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Candidato
Abdallah Maichine

Tutor:
Professor Abdelaziz Rhandi

Coordinatore:
Professor Roberto Scarpa

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Dedication

To my familly, especially my dear parents and my wife.

Contents

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Abstract

We consider the following matrix Schrödinger operator

$$\mathcal{A}u = \mathrm{div}(Q\nabla u) - Vu = \left(\mathrm{div}(Q\nabla u_j) - \sum_{k=1}^{m} v_{jk}u_k\right)_{1\le j\le m}$$

acting on vector valued functions $u : \mathbb{R}^d \to \mathbb{C}^m$, where Q is a symmetric real matrix-valued function which is supposed to be bounded and satisfy the ellipticity condition, and V is a measurable *unbounded* matrix-valued function.

We construct a realization A_p of $\mathcal{A}$ in the spaces $L^p(\mathbb{R}^d, \mathbb{C}^m)$, $1 \le p < \infty$, that generates a contractive strongly continuous semigroup. First, by using form methods, we obtain generation of holomorphic semigroups when the potential V is symmetric. In the general case, we use some other techniques of functional analysis and operator theory to get a m-dissipative realization. But in this case the semigroup is not, in general, analytic.

We characterize the domain of the operator A_p in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ by using firstly a non commutative version of the Dore-Venni theorem, see [**50**], and then a perturbation theorem due to Okazawa, see [**52, 53**].

We discuss some properties of the semigroup such as analyticity, compactness and positivity. We establish ultracontractivity and deduce that the semigroup is given by an integral kernel. Here, the kernel is actually a matrix whose entries satisfy Gaussian upper estimates.

Further estimates of the kernel entries are given for potentials with a diagonal of polynomial growth. Suitable estimates lead to the asymptotic behavior of the eigenvalues of the matrix Schrödinger operator when the potential is symmetric.

Introduction

Second-order elliptic differential operators with unbounded coefficients appear naturally as infinitesimal generators of diffusion processes; the associated parabolic equation is then the Kolmogorov equation for that process. While the *scalar* theory of such equations is by now well developed (see [**54**] and [**44**] for bounded and unbounded coefficients respectively, and the references therein), the literature on *systems* of parabolic equations with unbounded coefficients is still sparse.

Elliptic and parabolic systems with unbounded coefficients is then a new and fertile branch of Partial Differential Equations. The question arising is to extend the results of the theory of scalar elliptic equations to the vectorial ones. In particular, to associate a semigroup in L^p-spaces to a vector-valued elliptic operator with unbounded coefficients and to investigate further qualitative and quantitative properties.

Beside their own interests, such systems appear naturally in the study of backward-forward stochastic differential systems, in the study of Nash equilibria to stochastic differential games, in the analysis of the weighted $\overline{\partial}$-problem in $\mathbb{C}^d$, in the time-dependent Born–Openheimer theory and also in the study of Navier–Stokes equations. We refer the reader to [**2**, Section 6], [**31, 16, 11, 36, 35, 30, 27**] for further details.

Recently, it starts to appear some works in this direction, see for instance [**2**], [**3**], [**19**] and [**34**]. In the framework of semigroup theory, to our knowledge, one of the first articles dealing with systems of parabolic equations with unbounded coefficients is [**34**]. Here the diffusion coefficients were assumed to be strictly elliptic, bounded, the drift F and the potential V can grow as $|x|\log(1+|x|)$ and $\log(1+|x|)$ coupling respectively. It should be noted that for $V=0$ and a drift term growing as $|F(x)| \asymp |x|^{1+\varepsilon}$ one can not expect generation of a C_0-semigroup on L^p with respect to Lebesgue measure, even in the scalar case, see [**57**]. Due to the interaction between drift and potential terms, there are additional assumptions on the potential which in absence of a drift term are somehow restrictive. Indeed, for symmetric potentials, the assumptions made in [**34**] imply the boundedness of the potential term.

Subsequently, there were some other publications [**2, 3, 19**] where the coefficients of the differential operators are assumed to be locally Hölder continuous, and unbounded diffusion coefficients can be considered. The strategy in these references is quite different from that in [**34**]. Namely, in [**2, 3, 19**] solutions to the parabolic equation are at first constructed in the space of bounded and continuous functions. Afterwards the semigroup is extrapolated to the L^p-scale. Consequently, even though this approach more general coefficients are allowed, one obtains no information about the domain of the generator of the semigroup – a crucial information for applications. Moreover this approach cannot be applied for operators having only measurable singular or non smooth coefficients.

In this thesis we propose to study a particular vector-valued elliptic operator which is the vector–valued (matrix) Schrödinger operator, for which we adopt the following definition

$$\mathcal{A}u = div(Q\nabla u) - Vu, \tag{0.0.1}$$

acting on vector–valued functions $u = (u_1, \dots, u_m) : \mathbb{R}^d \to \mathbb{C}^m$, where

- $div(Q\nabla u) := (div(Q\nabla u_1), \dots, div(Q\nabla u_m))$ will be denoted simply by $\Delta_Q u$;
- $(Vu)(x) = V(x)u(x)$ to be understood as a matrix vector product;
- $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a measurable *bounded* matrix map satisfying, for every $x \in \mathbb{R}^d$, $Q(x)$ is symmetric and there exists $\eta_1 > 0$ such that
$$\langle Q(x)\xi, \xi\rangle \geq \eta_1|\xi|^2, \qquad x, \xi \in \mathbb{R}^d.$$
- $V : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ is a measurable matrix-valued function such that
$$\langle V(x)\xi, \xi\rangle \geq 0, \qquad x \in \mathbb{R}^d, \xi \in \mathbb{R}^m.$$

The above algebraic conditions on Q and V guarantee the dissipativity of the operator $\mathcal{A}$.

The theory of strongly continuous (or C_0-) semigroups, see Appendix A, allows to solve the parabolic system $\partial_t u = \mathcal{A}u$, once the operator $\mathcal{A}$ admits some realization that generates a strongly continuous semigroup in some function space.

The Schrödinger system $i\partial_t u = \mathcal{A}u$ can be solved in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ when the operator $\mathcal{A}$ is dissipative and self-adjoint, which can be the case only for symmetric potentials.

In the scalar case, Schrödinger operator has the form

$$\mathcal{A}_S u = div(Q\nabla u) - vu = \Delta_Q u - vu,$$

acting on smooth functions $u : \mathbb{R}^d \to \mathbb{C}$, where Q called *diffusion matrix*, is a symmetric matrix of size d and v a measurable function called *potential*. The scalar Schrödinger operators with real potential are widely studied in literature. The most studied Schrödinger operator in literature is the one of the form $\Delta - V$, where V is an unbounded positive potential (potential of sign changing is also considered in literature), see [**17**, Chapter 4], [**64, 60, 66**] and related references. Moreover, Schrödinger operators with magnetic field are also considered in literature, see [**6, 7, 8, 32**]. In quantum physics, the associated Schrödinger equation $i\partial_t u = \mathcal{A}_S u$

models the movement of a non-relativistic particle under the action of the electric field $\vec{E} = -\nabla v$. The term Δ_Q refers to the kinetic energy $(-\Delta_Q u, u)$ of the particle and v to its potential energy (vu, u).

The matrix Schrödinger operator, in non–relativistic mechanical quantum, appears as the *Hamiltonian* for a system of interacting adsorbate and substrate atoms. The entries of the potential matrix V represent the interparticle electrical interactions; namely, electron–electron repulsions, electron–nuclear attractions and nuclear–nuclear repulsions. For more details we refer to [**68, 69, 65**] and the references therein.
For *adiabatic* systems, an approximation called *Born-Oppenheimer Approximation* applies. Due to this approximation, the Hamiltonian (matrix Schrödinger operator) can be substituted by a diagonal operator with diagonal matrix potential. The *non adiabatic*, sometimes called also *diabatic* systems, is the case where such an approximation does not apply. Thereby, the necessity of the study of the matrix Schrödinger operator, taking in consideration all diagonal and off-diagonal entries of the potential matrix, see for instance [**68**].

Schrödinger equations with complex valued potential can be also transformed into a system of coupled real Schrödinger equations. For more details we refer to Section 3.5. We thus get another field where matrix Schrödinger operators can play a crucial role. Historically, Schrödinger operators with complex potentials were not an attracting topic as much as for real potentials. However, nowadays, these operators start to get attention. The particularity of such operators is that they are not self-adjoint, and thus spectral theory and techniques of self-adjoint operators are not applicable. Actually, one may not have a real spectrum; this is the case for some purely imaginary potentials, as in Example 2.16. Most of the bibliography about complex Schrödinger operators deal with spectral theory. However, in [**32**] the authors characterize the domain of a class of complex Schrödinger operators and give necessary conditions for the compactness of their resolvent which yields the discreteness of the spectrum. They have considered C^∞–potentials, which appears as a very strong regularity requirement. A series of papers dealing with spectral theory of complex Schrödinger operators appear recently by David Kreječiřik and his co-authors, see [**40, 25, 33**]. We also refer to [**13, 22, 26**] and references therein for more literature on complex Schrödinger operators.

In this thesis we want to use the semigroup approach in order to study qualitative behavior of solutions of a system of evolution equations involving a matrix Schrödinger operator of type (0.0.1). More precisely, we would like to construct realizations, in Lebesgue spaces $L^p(\mathbb{R}^d, \mathbb{C}^m)$ for $1 \leq p < \infty$, of the differential operator $\mathcal{A}$ that generate consistent strongly continuous semigroups.

