reasons, it is important to find a way to simplify the equations without loosing the characteristic features of the theory.
To this end, the symmetries have a key role in constructing simplified models while keeping the most important physical ingredients. In the framework oh GR, when a theory has a symmetry then a Killing vector field appears.

Let (M, g) a Riemannian manifold. A vector field X is a Killing field if the following equation holds:

$$L_X g = 0 \tag{A.1}$$

where L_X is the Lie derivative along X. This means that the vector field X leaves the metric g unchanged.
In a holonomous frame ($e_a = \frac{\partial}{\partial x^a}$, $X = X^a e_a$) these equations, thanks to the fact that ∇ is torsion free, can be rewritten in the following way:

$$\nabla^a X^b + \nabla^b X^a = 0. \tag{A.2}$$

Let consider a tensor metric and the vector field as

$$g = g_{\alpha\beta} dx^\alpha dx^\beta \quad X = X^i \frac{\partial}{\partial x_i}$$

by applying the Eq.(A.1) and introducing the quantities:

$$\xi_\alpha = g_{\alpha\iota} X^\iota$$

one trivially get the following equations:

$$\frac{\partial \xi_\alpha}{\partial x^\beta} + \frac{\partial \xi_\beta}{\partial x^\alpha} = 2\xi_\lambda \Gamma^\lambda_{\alpha\beta} \tag{A.3}$$

These equations, called *Killing equations*, provide a characterization of metric's symmetries.

The transformation leaving the metric unchanged, also called isometries, form a group. More generally, it can be shown that Killing fields are the infinitesimal generators of a (continuous) symmetry group of (M, g) and furthermore, Killing fields of a given metric manifold form a Lie algebra. Roughly speaking, this is due to the fact that the continuous symmetries of a metric manifold form a Lie group. The Lie algebra of all Killing fields of a given spacetime (M, g) will be denoted by $Kil(g)$, while a Killing subalgebra will be denoted by G. A Lie algebra can be identified with the tangent space at the origin of a Lie group G_L and, by exponentiation, the Lie algebra covers the simply connected part (that is again a Lie group) of the Lie group G_L. The group corresponding to a subalgebra G of $Kil(g)$ is called the group of motions or group of isometries and denoted by G_r where r, the order of the group, is the number of generators. If $G = Kil(g)$, then the corresponding group is called complete group of motions.

Finally, it can be proven that the maximum number of isometries of a metric, i.e. the maximum number of Killing vector fields appearing in a n-dimensional manifold is given by:

$$N = \frac{n(n+1)}{2}$$

where n represent the dimension of manifold.

Appendix B

The Kerr-Newman-de Sitter solution

The Kerr-Newmann-de Sitter solution, which describe a charged rotating black hole in de Sitter background, can be written as follows, [41]:

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_r}{\Xi^2\Sigma}(dt - a\sin^2\vartheta d\varphi)^2 + \frac{\Delta_\vartheta \sin^2\vartheta}{\Xi^2\Sigma}\left[adt - (r^2+a^2)d\varphi\right]^2 + \\
&\quad + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\vartheta}d\vartheta^2
\end{aligned}
$$

where:

$$
\begin{aligned}
\Delta_r &= r^2 - 2Mr + a^2 - \frac{\Lambda r^2(r^2+a^2)}{3} + q^2 \\
\Delta_\vartheta &= 1 + \frac{\Lambda a^2\cos^2\vartheta}{3} \\
\Xi &= 1 + \frac{\Lambda a^2}{3} \\
\Sigma &= r^2 + a^2\cos^2\vartheta
\end{aligned}
$$

and:

q is the charge, M is the mass, $a = J/M$ and Λ is the cosmological constant.

Now it is straightforward to deduce the different kinds of solutions depending on what parameters are different to zero.

$$\begin{aligned}
\text{if} \quad a &= 0, \quad q=0, \quad M=0 \quad \rightarrow \quad \text{de Sitter metric} \\
\text{if} \quad a &= 0, \quad q=0, \quad \Lambda=0 \quad \rightarrow \quad \text{Schwarzschild metric} \\
\text{if} \quad a &= 0, \quad q=0 \quad \rightarrow \quad \text{Schwarzschild-de Sitter metric} \\
\text{if} \quad q &= 0, \quad \Lambda=0 \quad \rightarrow \quad \text{Kerr metric} \\
\text{if} \quad q &= 0 \quad \rightarrow \quad \text{Kerr-de Sitter metric} \\
\text{if} \quad q &= 0 \quad M=0 \quad \rightarrow \quad \textit{rotating de Sitter metric} \\
\text{if} \quad a &= 0 \quad \rightarrow \quad \text{Newman-de Sitter metric.}
\end{aligned} \tag{B.1}$$

Appendix C

Dynamical Systems

Dynamical system technics have proved to be very useful tools in dealing with Cosmology. Indeed, it is customary to recast the evolution equations of Cosmology as an autonomous dynamical system in order to characterize the relevant features of cosmological models (e.g. the late time behaviour of the universe, the attractor solutions etc). An introduction to dynamical systems with extended application to Cosmology is provided in [101].

