Elliptic Operators with Unbounded Coefficients

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Abstract:

Aim of this manuscript is to give generation results and some Hardy inequalities concerning elliptic operators with unbounded coefficients of the form

$$\mathcal{A}u = \operatorname{div}(aDu) + F \cdot Du + Vu$$

where V is a real valued function, $a(x) = (a_{kl}(x))$ is symmetric and satisfies the ellipticity condition and a and F grow to infinity. In particular, we mainly deal with Scrhödinger type operators, i.e., operators with vanishing drift term, $\nabla a + F = 0$. The case of the whole operator is also considered in the sense that a weighted Hardy inequality for these operators is provided. Finally we will consider the higher order elliptic operator perturbed by a singular potential $A = \Delta^2 - c|x|^{-4}$.

Due to their importance for the strong relation with Schrödinger operators, we provide a survey on the most significant proofs of Hardy's inequalities appeared in literature. Furthermore, we generalise Hardy inequality proving a weighted inequality with respect to a measure $d\mu = \mu(x)\, dx$ satisfying suitable local integrability assumptions in the weighted spaces $L^2_\mu(\mathbb{R}^N) = L^2(\mathbb{R}^N, d\mu)$. We claim that for all $u \in H^1_\mu(\mathbb{R}^N)$, $c \leq c_{0,\mu}$

$$c \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, d\mu \leq \int_{\mathbb{R}^N} |\nabla u|^2\, d\mu + C_\mu \int_{\mathbb{R}^N} u^2\, d\mu$$

holds with $c_{0,\mu}$ optimal constant. The interest in studying such an inequality is the relation with the parabolic problem associated to the Kolmogorov operator perturbed by a singular potential

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u + \frac{c}{|x|^2} u.$$

Moreover, we consider the Schrödinger type operator L_0 with unbounded diffusion

$$L_0 u = Lu + Vu = (1 + |x|^\alpha)\Delta u + \frac{c}{|x|^2} u$$

with $\alpha \geq 0$ and $c \in \mathbb{R}$. The aim is to obtain sufficient conditions on the parameters ensuring that L_0 with a suitable domain generates a quasi-contractive and positivity preserving C_0-semigroup in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. The proofs are based on some L^p-weighted Hardy inequality and perturbation techniques. In fact we treat the operator L_0 as a perturbation of the elliptic operator $L = (1 + |x|^\alpha)\Delta$ which has already been studied in literature.

Finally, we study the biharmonic operator perturbed by an inverse fourth-order potential

$$A = A_0 - V = \Delta^2 - \frac{c}{|x|^4},$$

where c is any constant such that $c < C^* := \left(\frac{N(N-4)}{4}\right)^2$. Making use of the Rellich inequality, multiplication operators and off-diagonal estimates, we prove that the semigroup generated by $-A$ in $L^2(\mathbb{R}^N)$, $N \geq 5$, extrapolates to a bounded holomorphic C_0-semigroup on $L^p(\mathbb{R}^N)$ for all $p \in [p_0', p_0]$, where $p_0 = \frac{2N}{N-4}$ and p_0' is its dual exponent. Furthermore, we study the boundedness of the Riesz transform

$$\Delta A^{-1/2} := \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \Delta e^{-tA}\, dt$$

on $L^p(\mathbb{R}^N)$ for all $p \in (p_0', 2]$. The boundedness of $\Delta A^{-1/2}$ on $L^p(\mathbb{R}^N)$ implies that the domain of $A^{1/2}$ is included in the Sobolev space $W^{2,p}(\mathbb{R}^N)$. Thus, we obtain $W^{2,p}$-regularity of the solution to the evolution equation with initial datum in $L^p(\mathbb{R}^N)$ for $p \in (p_0', 2]$, i.e., $u(t) \in W^{2,p}(\mathbb{R}^N)$.

Publications

[1] A. Canale, F. Gregorio, A. Rhandi, C. Tacelli: *Weighted Hardy inequalities and Kolmogorov-type operators*, preprint.

[2] S. Fornaro, F. Gregorio, A. Rhandi: *Elliptic operators with unbounded diffusion coefficients perturbed by inverse square potentials in L^p-spaces*, Comm. on Pure and Appl. Anal. **15** (2016), no. 6, 2357-2372.

[3] F. Gregorio, S. Mildner: *Fourth-order Schrödinger type operator with singular potentials*, Archiv der Mathematik **107** (2016), no. 3, 285-294.