Since matrix Schrödinger operator is classified as an elliptic operator, we thought about applying form methods, which is an easiest way to get generation of semigroups and requires minimal regularity conditions on the coefficients of the operator. In the case where the potential matrix V is symmetric, the form methods apply and one obtains a dissipative self adjoint realization in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ of $\mathcal{A}$ which generates a strongly continuous semigroup. This semigroup can be extended to an analytic semigroup on $S_{\pi/2}$ (the right half plan of $\mathbb{C}$). However, we discovered, via a counter example, that we cannot associate a continuous form to the matrix Schrödinger operator with some *unbounded* antisymmetric potentials. Later on, we proved that the semigroup (constructed otherwise) associated to such operators is not analytic. We then became convinced that form methods work only for symmetric potentials and we though otherwise for nonsymmetric ones. We then decide to dedicate the first chapter of this thesis to study symmetric Schrödinger operators.

In Chapter 1, we consider a symmetric Schrödinger operator and associate it to a sesquilinear form assuming that Q is only bounded (no more regularity is required) and the entries v_{ij}, $i, j \in \{1, \dots, m\}$, of V are locally integrable. In the first section we introduce the sesquilinear form

$$a(f, g) = \int_{\mathbb{R}^d} \sum_{j=1}^{m} \langle Q(x)\nabla f_j(x), \nabla g_j(x)\rangle_{\mathbb{C}^d}\, dx + \int_{\mathbb{R}^d} \langle V(x)f(x), g(x)\rangle_{\mathbb{C}^m}\, dx.$$

defined on the domain

$$D(a) = \{f = (f_1, \dots, f_m) \in H^1(\mathbb{R}^d, \mathbb{C}^m) : \int_{\mathbb{R}^d} \langle V(x)f(x), f(x)\rangle_{\mathbb{C}^m}\, dx < +\infty\}.$$

Note that if f and g are two elements of $D(a)$, then $x \mapsto \langle V(x)f(x), g(x)\rangle$ is integrable over $\mathbb{R}^d$. This can seen by rewriting $\langle V(x)f(x), g(x)\rangle$ as

$$\langle V^{1/2}(x)f(x), V^{1/2}(x)g(x)\rangle$$

and applying the Cauchy-Schwartz inequality. Such an argumentation is not valid when V is antisymmetric.

Therefore, we check that a is an accretive, densely defined, closed and continuous sesquilinear form and conclude that it is associated to a closed self adjoint operator $-A$. Then A generates a strongly continuous semigroup $\{T(t) : t \geq 0\}$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. Of course no concrete information about the domain of A is obtained. Afterwards we extrapolate this semigroup to the L^p-scale. For that end, and since the semigroup $\{T(t) : t \geq 0\}$ is symmetric, it is enough for $\{T(t) : t \geq 0\}$ to be L^∞-contractive, that is

$$\|T(t)f\|_\infty \leq \|f\|_\infty, \qquad \forall f \in L^2(\mathbb{R}^d, \mathbb{C}^m) \cap L^\infty(\mathbb{R}^d, \mathbb{C}^m),$$

which allows to extended every operator $T(t)$ to a bounded linear operator in $L^\infty(\mathbb{R}^d, \mathbb{C}^m)$, and by duality in $L^1(\mathbb{R}^d, \mathbb{C}^m)$. Then, by using the *Riesz-Thorin interpolation theorem*, the semigroup is extended to all spaces $L^p(\mathbb{R}^d, \mathbb{C}^m)$, $1 \leq p < \infty$. As a consequence of the *Stein interpolation theorem*, see [**17**, Section 1.1.6], the extrapolation $\{T_p(t) : t \geq 0\}$ is actually a holomorphic (analytic) semigroup for $1 < p < \infty$.

It remains only to establish the L^∞-contractivity property to get a consistent semigroup in all L^p-spaces associated to the realization A_p of $\mathcal{A}$ in $L^p(\mathbb{R}^d, \mathbb{C}^m)$. The L^∞-contractivity property is equivalent to the invariance of the restriction to $L^2(\mathbb{R}^d, \mathbb{C}^m)$ of the L^∞–unit ball under the semigroup $\{T(t) : t \geq 0\}$. Denoting $\mathcal{B}_\infty$ this restriction. One has

$$\mathcal{B}_\infty := \{f \in L^2(\mathbb{R}^d, \mathbb{C}^m) : \|f\|_\infty \leq 1\}.$$

$\mathcal{B}_\infty$ is a closed convex subset of $L^2(\mathbb{R}^d, \mathbb{C}^m)$. It thus suffices to identify the (unique) projection over $\mathcal{B}_\infty$ and apply the generalized *Beurling-Deny* criterion of invariance of convex subsets by the semigroups. For general *Beurling-Deny* criteria, we refer to the book by E. M. Ouhabaz [**54**, Chapter 2] and for the L^∞-contractivity criterion for vector valued functions we refer to [**55**] of the same author.

The conditions of the *Beurling-Deny* criterion of the L^∞-contractivity for semigroups are recalled in this manuscript in Theorem B.8.

In the above construction everything is similar to the scalar Schrödinger operator, see [**17**, Section 4.2]. The difference starts to appear when talking about (componentwise) positivity of the semigroup. In the scalar case, Schrödinger operators with nonnegative locally integrable potential are always generators of positive semigroups. However, the matrix Schrödinger semigroup $\{T(t) : t \geq 0\}$ is positive if, and only if, the off-diagonal entries of the potential matrix V are nonpositive, i.e. $v_{ij} \leq 0$, for all $i \neq j$.

An investigation on compactness of the semigroup $\{T(t) : t \geq 0\}$ is also done. More precisely, we show that a sufficient condition is that the lowest eigenvalue of V blows up at infinity, i.e.

$$\langle V(x)\xi, \xi\rangle \geq \mu(x)|\xi|^2, \qquad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^m,$$

where μ is a nonnegative locally bounded function which goes to ∞ when $|x|$ tends to ∞. We end Chapter 1 by an example where the compactness condition is not satisfied and the semigroup $\{T(t) : t \geq 0\}$ is not compact even if all entries of the potential matrix blow up at infinity.

The question arising now is that if we can associate a semigroup in L^p–spaces to the operator $\mathcal{A}$ in the nonsymmetric case. This is the topic of Chapter 2. In this chapter, we first construct a realization of $\mathcal{A}$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ by following the same strategy as in [**37**], where the author constructed a m-accretive realization in $L^2(\mathbb{R}^d, \mathbb{C})$ of a scalar Schrödinger operator with complex potential. In [**37**], Kato considered complex potentials v with nonnegative real part such that $v \in L^s_{loc}(\mathbb{R}^d)$, where the exponent $s = 2d(d-2)^{-1}$ if $d \geq 3$, $s > 1$ if $d = 2$ and $p = 1$ if $d = 1$. In our construction, see Section 2.3, we consider locally bounded potentials, i.e.

$v_{ij} \in L^\infty_{loc}(\mathbb{R}^d)$, for each $i, j \in \{1, \dots, m\}$. Of course, weaker condition as the one of [**37**] can be considered. Nevertheless, we are concerned with more than generation of semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. Since we aim to extrapolate the semigroup to all L^p–spaces, we need to have the space of test functions as a core for $\mathcal{A}$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$, which requires local elliptic regularity for $\mathcal{A}$. So, this is why one needs locally bounded coefficients.
Considering the operator A as the realization of $\mathcal{A}$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ with domain

$$D(A) = \{u \in H^1(\mathbb{R}^d, \mathbb{C}^m) : \mathcal{A}u \in L^2(\mathbb{R}^d, \mathbb{C}^m)\},$$

we prove that $(A, D(A))$ is m-dissipative. Hence, it generates a contractive strongly continuous semigroup $\{T(t) : t \geq 0\}$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. This can be obtained by considering an auxiliary operator L which is the realization of $\mathcal{A}$ acting from the Sobolev space $H^1(\mathbb{R}^d, \mathbb{C}^m)$ into its dual $H^{-1}(\mathbb{R}^d, \mathbb{C}^m)$, endowed with the (maximal) domain

$$D(L) = \{u \in H^1(\mathbb{R}^d, \mathbb{C}^m) : \mathcal{A}u \in H^{-1}(\mathbb{R}^d, \mathbb{C}^m)\}.$$

We show that $-L$ is a maximal monotone operator, which implies that $-A$ is m-accretive. The main ingredient of the proof is the following *Kato type inequality*, established for the operator Δ_Q acting on vector valued functions:

$$\Delta_Q |u| \geq \mathbb{1}_{\{u \neq 0\}} \frac{1}{|u|} \sum_{j=1}^m u_j \Delta_Q u_j, \qquad u \in H^1_{loc}(\mathbb{R}^d, \mathbb{C}^m).$$

Subsequently, we prove that the space of test functions is a core for A and consequently the semigroup $\{T(t) : t \geq 0\}$ is given by the Trotter-Kato product (Chernoff) formula

$$T(t) = \lim_{n \to \infty} \left[e^{\frac{t}{n}\Delta_Q} e^{-\frac{t}{n}V} \right]^n,$$

where $\{e^{t\Delta_Q}\}$ is the semigroup generated by the operator Δ_Q and $\{e^{-tV}\}$ the multiplication semigroup generated by the dissipative multiplication operator $-V$. Since both $\{e^{t\Delta_Q}\}$ and $\{e^{-tV}\}$ are contractive semigroups in L^p–spaces, $1 < p < \infty$, it follows that $T(t)$ is L^p-contractive, for every $1 < p < \infty$. Then we extrapolate the semigroup $\{T(t) : t \geq 0\}$ to all L^p–spaces, $1 < p < \infty$.
After constructing the semigroups $\{T_p(t) : t \geq 0\}$, $p \in (1, \infty)$, we then look for the domain of the generators A_p. Similarly to the case $p = 2$, we show that the space of test functions $C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$ is a core for A_p and consequently, the domain of A_p coincides with the maximal domain

$$D_{p,\max}(\mathcal{A}) = \{u \in L^p(\mathbb{R}^d, \mathbb{C}^m) \cap W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}^m) : \mathcal{A}u \in L^p(\mathbb{R}^d, \mathbb{C}^m)\}.$$

Note that the construction of realizations and semigroups made in this chapter is compatible with the one of Chapter 1 for symmetric potential. Moreover, in Chapter 2, we obtain further information about the domain of the operators A_p. We establish a semigroup in $L^1(\mathbb{R}^d, \mathbb{R}^m)$ which is consistent with the semigroups $\{T_p(t) : t \geq 0\}$ and its generator is an L^1-realization of $\mathcal{A}$.