A very general and abstract definition of Dynamical System is the following.
Let $\mathbf{X}$ be a metric space. A (continuous) dynamical system is a one-parameter family of invertible maps $\phi_t : \mathbf{X} \to \mathbf{X}$, $t \in \mathbf{R}$, such that:

- $\phi_0 = Id$
- $\phi_{t_1+t_1} = \phi_{t_1} + \phi_{t_2}, \quad \forall t_1, t_1 \in \mathbf{R}$
- $\phi_{-t} = (\phi_t)^{-1}, \quad \forall t \in \mathbf{R}$.

Dynamical System of practical interest are generally in the form of vector field on a state space $\mathbf{X}$

$$\dot{x} = f(x). \tag{C.1}$$

The *sate space* or *phace space* $\mathbf{X}$ can be a differential manifold (e.g. a sphere, a torus etc.). More simply one can have a vector space $\mathbf{X} = \mathbf{R^n}$, $x = (x_1, .., x_n)$, in which case Eq. (C.1) represents a system of ordinary differential equations (e.g linear differential equations, gradient differential equations, Hamiltonian differential equations). Then the one parameter maps considered below are naturally interpreted as the flow of Eq. (C.1).

An equilibrium solution or *fixed point* or *critical point* is a point $x^* \in \mathbf{X}$ such that $f(x^*) = 0$, that is a solution which doesn't change with time.

One of the main goal of Dynamical System theory is to determine the future asymptotic behavior (i.e. $t \to \infty$), because one is interested in the long-term evolution of the corresponding physical system. In cosmology one is also interested in the past asymptotic behavior (near the initial singularity). From the pure mathematical point of view, two important definition can be given. A solution $x^*(t)$ of a dynamical system is said to be (Liapunov) *stable* if, given $\varepsilon > 0$, there exist a $\delta = \delta(\varepsilon)$ such that, for any other solution, $y(t)$, satisfying $|x^*(t_0) - y(t_0)| < \delta$, then $|x^*(t) - y(t)| < \varepsilon$ for $t > t_0$, $t_0 \in \mathbf{R}$. A solution which is not stable is said to be *unstable*. A solution $x^*(t)$ of a dynamical system is said to be *asymptotically stable* if it is Liapunov stable and if there exist a constant $b > 0$ such that, if $|x^*(t_0) - y(t_0)| < b$ then $\lim_{t\to\infty} |x^*(t) - y(t)| = 0$.

These definitions do not actually provide us with a method for determining whether or not a given solution is stable. The first step in obtaining qualitative information about the solutions of a differential equation is to study the local properties of the flow in the neighborhood of the equilibrium points. Once that a differential equation have been linearized around any of its equilibrium points and their stability have been determined, the behavior of the resulting linear system is related to the original non-linear system by the Hartman-Grobman theorem.

Given a linear differential equation $\dot{x} = Ax$ on $\mathbf{R^n}$, where A is an $n \times n$ matrix of real numbers, three subspaces of $\mathbf{R^n}$ are defined:

$$\begin{array}{ll} \textit{the stable subspace} & E^s = span(s_1, ..., s_{n_s}), \\ \textit{the unstable subspace} & E^u = span(u_1, ..., u_{n_n}), \\ \textit{the center subspace} & E^c = span(c_1, ..., c_{n_c}), \end{array}$$

where $s_1, ..., s_{n_s}$ are the generalized eigenvectors of A whose eigenvalues have negative real parts, $u_1, ..., u_{n_u}$ are those whose eigenvalues have positive real parts and $c_1, ..., c_{n_c}$ are those whose eigenvalues have zero real parts. Clearly $E^s \oplus E^u \oplus E^c = \mathbf{R^n}$ and

$$\begin{array}{lcl} x \in E^s & \Rightarrow & \lim\limits_{t\to+\infty} \exp(At)x = 0, \\ x \in E^u & \Rightarrow & \lim\limits_{t\to-\infty} \exp(At)x = 0. \end{array}$$

This is a description of the asymptotic behavior of the linear system: all initial states in the stable subspace are attracted to the equilibrium point the 0 vector, while all initial states in the unstable subspace are repelled by 0. Let us turn to nonlinear systems represented by Eq.(C.1). The linearization of Eq.(C.1) at an equilibrium point $\overline{x}$ is given by

$$\dot{x} = Df(\overline{x})(x - \overline{x}) \tag{C.2}$$

where D is the derivative of f. When an equilibrium point $\overline{x}$ is *hyperbolic*, that is, when all the eigenvalues of $Df(\overline{x})$ have non-vanishing real part, the *Hartman-Grobman theorem* ensures that in a neighborhood of $\overline{x}$ exists a homomorphism which maps orbits of the flow generated by the original nonlinear differential equation onto orbits of the corresponding linearized system preserving their orientation (those orbits are said *topologically equivalent*). Thus the Hartman-Grobman theorem tells that the stability properties of a nonlinear dynamical system near a hyperbolic fixed point are qualitatively described by its linearization at that point.

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