We end this chapter by establishing some properties of the semigroups $\{T_p(t) : t \geq 0\}$. We again obtain the same result for positivity of $\{T(t) : t \geq 0\}$ as in Chapter 1. Namely, $\{T(t) : t \geq 0\}$ is positive if, and only if, $v_{ij} \leq 0$, for all $i \neq j \in \{1, \dots, m\}$. We first apply the so-called *positive minimum principle* to get the necessary condition and show that is actually sufficient by using the Trotter-Kato product formula. For the analyticity, we first give an example where the semigroup $\{T(t) : t \geq 0\}$ is not analytic and then give a sufficient condition to obtain analyticity, which is

$$\mathrm{Re}\,\langle V(x)\xi, \xi\rangle \geq C\,|\mathrm{Im}\,\langle V(x)\xi, \xi\rangle|\,,$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^m$ and some $C > 0$. This condition means that the numerical range of $V(x)$, then its spectrum, is included in a sector of angle $\theta = \arctan(1/C) < \pi/2$, uniformly with respect to x. Such condition is never satisfied for antisymmetric matrix potentials, since their spectrum lie on the imaginary axis.

Now, after obtaining a semigroup $\{T_p(t) : t \geq 0\}$ associated to the realization A_p of $\mathcal{A}$ with maximal domain, we ask if the domain $Dp, \max(\mathcal{A})$ may coincide with the so-called *natural domain* of A_p, $1 < p < \infty$, which is $W^{2,p}(\mathbb{R}^d, \mathbb{C}^m) \cap D(V_p)$. Obviously, the natural domain is a subset of the maximal one. On the other hand, it contains the space of test functions, which is a core for A_p. The last statement means that A_p is the closure of the realization of $\mathcal{A}$ defined over $C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$. Therefore, if $\mathcal{A}$ is closed on the natural domain $W^{2,p}(\mathbb{R}^d, \mathbb{C}^m) \cap D(V_p)$, then it follows that the domains coincide.

Chapter 3 is devoted to investigate the closure of A_p on the natural domain. As a consequence we get the so-called *maximal inequality*

$$\|u\|_{2,p} + \|Vu\|_p \leq C(\|u\|_p + \|\Delta_Q u - Vu\|_p)$$

for some positive constant C, where $\|\cdot\|_{2,p}$ denotes the norm of $W^{2,p}(\mathbb{R}^d, \mathbb{C}^m)$. The above inequality implies the equivalence between the graph norm of A_p and the norm $|||\cdot||| : u \mapsto \|u\|_{2,p} + \|Vu\|_p$.

Our strategy is to apply a noncommutative version of the Dore-Venni theorem proved by Monniaux and Prüss in [**50**]. To that purpose we consider in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ for $p \in (1, \infty)$, the operators D_p defined as $D_p u = \Delta_Q u - u$ for $u \in W^{2,p}(\mathbb{R}^d, \mathbb{C}^m)$ and V_p as the multiplication operator by V. We assume that $V(x)$ is an injective matrix for all $x \in \mathbb{R}^d$ likewise, the operator V_p becomes injective. We can also make a rescaling for V_p in order to avoid the requirement of injectivity. Here we impose injectivity since we are going to deal with imaginary powers and general functional calculus for the operators D_p and V_p. So it is just a technical requirement.
In addition to the hypotheses on Q and V considered in Chapter 2, we assume that V has locally bounded first-order derivatives, i.e. $V \in W^{1,\infty}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ and

satisfy

$$\sup_{x\in\mathbb{R}^d} |\partial_j V(x)(V(x))^{-\alpha}| < \infty \qquad \text{or}$$
$$\sup_{x\in\mathbb{R}^d} |(V(x))^{-\alpha}\partial_j V(x)| < \infty,$$

for all $j \in \{1, \dots, m\}$ and some $\alpha \in [0, \frac{1}{2})$. The above condition allows Lipschitz potentials (by taking $\alpha = 0$). For particular potentials of the form $V(x) = v(x)V_0$, with v a locally bounded nonnegative real function and V_0 a constant accretive matrix, the condition becomes

$$|\nabla v| \le M v^\alpha, \qquad M > 0,\ \alpha \in [0, 1/2).$$

Such a condition is satisfied for $v(x) = |x|^r$ with $r \in [1, 2)$. However, for scalar Schrödinger operators, the maximal inequality hold for all polynomial radial potentials, see [**60**] and [**52**]. We thus think to get maximal inequality for potentials that have a same diagonal entry which can be a radial polynomial function, or more generally, nonnegative functions v such that $\log(v)$ is Lipschitz continuous. For this purpose we apply Okazawa's perturbation theorem, see Theorem A.20.

A compactness condition is also established for potential satisfying the maximal inequality. Such condition coincides with the one of Chapter 1 for symmetric potentials.

Last, we showed how scalar Schrödinger operators with complex potentials can be seen as a particular matrix Schrödinger operator and showed that the compactness of the resolvent and then discreteness of the spectrum of the scalar Schrödinger operators with complex potential is obtained whence either real or imaginary part of the potential blow up at infinity.

The last chapter of this thesis is concerned with regularity properties of the matrix Schrödiner semigroup $\{T(t) : t \ge 0\}$. We first state ultracontractivity property of $\{T(t) : t \ge 0\}$. We prove that, for every $t > 0$, $T(t)$ maps $L^1(\mathbb{R}^d, \mathbb{C}^m)$ into $L^\infty(\mathbb{R}^d, \mathbb{C}^m)$ continuously, with continuity norm of order $t^{-d/2}$. As a consequence, the semigroup $\{T(t) : t \ge 0\}$ is then given by a matrix integral kernel

$$T(t)f(x) = \int_{\mathbb{R}^d} K(t, x, y)f(y)dy, \qquad t > 0\ x \in \mathbb{R}^d.$$

For every $t > 0$, $K(t, \cdot, \cdot)$ is a bounded matrix valued function. This is obtained by adapting the Dunford–Pettis theorem, which holds basically for scalar functions, see [**4**], to vector-valued functions. Afterward, we follow the classical method of establishing ultracontractivity of the twisted semigroup and we get upper Gaussian estimate for all entries of the matrix kernel:

$$|k_{ij}(t, x, y)| \le Ct^{-\frac{d}{2}} \exp\{-\tau \frac{|x-y|^2}{4t}\}, \qquad t > 0,\ x, y \in \mathbb{R}^d,$$

for some $C > 0$ and $\tau \in \mathbb{R}$ and all $i, j \in \{1, \dots, m\}$.

Noting that, for each $i \in \{1, \dots, m\}$, k_{ii} coincides with the kernel associated to the scalar Schrödinger operator $\Delta_Q - v_{ii}$, and one knows several bounds for k_{ii} from the literature about kernel estimates of scalar Schrödinger operators, cf. [**45, 48, 49, 56, 61**] and [**17**, Section 4.5].

Furthermore, in the symmetric case, one can dominate the off–diagonal entries of the kernel matrix by the diagonal ones as follows

$$|k_{ij}(t,x,y) + k_{ij}(t,y,x)| \leq 2\sqrt{k_{ii}(t,x,y)}\sqrt{k_{jj}(t,x,y)}$$

for all $i \neq j \in \{1, \dots, m\}$, every $t > 0$ and almost every $x, y \in \mathbb{R}^d$.

We then focus on the so-called *diagonal estimates*, which means estimates of $k_{ii}(t,x,x)$ for all $x \in \mathbb{R}^d$ and $i \in \{1, \dots, m\}$. The importance of such estimates is that for symmetric potentials, they permit to deduce the behavior of the *trace* of the matrix Schrödinger semigroup, in the case of compactness. Actually, in the symmetric case, we prove that the trace of $T(t)$ is given by

$$\operatorname{tr}(T(t)) = \int_{\mathbb{R}^d} \sum_{i=1}^{m} k_{ii}(t,x,x)\, dx, \qquad t > 0.$$

We recall that, under the compactness condition, the eigenvalues of $T(t)$ are $e^{-\lambda_n t}$, $n \in \mathbb{N}$, where $\{\lambda_n : n \in \mathbb{N}\}$ is the discrete spectrum of $-A_p$ ($-A_p$ is accretive and has nonnegative eigenvalues) and the trace of $T(t)$ is the sum of $e^{-\lambda_n t}$, which may be finite or infinite. In the case of finite trace, $T(t)$ is called a Hilbert-Schmidt operator.

Estimates of Sikora's type, see [**61, 49**], yields the behavior near 0 of the trace of $T(t)$. From which we deduce, by using a Karamata's theorem, the asymptotic distribution of the eigenvalues of the Schrödinger operator $A = A_2$. This is done in the particular case when diagonal entries v_{ii}, $i \in \{1, \dots, m\}$, of the potential V have the same behavior when $|x|$ goes to infinity. This ends the main matter of this thesis.

This thesis contains also some appendices where we summarized briefly some mathematical background used when dealing with matrix Schrödinger operators.

In Appendix A we recall some results and terminology on operators and semigroup theory. Appendix B deals with the topic of sesquilinear forms and associated operators and semigroups. In Appendix C we introduce the theory of functional calculus for sectorial operators as it is presented in [**29**]. This appendix helps in elaborating many results of Chapter 3. Finally, in Appendix D we give some basic results on the operator multiplication by a matrix–valued function in L^p–spaces.

CHAPTER 1

Symmetric matrix Schrödinger operators

The easiest way to get generation of semigroup in L^p-spaces is the form methods; which consists in associating a sesquilinear form to the operator in question in the Hilbert space L^2. Establishing some suitable properties of the sesquilinear form allows to associate an **analytic** semigroup, in L^2, to the operator and by extrapolation techniques one could extend the semigroup to L^p-spaces, $1 < p < \infty$.

As we get always an analytic semigroup from form methods, we can apply such a method only for sectorial (quasi-sectorial) operators with numerical range in a sector of angle less than $\pi/2$. In particular, for symmetric operators. However, matrix Schrödinger operators, unless the matrix potential is symmetric, is not in general symmetric operators. Moreover, it may happen that a matrix Schrödinger operator generates a strongly continuous semigroup which is NOT analytic as we will see in Example 2.16. This is the reason why in this chapter we limit ourself to matrix Schrödinger operators with symmetric potential.

In this chapter we consider the matrix Schrödinger operator

$$\mathcal{A} = \mathrm{div}(Q\nabla\cdot) - V = \Delta_Q - V,$$

where $V = (v_{ij})_{1\le i,j\le m}$ is a symmetric semi-definite positive matrix-valued function and Q a bounded symmetric matrix satisfying the ellipticity condition (1.1.1). Similarly to the scalar case, see [**17**, Section 1.8], under the weakest condition on V, $v_{ij} \in L^1_{loc}(\mathbb{R}^d)$ for all $i,j \in \{1,\dots,m\}$, we associate a symmetric sesquilinear form to $-\mathcal{A}$ and prove that $\mathcal{A}$ admits a dissipative self-adjoint realization in $L^2(\mathbb{R}^d,\mathbb{C}^m)$ that generates a contractive strongly continuous semigroup. Using the 'Beurling-Deny' criterion of L^∞-contractivity in its vectorial version, one deduces that this semigroup can be extrapolated to the spaces $L^p(\mathbb{R}^d,\mathbb{C}^m)$, $1 < p < \infty$. We also investigate on positivity and compactness of this semigroup.

This chapter is structured as follow: In Section 1.1 we study the associated form to $\mathcal{A}$ and show that $\mathcal{A}$ has a self-adjoint realization A that generates an analytic strongly continuous semigroup $\{T(t) : t \ge 0\}$ in $L^2(\mathbb{R}^d,\mathbb{C}^m)$. In Section 1.2, we apply a 'Beurling-Deny' criterion type, see [**55**, Theorem 2], to establish L^∞-contractivity of the semigroup $\{T(t) : t \ge 0\}$ and thus extrapolate it to $L^p(\mathbb{R}^d,\mathbb{C}^m)$. Section 1.3 is devoted to characterize positivity, study compactness of the semigroup and analyze the spectrum of A.

The contents of this chapter are taken from the paper [**46**].

1.1. Generation of semigroup in L^2

Throughout this chapter we assume the following hypotheses:

1.1.1. Hypotheses.

(a) Let $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a symmetric matrix-valued function. Assume that there exist positive constants η_1 and η_2 such that

$$\eta_1|\xi|^2 \le \langle Q(x)\xi, \xi\rangle \le \eta_2|\xi|^2, \qquad x, \xi \in \mathbb{R}^d. \tag{1.1.1}$$

(b) Let $V : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ be a matrix-valued operator such that $v_{ij} = v_{ji} \in L^1_{loc}(\mathbb{R}^d)$ for all $i, j \in \{1, \ldots, m\}$ and

$$\langle V(x)\xi, \xi\rangle \ge 0, \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^m. \tag{1.1.2}$$

We introduce, for $x \in \mathbb{R}^d$, the inner–product $\langle\cdot,\cdot\rangle_{Q(x)}$ given, for $y, z \in \mathbb{R}^d$, by $\langle y, z\rangle_{Q(x)} := \langle Q(x)y, z\rangle$ and its associated norm $|z|_{Q(x)} := \sqrt{\langle Q(x)z, z\rangle}$ for each $z \in \mathbb{R}^d$.

1.1.2. The L^2-sesquilinear form. Let us define the sesquilinear form

$$a(f, g) := \int_{\mathbb{R}^d} \sum_{j=1}^m \langle Q(x)\nabla f_j(x), \nabla g_j(x)\rangle dx + \int_{\mathbb{R}^d} \langle V(x)f(x), g(x)\rangle dx, \tag{1.1.3}$$

for $f, g \in D(a)$. Here $D(a)$ denotes the domain of a and is defined by

$$D(a) := \{f = (f_1, \ldots, f_m) \in H^1(\mathbb{R}^d, \mathbb{C}^m) : \int_{\mathbb{R}^d} \langle V(x)f(x), f(x)\rangle dx < +\infty\}. \tag{1.1.4}$$

We endow $D(a)$ with the norm

$$\begin{aligned}\|f\|_a &= \left(\|f\|^2_{H^1(\mathbb{R}^d,\mathbb{C}^m)} + \int_{\mathbb{R}^d} \langle V(x)f(x), f(x)\rangle dx\right)^{1/2} \\ &= \left(\|f\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^m)} + \sum_{j=1}^m \|\nabla f_j\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^m)} + \int_{\mathbb{R}^d} \langle V(x)f(x), f(x)\rangle dx\right)^{1/2}.\end{aligned}$$

We now give some properties of a.

Proposition 1.1. *Assume Hypotheses 1.1.1 are satisfied. Then,*

(i) a is densely defined, i.e. $D(a)$ is dense in $L^2(\mathbb{R}^d, \mathbb{C}^m)$.
(ii) a is accretive.
(iii) a is continuous, i.e. exists $M > 0$ such that

$$|a(f, g)| \le M\|f\|_a\|g\|_a, \qquad f, g \in D(a).$$

(iv) a is closed, i.e. $(D(a), \|.\|_a)$ *is a complete space.*

PROOF. (i) It is easy to see that $C_c^\infty(\mathbb{R}^d,\mathbb{C}^m) \subset D(a)$. Indeed, $C_c^\infty(\mathbb{R}^d,\mathbb{C}^m) \subset H^1(\mathbb{R}^d,\mathbb{C}^m)$ and, for $f \in C_c^\infty(\mathbb{R}^d,\mathbb{C}^m)$

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\langle V(x)f(x), f(x)\rangle dx\right| &\le \int_{\mathbb{R}^d}|V(x)f(x)||f(x)|dx\\ &\le \int_{\mathbb{R}^d}|V(x)||f(x)|^2dx\\ &\le \|f\|_\infty^2\int_{supp(f)}|V(x)|dx < \infty.\end{aligned}$$

Hence, $D(a)$ is dense in $L^2(\mathbb{R}^d,\mathbb{C}^m)$.
(ii)*Accretivity:* For $f \in D(a)$ one has

$$\text{Re } a(f) = \int_{\mathbb{R}^d}\sum_{j=1}^m|Q^{1/2}(x)\nabla f_j(x)|^2dx + \int_{\mathbb{R}^d}\text{Re}\,\langle V(x)f(x), f(x)\rangle dx \ge 0.$$

(iii) *Continuity:* Let $f, g \in D(a)$. By application of Cauchy–Schwartz and Young inequalities one gets

$$\begin{aligned}|a(f,g)| &\le \eta_2\sum_{j=1}^m\int_{\mathbb{R}^d}|\nabla f_j(x)||\nabla g_j(x)|dx + \int_{\mathbb{R}^d}|\langle V(x)^{1/2}f(x), V(x)^{1/2}g(x)\rangle|dx\\ &\le \eta_2\sum_{j=1}^m\|\nabla f_j\|_2\|\nabla g_j\|_2 + \left(\int_{\mathbb{R}^d}|V(x)^{1/2}f(x)|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^d}|V(x)^{1/2}g(x)|^2dx\right)^{1/2}\\ &\le \eta_2\left(\sum_{j=1}^m\|\nabla f_j\|_2^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^m\|\nabla g_j\|_2^2\right)^{\frac{1}{2}}\\ &+ \left(\int_{\mathbb{R}^d}\langle V(x)f(x), f(x)\rangle dx\right)^{1/2}\left(\int_{\mathbb{R}^d}\langle V(x)g(x), g(x)\rangle dx\right)^{1/2}\\ &\le (1+\eta_2)\|f\|_a\|g\|_a.\end{aligned}$$

(iv) *Closedness:* Let $(f_n)_{n\in\mathbb{N}} \subset D(a)$ be a Cauchy sequence in $(D(a), \|\cdot\|_a)$. Then

$$\|f_n - f_l\|_{H^1(\mathbb{R}^d,\mathbb{C}^m)} + \int_{\mathbb{R}^d}\langle V(x)(f_n(x)-f_l(x)), (f_n(x)-f_l(x)\rangle dx \underset{n,l\to\infty}{\longrightarrow} 0.$$

This yields

$$\begin{cases} f_n - f_l \longrightarrow 0 \quad in \quad H^1(\mathbb{R}^d,\mathbb{C}^m)\\ \int_{\mathbb{R}^d}|V^{1/2}(f_n-f_l)|^2 \longrightarrow 0\end{cases}.$$

Hence, $(f_n)_{n\in\mathbb{N}}$ and $(V^{1/2}f_n)_{n\in\mathbb{N}}$ are Cauchy sequences respectively in $H^1(\mathbb{R}^d,\mathbb{C}^m)$ and $L^2(\mathbb{R}^d,\mathbb{C}^m)$. Therefore

$$\begin{cases} f_n \longrightarrow f \quad in \quad H^1(\mathbb{R}^d,\mathbb{C}^m) \\ V^{1/2}f_n \longrightarrow g \quad in \quad L^2(\mathbb{R}^d,\mathbb{C}^m) \end{cases}.$$

The pointwise convergence of subsequences implies that

$$V^{1/2}f = g \in L^2(\mathbb{R}^d,\mathbb{C}^m).$$

Then $f \in D(a)$ and

$$a(f_n - f) = \|f_n - f\|^2_{H^1(\mathbb{R}^d,\mathbb{C}^m)} + \int_{\mathbb{R}^d} |V^{1/2}(x)(f_n - f)(x)|^2 dx \underset{n\to\infty}{\longrightarrow} 0,$$

which ends the proof. □

Now, let $\mathcal{A}$ defined in (0.0.1), i.e.

$$\mathcal{A} = \mathrm{div}(Q\nabla\cdot) - V = \Delta_Q - V. \tag{1.1.5}$$

Thanks to Proposition 1.1 and applying Theorem B.6, we obtain

Corollary 1.2. *$\mathcal{A}$ admits a realization A in $L^2(\mathbb{R}^d,\mathbb{C}^m)$ that generates a bounded strongly continuous and analytic semigroup $\{T(t) : t \geq 0\}$. Moreover, A is self–adjoint and $-A$ is the linear operator associated to the form a.*

Remark 1.3. The form method does not apply for non symmetric potentials. In fact, semigroups associated to continuous sesquilinear forms are always analytic semigroups. However, matrix Schrödinger semigroups are not always analytic, as Example 2.16 shows, where it has been proved that the semigroup associated to matrix Schrödinger operator with matrix potential

$$\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}, \qquad x \in \mathbb{R},$$

is not analytic. Otherwise, we show by a direct computation that the continuity property of the associated form fails when we take, instead of a symmetric potential, the same antisymmetric potential

$$V(x) = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be such that $\chi_{B(1)} \leq \varphi \leq \chi_{B(2)}$. Consider, for $n \geq 1$,

$$f_n(x) = \frac{\varphi(x/n)}{\sqrt{1+|x|^2}}e_1 \quad and \quad g_n(x) = \frac{\varphi(x/n)}{\sqrt{1+|x|^2}}e_2,$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$, and take, for simplicity, $Q = I$, the identity matrix in $\mathbb{R}^d$. Since $V = -V^*$ then $\langle V(x)\xi, \xi\rangle = 0$, for all $\xi \in \mathbb{R}^2$. Thus,

$$a(f_n) = a(g_n) = \int_{\mathbb{R}} \left| -\frac{\varphi(x/n)}{(1+|x|^2)^{\frac{3}{2}}} x + \frac{1}{n}\frac{1}{\sqrt{1+|x|^2}} \nabla\varphi(x/n)\right|^2 dx$$

and

$$|a(f_n, g_n)| = \int_{\mathbb{R}^d} \frac{|x|}{(1+|x|^2)} \varphi(x/n) dx.$$

If the continuity property of the form were satisfied, then there will exist $C > 0$ such that

$$|a(f_n, g_n)| \leq C\|f_n\|_a\|g_n\|_a = C(\|f_n\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^2)} + a(f_n)).$$

By the Lebesgue dominated convergence theorem one can let n tends to ∞ and obtains

$$\int_{\mathbb{R}^d} \frac{|x|}{1+|x|^2} dx \leq C\left(\int_{\mathbb{R}^d} \frac{|x|^2}{(1+|x|^2)^3} dx + \int_{\mathbb{R}^d} \frac{1}{1+|x|^2} dx\right) < \infty.$$

However, the integral of the left-hand side is infinite.

1.2. Extension to L^p

In this section we will show that $\mathcal{A}$ has a L^p-realization which generates a holomorphic semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^m)$, $1 < p < \infty$. In order to do so we prove that, for every $t > 0$, the restriction $T(t)_{|L^2\cap L^\infty}$ of $T(t)$ to $L^2(\mathbb{R}^d, \mathbb{C}^m) \cap L^\infty(\mathbb{R}^d, \mathbb{C}^m)$ can be extended to a bounded operator $T_p(t)$ in $L^p(\mathbb{R}^d, \mathbb{C}^m)$, $2 < p < \infty$. Then, we show that $\{T_p(t) : t \geq 0\}$ is strongly continuous. Moreover, since $\{T(t) : t \geq 0\}$ is self-adjoint, the semigroups $\{T_p(t) : t \geq 0\}$, for $1 < p < 2$ is the adjoint of $\{T_{p'}(t) : t \geq 0\}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and so $p' > 2$. For this aim it suffices that $\{T(t) : t \geq 0\}$ satisfies the L^∞-contractivity property:

$$\|T_2(t)f\|_\infty \leq \|f\|_\infty, \qquad \forall f \in L^2(\mathbb{R}^d, \mathbb{C}^m) \cap L^\infty(\mathbb{R}^d, \mathbb{C}^m). \tag{1.2.1}$$

A characterization of (1.2.1) via the associated form is given in [**55**], see Theorem B.8. According to this characterization, (1.2.1) holds true when the following are satisfied:

(i) $f \in D(a) \Longrightarrow (1 \wedge |f|)sign(f) \in D(a)$,

(ii)
$$a((1 \wedge |f|)sign(f)) \leq a(f), \qquad \forall f \in D(a),$$

where $sign(f) := \frac{f}{|f|}\chi_{\{f\neq 0\}}$ for $f \in L^2(\mathbb{R}^d, \mathbb{C}^m)$.

We start by the following lemma where we establish (i).

Lemma 1.4. *a) Assume $f \in H^1(\mathbb{R}^d, \mathbb{C}^m)$. Then, $|f| \in H^1(\mathbb{R}^d)$ and*

$$\nabla |f| = \frac{\sum_{j=1}^m f_j \nabla f_j}{|f|} \chi_{\{f \neq 0\}}. \tag{1.2.2}$$

b) Let $f \in D(a)$. Then $(1 \wedge |f|) sign(f) \in D(a)$. In particular,

$$\nabla((1 \wedge |f|) sign(f))_j = \frac{1 + sign(1 - |f|)}{2} \frac{f_j}{|f|} \chi_{\{f \neq 0\}} \nabla |f| + \frac{1 \wedge |f|}{|f|} (\nabla f_j - \frac{f_j}{|f|} \nabla |f|) \chi_{\{f \neq 0\}} \tag{1.2.3}$$

for every $j \in \{1, ..., m\}$.

Proof. a) Let $f \in H^1(\mathbb{R}^d, \mathbb{C}^m)$. Define, for $\varepsilon > 0$, $f_\varepsilon = \left(\sum_{j=1}^m |f_j|^2 + \varepsilon^2 \right)^{\frac{1}{2}} - \varepsilon$. One has

$$0 \leq f_\varepsilon = \frac{|f|^2}{\left(\sum_{j=1}^m |f_j|^2 + \varepsilon^2 \right)^{\frac{1}{2}} + \varepsilon} \leq |f|.$$

Hence, by dominated convergence theorem $f_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} |f|$ in $L^2(\mathbb{R}^d)$. On the other hand, $f_\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ and

$$\nabla f_\varepsilon = \frac{\sum_{j=1}^m f_j \nabla f_j}{\left(\sum_{j=1}^m |f_j|^2 + \varepsilon \right)^{\frac{1}{2}}} \underset{\varepsilon \to 0}{\longrightarrow} \frac{\sum_{j=1}^m f_j \nabla f_j}{|f|} \chi_{\{f \neq 0\}}.$$

Again, the dominated convergence theorem yields $|f| \in H^1(\mathbb{R}^d)$ and (1.2.2).
b) Let $f \in D(a)$, i.e. $f \in H^1(\mathbb{R}^d, \mathbb{C}^m)$ and $V^{1/2} f \in L^2(\mathbb{R}^d, \mathbb{C}^m)$. One has

$$\begin{aligned} \int_{\mathbb{R}^d} \langle V(x)(1 \wedge |f|) sign(f), (1 \wedge |f|) sign(f) \rangle dx &\leq \int_{\{f \neq 0\}} \left(\frac{1 \wedge |f|}{|f|} \right)^2 \langle V(x) f(x), f(x) \rangle dx \\ &\leq \int_{\mathbb{R}^d} \langle V(x) f(x), f(x) \rangle dx < \infty. \end{aligned}$$

It remains now to show that $(1 \wedge |f|) sign(f) \in H^1(\mathbb{R}^d, \mathbb{C}^m)$. Set

$$Pf := (1 \wedge |f|) sign(f) = (1 \wedge |f|) \frac{f}{|f|} \chi_{\{f \neq 0\}},$$

and

$$P_\varepsilon f := (1 \wedge |f|) \frac{f}{|f| + \varepsilon} = \frac{1 + |f| - |1 - |f||}{2} \frac{f}{|f| + \varepsilon}$$

for $\varepsilon > 0$. Since $|P_\varepsilon f| \leq (1 \wedge |f|) \leq |f|$ and $P_\varepsilon f \underset{\varepsilon\to 0}{\longrightarrow} Pf \quad a.e.$, It follows, be dominated convergence theorem, that $P_\varepsilon f \underset{\varepsilon\to 0}{\longrightarrow} Pf$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$.

On the other hand, $P_\varepsilon f \in H^1_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ and

$$\begin{aligned}\nabla (P_\varepsilon f)_j &= \nabla \left(\frac{1 + |f| - |1 - |f||}{2} \frac{f_j}{|f| + \varepsilon} \right) \\ &= \frac{1 + |f| - |1 - |f||}{2} \left(\frac{\nabla f_j}{|f| + \varepsilon} - \frac{f_j}{(|f| + \varepsilon)^2} \nabla |f| \right) \\ &+ \frac{1}{2} \frac{f_j}{|f| + \varepsilon} \left(\nabla |f| + sign(1 - |f|) \nabla |f| \right) \\ &= \frac{1 \wedge |f|}{|f| + \varepsilon} \left(\nabla f_j - \frac{f_j}{|f| + \varepsilon} \nabla |f| \right) \\ &+ \frac{1}{2} \frac{f_j}{|f| + \varepsilon} \left(1 + sign(1 - |f|) \right) \nabla |f|.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0} \nabla (P_\varepsilon f)_j = &\frac{1 \wedge |f|}{|f|} \left(\nabla f_j - \frac{f_j}{|f|} \nabla |f| \right) \chi_{\{f \neq 0\}} \\ &+ \frac{1}{2} \frac{f_j}{|f|} \left(1 + sign(1 - |f|) \right) \chi_{\{f \neq 0\}} \nabla |f|,\end{aligned}$$

and

$$|\nabla (P_\varepsilon f)_j| \leq \frac{1 \wedge |f|}{|f|} (|\nabla f_j| + |\nabla |f||) + |\nabla |f|| \leq |\nabla f_j| + 2|\nabla |f|| \ \in L^2(\mathbb{R}^d).$$

By the dominated convergence theorem we conclude that $Pf = \lim_{\varepsilon\to 0} P_\varepsilon f \in H^1(\mathbb{R}^d, \mathbb{C}^m)$, and

$$\begin{aligned}\nabla (Pf)_j &:= \lim_{\varepsilon\to 0} \nabla (P_\varepsilon f)_j \\ &= \frac{1 \wedge |f|}{|f|} \left(\nabla f_j - \frac{f_j}{|f|} \nabla |f| \right) \chi_{\{f \neq 0\}} + \frac{1}{2} \frac{f_j}{|f|} \left(1 + sign(1 - |f|) \right) \chi_{\{f \neq 0\}} \nabla |f|.\end{aligned}$$

□

In the following we state another lemma where we prove (ii).

Lemma 1.5. *Let $f \in D(a)$. Then*

$$a((1 \wedge |f|) sign(f)) \leq a(f). \tag{1.2.4}$$

PROOF. Let $f \in D(a)$. One has,

$$
\begin{aligned}
\alpha_f &:= \langle Q\nabla((1\wedge|f|)sign(f)), \nabla((1\wedge|f|)sign(f))\rangle \\
&= \sum_{j=1}^{m}\left|\frac{1\wedge|f|}{|f|}\left(\nabla f_j - \frac{f_j}{|f|}\nabla|f|\right)\chi_{\{f\neq 0\}} + \frac{1}{2}\frac{f_j}{|f|}(1+sign(1-|f|))\chi_{\{f\neq 0\}}\nabla|f|\right|_Q^2 \\
&= \frac{(1+sign(1-|f|))^2}{4}\chi_{\{f\neq 0\}}|\nabla|f||_Q^2 + \frac{(1\wedge|f|)^2}{|f|^2}\chi_{\{f\neq 0\}}\sum_{j=1}^{m}|\nabla f_j - \frac{f_j}{|f|}\nabla|f||_Q^2 \\
&+ (1+sign(1-|f|))\frac{1\wedge|f|}{|f|}\chi_{\{f\neq 0\}}\sum_{j=1}^{m}\langle Q\nabla|f|, (\nabla f_j - \frac{f_j}{|f|}\nabla|f|)\rangle\frac{f_j}{|f|} \\
&= \frac{(1+sign(1-|f|))^2}{4}\chi_{\{f\neq 0\}}|\nabla|f||_Q^2 \\
&+ \frac{(1\wedge|f|)^2}{|f|^2}\chi_{\{f\neq 0\}}\left(\sum_{j=1}^{m}\langle Q\nabla f_j, \nabla f_j\rangle + |\nabla|f||_Q^2 - \langle\nabla|f|^2, \frac{\nabla|f|}{|f|}\rangle_Q\right) \\
&+ (1+sign(1-|f|))\frac{1\wedge|f|}{|f|}\chi_{\{f\neq 0\}}\underbrace{\left(\frac{1}{2}\langle Q\nabla|f|, \nabla|f|^2\rangle - |f|\langle Q\nabla|f|, \nabla|f|\rangle\right)}_{0} \\
&= \frac{(1+sign(1-|f|))^2}{4}\chi_{\{f\neq 0\}}|\nabla|f||_Q^2 + \frac{(1\wedge|f|)^2}{|f|^2}\chi_{\{f\neq 0\}}\left(\sum_{j=1}^{m}\langle Q\nabla f_j, \nabla f_j\rangle - |\nabla|f||_Q^2\right) \\
&= \left(\frac{(1+sign(1-|f|))^2}{4} - \frac{(1\wedge|f|)^2}{|f|^2}\right)\chi_{\{f\neq 0\}}|\nabla|f||_Q^2 + \frac{(1\wedge|f|)^2}{|f|^2}\chi_{\{f\neq 0\}}\sum_{j=1}^{m}\langle Q\nabla f_j, \nabla f_j\rangle.
\end{aligned}
$$

Discussing the different cases $|f| < 1$, $|f| = 1$ and $|f| > 1$, one can easily see that

$$
\frac{(1+sign(1-|f|))^2}{4} - \frac{(1\wedge|f|)^2}{|f|^2} \leq 0.
$$

Thus,

$$
\begin{aligned}
\alpha_f &\leq \frac{(1\wedge|f|)^2}{|f|^2}\chi_{\{f\neq 0\}}\sum_{j=1}^{m}\langle Q\nabla f_j, \nabla f_j\rangle \\
&\leq \sum_{j=1}^{m}\langle Q\nabla f_j, \nabla f_j\rangle.
\end{aligned}
$$

Integrating over $\mathbb{R}^d$, one gets

$$a_0((1\wedge|f|)sign(f)) := \int_{\mathbb{R}^d}\langle Q\nabla((1\wedge|f|)sign(f)), \nabla((1\wedge|f|)sign(f))\rangle dx$$
$$\leq \int_{\mathbb{R}^d}\sum_{j=1}^m \langle Q\nabla f_j, \nabla f_j\rangle dx := a_0(f).$$

Therefore,

$$a((1\wedge|f|)sign(f)) = a_0((1\wedge|f|)sign(f)) + \int_{\mathbb{R}^d}\langle V(x)(1\wedge|f|)sign(f), (1\wedge|f|)sign(f)\rangle dx$$
$$\leq a_0(f) + \int_{\mathbb{R}^d}\langle Vf, f\rangle dx = a(f).$$

□

Consequently, we get

Corollary 1.6. *The semigroup $\{T(t) : t \geq 0\}$ is L^∞-contractive.*

Now, we are able to state the main theorem of this section.

Theorem 1.7. *Let $1 < p < \infty$. Then, $\mathcal{A}$ admits a realization A_p in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ that generates a bounded strongly continuous and analytic semigroup $\{T_p(t) : t \geq 0\}$.*

Proof. Let $2 < p < \infty$. According to Corollary 1.2 and Corollary 1.6 $\{T(t) : t \geq 0\}$ is self–adjoint and L^∞–contractive. Hence, by the interpolation theorems of *Riesz-Thorin* and *Stein*, see [**17**, Section 1.1.5; Section 1.1.6], $\{T(t) : t \geq 0\}$ admits a unique analytic contractive extension $\{T_p(t) : t \geq 0\}$ to $L^p(\mathbb{R}^d, \mathbb{C}^m)$. Moreover, for every $f \in L^2(\mathbb{R}^d, \mathbb{C}^m) \cap L^\infty(\mathbb{R}^d, \mathbb{C}^m)$,

$$\|T(t)f - f\|_p \leq \|T(t)f - f\|_2^\theta \|T(t)f - f\|_\infty^{1-\theta} \leq 2^{1-\theta}\|f\|_\infty^{1-\theta}\|T(t)f - f\|_2^\theta,$$

where $\theta = \frac{2}{p}$. This shows how $\{T_p(t) : t \geq 0\}$ is strongly continuous.

Concerning the case $1 < p < 2$, we prove by duality argument that (1.2.1) implies $\|T(t)f\|_1 \leq \|f\|_1$, for every $t > 0$ and $f \in L^1(\mathbb{R}^d, \mathbb{C}^m)$. Applying again the Riesz–Thorin interpolation theorem, and arguing similarly, we obtain an analytic extrapolation of $\{T(t) : t \geq 0\}$ to $L^p(\mathbb{R}^d, \mathbb{C}^m)$; such extrapolation is always strongly continuous.

□

Remark 1.8. We can extrapolate the semigroup $\{T(t); t \geq 0\}$ to a strongly continuous one in $L^1(\mathbb{R}^d, \mathbb{C}^m)$. This follows by consistency and L^p-contractivity of the semigroup $\{T(t) : t \geq 0\}$ and according to [**67**]. The detailed proof can be found in Section 2.6.

1.3. Further properties of the semigroup

In this section we study positivity, compactness of $\{T(t)\}_{t\geq 0}$ and spectrum of A.

1.3.1. Positivity. In this subsection we give a necessary and sufficient condition for positivity of the semigroup $\{T(t) : t \geq 0\}$. We use the form characterization of the invariance of convex subsets via semigroups. For this purpose we introduce the closed convex subset of $L^2(\mathbb{R}^d, \mathbb{R}^m)$

$$C^+ := \{f = (f_1, \ldots, f_m) \in L^2(\mathbb{R}^d, \mathbb{R}^m) : f \geq 0 \qquad a.e.\}.$$

The projection P_+ on C^+ is given by

$$P_+ f := f^+ = (f_j \wedge 0)_{1\leq j\leq m}, \qquad \forall f \in L^2(\mathbb{R}^d, \mathbb{R}^m).$$

One knows that the projection on a closed convex subset of a Banach space is uniquely defined and it is easy to check that P_+ defined above is the right one for C^+. We recall that $\{T(t) : t \geq 0\}$ is positive if, and only if, for every $f \in L^2(\mathbb{R}^d, \mathbb{R}^m)$ such that $f \geq 0$, $T(t)f \geq 0$ for all $t > 0$, see Section A.4 for more details. According to Corollary B.9, $\{T(t) : t \geq 0\}$ is positive if, and only if,

- $f^+ \in D(a)$ for all $f \in D(a)$;
- $a(f^+, f^-) \leq 0$ for all $f \in D(a)$.

Now, applying the above criterion, we get the following characterisation of positivity of $\{T(t)\}_{t\geq 0}$ in term of entries of the potential matrix V as follows

Theorem 1.9. *The semigroup $\{T(t) : t \geq 0\}$ is positive if and only if $v_{ij} \leq 0$, for all $i \neq j \in \{1, \ldots, m\}$.*

PROOF. Suppose that $\{T(t) : t \geq 0\}$ is positive. Let $i \neq j \in \{1, \ldots, m\}$ and consider $f = \varphi(e_i - e_j)$ where $0 \leq \varphi \in C_c^\infty(\mathbb{R}^d)$ is arbitrarily chosen. One has $f^+ = \varphi e_i$, $f^- = \varphi e_j$ and $\langle Q\nabla f^+, \nabla f^-\rangle = 0$.

Applying Corollary B.9, we obtain

$$0 \geq a(f^+, f^-) = \sum_{k=1}^m \int_{\mathbb{R}^d} \langle Q\nabla u_k^+, \nabla f_k^-\rangle dx + \int_{\mathbb{R}^d} \langle V f^+, f^-\rangle dx = \int_{\mathbb{R}^d} v_{ij}\varphi^2 dx,$$

which yields $v_{ij} \leq 0$ a.e. Conversely, assume that the off-diagonal coefficients v_{ij}, $i \neq j$, are less than or equal to 0 and let $f \in D(a)$. Let us show first that $f^+ \in D(a)$. According to [**28**, Lemma 7.6] one has, $\nabla f_k^+ = \chi_{\{f_k>0\}}\nabla f_k$ and $\nabla f_k^- = \chi_{\{f_k<0\}}\nabla f_k$. Therefore, $f^+ \in H^1(\mathbb{R}^d, \mathbb{R}^m)$ and $\langle Q\nabla f_k^+, \nabla f_k^-\rangle = 0$. On the other hand,

$$\begin{aligned}
\langle Vf, f\rangle &= \langle V(f^+ - f^-), (f^+ - f^-)\rangle \\
&= \langle Vf^+, f^+\rangle + \langle Vf^-, f^-\rangle - 2\langle Vf^+, f^-\rangle \\
&= \langle Vf^+, f^+\rangle + \langle Vf^-, f^-\rangle - 2\sum_{i,j=1}^{m} v_{ij} f_i^+ f_j^- \\
&= \langle Vf^+, f^+\rangle + \langle Vf^-, f^-\rangle - 2\underbrace{\sum_{i\neq j} v_{ij} f_i^+ f_j^-}_{\leq 0} \\
&\geq \langle Vf^+, f^+\rangle + \langle Vf^-, f^-\rangle \\
&\geq \langle Vf^+, f^+\rangle.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}^d} \langle Vf^+, f^+\rangle dx \leq \int_{\mathbb{R}^d} \langle Vf, f\rangle dx < \infty.$$

Consequently $f^+ \in D(a)$ and

$$\begin{aligned}
a(f^+, f^-) &= \sum_{k=1}^{m} \int_{\mathbb{R}^d} \langle Q\nabla f_k^+, \nabla f_k^-\rangle dx + \int_{\mathbb{R}^d} \langle Vf^+, f^-\rangle dx \\
&= \int_{\mathbb{R}^d} \sum_{i,j=1}^{m} v_{ij} f_i^+ f_j^- dx \\
&= \int_{\mathbb{R}^d} \sum_{i=1}^{m} v_{ii} f_i^+ f_i^- dx + \int_{\mathbb{R}^d} \sum_{i\neq j} v_{ij} f_i^+ f_j^- dx \\
&= \int_{\mathbb{R}^d} \sum_{i\neq j} v_{ij} f_i^+ f_j^- dx \leq 0.
\end{aligned}$$

□

Remark 1.10. In Chapter 2, we get the same characterization of positive matrix Schrödinger semigroup, even if the matrix potential is not symmetric. The result was obtained by applying the positive minimum principle, see Theorem A.4.1, which gives a necessary condition, but in our situation will be also a sufficient condition.

1.3.2. Compactness. In this subsection we give a necessary condition for compactness of the resolvent of the operator A in $L^2(\mathbb{R}^d, \mathbb{R}^m)$ and give counter example when the condition is not satisfied. Our assumption is that the smallest eigenvalue $\mu(x)$ of $V(x)$ blow up at infinity, which we rewrite as follows.

There exists $\mu : \mathbb{R}^d \to \mathbb{R}^+$ locally integrable such that $\lim_{|x|\to\infty} \mu(x) = +\infty$, and

$$\langle V(x)\xi, \xi\rangle \geq \mu(x)|\xi|^2, \qquad \forall \xi \in \mathbb{R}^m, \forall x \in \mathbb{R}^d. \tag{1.3.1}$$

Proposition 1.11. *Assume that* (1.3.1) *is satisfied. Then, $T_p(t)$ is compact in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, for every $t > 0$. Consequently, the spectrum of A_p is independent of $p \in (1, \infty)$, countable and consists of negative eigenvalues that accumulate at $-\infty$.*

Proof. It suffices to prove $D(a)$ is compactly embedded in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. Indeed, this implies that A has a compact resolvent and, by analyticity, $T(t)$ is compact in $L^2(\mathbb{R}^d, \mathbb{R}^m)$, for every $t > 0$. The compactness in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, $1 < p < \infty$, is a consequence of the consistency of the semigroups $\{T_p(t) : t \geq 0\}$ and the compactness in $L^2(\mathbb{R}^d, \mathbb{R}^m)$, see for instance [**17**, Theorem 1.6.1]. The p-independence of the spectrum is a consequence of [**17**, Corollary 1.6.2].

Let us consider the 'diagonal' sesquilinear form

$$a_\mu(f, g) = \int_{\mathbb{R}^d} \sum_{j=1}^{m} \langle Q(x)\nabla f_j(x), \nabla g_j(x)\rangle dx + \int_{\mathbb{R}^d} \sum_{j=1}^{m} \mu(x) f_j(x) g_j(x) dx$$

with domain

$$D(a_\mu) = \{f \in H^1(\mathbb{R}^d, \mathbb{C}^m) : \int_{\mathbb{R}^d} \sum_{j=1}^{m} \mu(x)|f_j(x)|^2 dx < +\infty\}.$$

By (1.3.1) we deduce that $D(a) \subseteq D(a_\mu)$ and $a_\mu(f) \leq a(f)$ for all $f \in D(a)$. That is, $D(a)$ is continuously embedded in $D(a_\mu)$. It thus suffices to prove that $D(a_\mu)$ is compactly embedded in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. This holds actually, since $\lim_{|x|\to\infty} \mu(x) = +\infty$. Indeed, let us show that the closed unit ball of $D(a_\mu)$ is compact in $L^2(\mathbb{R}^d; \mathbb{R}^m)$. Let f belongs to the unit ball of $D(a_\mu)$ so that in particular $\|\mu f\|_2 \leq a_\mu(f) \leq 1$. Given $\varepsilon > 0$ we fix $R > 0$ sufficiently large so that $1 \leq \mu\varepsilon$ outside the ball $B_R := \{x \in \mathbb{R}^d : |x| < R\}$. Then,

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B_R} |f(x)|^2\, dx \leq& \varepsilon^2 \int_{\mathbb{R}^d\setminus B_R} \mu(x)^2 |f(x)|^2\, dx \\ \leq& \varepsilon^2 \int_{\mathbb{R}^d} \mu(x)^2 |f(x)|^2\, dx \leq \varepsilon^2 \|\mu f\|_2^2 \leq \varepsilon^2.\end{aligned}$$

Since the set of restriction to B_R of functions in $D(a_\mu)$ is embedded in $H^1(B_R; \mathbb{R}^m)$ which is compactly embedded into $L^2(B_R; \mathbb{R}^m)$ by Sobolev embedding theorem, see [**?**, Theorem 6.2]. Thus, we can find finitely many functions $g_1, \ldots, g_k \in L^2(B_R; \mathbb{R}^m)$ such that, for every f in the unit ball of $D(a_\mu)$, there exists $j \in \{1, \ldots, k\}$ such that

$$\int_{B_R} |f(x) - g_j(x)|^2\, dx \leq \varepsilon^2.$$

Extending by the function g_j by 0 to $\mathbb{R}^d$ and denoting by $\tilde{g}_j$ this extension, we get

$$\int_{\mathbb{R}^d} |f(x) - \tilde{g}_j(x)|^2 \, dx = \int_{B_R} |f(x) - g_j(x)|^2 \, dx + \int_{\mathbb{R}^d \setminus B_R} |f(x)|^2 \, dx \leq 2\varepsilon^2.$$

This shows that the unit ball of $D(a_\mu)$ is covered by the balls of $L^2(\mathbb{R}^d; \mathbb{R}^m)$ centered at $\tilde{g}_j$ of radius $\sqrt{2}\varepsilon$. By arbitrariness of $\varepsilon > 0$, it follows that the unit ball of $D(a_\mu)$ is totally bounded in $L^2(\mathbb{R}^d; \mathbb{R}^m)$.
Therefore, the embedding $D(a_\mu) \hookrightarrow L^2(\mathbb{R}^d, \mathbb{R}^m)$ is compact, and thus so is $D(a) \hookrightarrow L^2(\mathbb{R}^d, \mathbb{R}^m)$.

The discreteness of the spectrum follows now by the spectral mapping theorem, since A has compact resolvent. □

Example 1.12. Here we give a counter-example where Proposition 1.11 cannot apply and the compactness result fails even if all entries of the matrix potential blow up at infinity. We even have a spectrum which is not reduced only to eigenvalues. Let us consider the following two-size matrix-valued function

$$x \mapsto V(x) := v(x) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = v(x)J,$$

where $v \in L^1_{loc}(\mathbb{R}^d)$ can be any nonnegative function satisfying $\lim\limits_{|x| \to \infty} v(x) = +\infty$. Obviously V is symmetric and satisfies (1.1.2). Diagonalizing V, we obtain

$$V(x) = P^{-1} \begin{pmatrix} 2v(x) & 0 \\ 0 & 0 \end{pmatrix} P,$$

where $P := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The Schrödinger operator $\mathcal{A}$ with $Q = I_2$ becomes

$$\mathcal{A} = \Delta - V = P^{-1} \begin{pmatrix} \Delta - 2v(x) & 0 \\ 0 & \Delta \end{pmatrix} P.$$

Since the Laplacian operator Δ has no compact resolvent on $L^2(\mathbb{R}^d)$, thus $\begin{pmatrix} \Delta - 2v(x) & 0 \\ 0 & \Delta \end{pmatrix}$ also has no compact resolvent and then A also. Moreover, the spectrum of A, $\sigma(A) = \sigma(\Delta) \cup \sigma(\Delta - 2v) =]-\infty, 0]$ is not discrete. However, the punctual spectrum $\sigma_p(A) = \sigma(\Delta - 2v)$ is actually countable, since the scalar potential $2v$ blows up at infinity.

Such potentials can be constructed even for higher dimensions, $m \geq 3$. One can consider $V(x) = v(x)J_m$, where v is any nonnegative locally integrable function that blows up at infinity and J_m a symmetric semi–definite $(m \times m)$-matrix having

0 as an eigenvalue. For example, one can take

$$J_m = \begin{pmatrix} m-1 & -1 & \cdots & -1 \\ -1 & m-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & m-1 \end{pmatrix}.$$

Remark 1.13. Under the condition (1.3.1) which guaranties compactness of the resolvent of A, one can get more information about the spectrum $\sigma(A)$ of A by application of the min–max principle.

Indeed, let $\mu, \nu : \mathbb{R}^d \to \mathbb{R}^+$ be locally integrable such that μ blows up at infinity and

$$\mu(x)|\xi|^2 \leq \langle V(x)\xi, \xi\rangle \leq \nu(x)|\xi|^2, \tag{1.3.2}$$

for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^m$. Denotes by $\{\lambda_1 < \lambda_2 < \dots\}$ the increasing sequence of eigenvalues of $-A$. By $\{\lambda_1^\mu < \lambda_2^\mu < \dots\}$ we denote the eigenvalues of the scalar operator $-\Delta_Q + \mu$ acting on $L^2(\mathbb{R}^d)$; we use the same notation for ν. According to the min-max principle, one has $\lambda_n^\nu \leq \lambda_n \leq \lambda_n^\mu$, for all $n \in \mathbb{N}$.

We recall that the min-max principle is a way to express eigenvalues of an operator via its associated form. Applying it for A one obtains

$$\lambda_n = \max_{F_1,\dots,F_{n-1}\in H} \inf\{a(f) : f \in \{F_1, \dots, F_{n-1}\}^\perp \cap D(a) \textit{ with } \|f\| = 1\}.$$

We particularly can choose, for almost every $x \in \mathbb{R}^d$, $\mu(x)$ as the smallest eigenvalue of $V(x)$ and $\nu(x)$ to be the greatest eigenvalue of $V(x)$. If, it happened that μ and ν behaves (especially asymptotically) in a way such that one gets the same asymptotic distribution of eigenvalues of $\Delta_Q - \nu$ and $\Delta_Q - \mu$, then the asymptotic distribution of the eigenvalues of $\Delta_Q - V$ will be the same as $\Delta_Q - \nu$ and $\Delta_Q - \mu$. This will be the subject of Section 4.5.

CHAPTER 2

Semigroup associated to matrix Schrödinger operators

In this chapter we analyze the paper [**42**]. The question arising is to associate a strongly continuous semigroup for a realization of $\mathcal{A}$ in L^p-spaces, in the case where V is not symmetric. Actually, in the scalar case, the Schrödinger operator is always symmetric when the potential is of real–valued. However, this is not the case for complex–valued potentials. On the other hand, there exists a correspondence between scalar Schrödinger operators with complex potentials and some matrix Schrödinger operators with real nonsymmetric potentials. Such correspondence is well clarified in Section 2.8.

The literature on Schrödinger operators with complex potential is not as rich as for real potentials. However, there exist some references on this direction. We particularly mention [**37**], which is an earlier publication dealing with this topic, and [**32, 43**] and the references therein.

In this Chapter we use the same approach applied by T. Kato in [**37**] to construct an m-dissipative realization A of $\mathcal{A}$ in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. Thus one obtains a strongly continuous semigroup in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. Afterward, we call again tools from semigroup theory to extrapolate the semigroup to all L^p–spaces, $p \in (1, \infty)$. More precisely, we first prove that the space of test functions is a core for A. Moreover the Trotter–Kato product formula, see (2.4.1), for the semigroup in $L^2(\mathbb{R}^d, \mathbb{R}^m)$ holds. This formula permits to extrapolate the semigroup to the L^p–scale.

Once we obtain a consistent semigroup in all L^p-spaces, $1 < p < \infty$, we investigate on further properties. We start by identifying the domain of A_p, the realization of $\mathcal{A}$ in $L^p(\mathbb{R}^d, \mathbb{R}^m)$ with the maximal domain $D_{p,\max}(\mathcal{A})$. We then study the analyticity and positivity of the semigroup. We first show that the space of test functions is a core for A_p, $1 < p < \infty$, and then by using the local elliptic regularity of Δ_Q, we prove that the domain $D(A_p)$ of A_p is actually the maximal domain. For the analyticity, we suppose furthermore that the potential V has numerical range (hence spectrum) included in a sector of angle less than $\pi/2$ and that this hold uniformly with respect to x. Likewise, the spectrum of A_p will be included in a sector of angle less than $\pi/2$ and $-A_p$ is then sectorial of angle less than $\pi/2$. Finally, we characterize positivity by applying the *positive minimum principle* for generators of semigroups to get the necessary condition, which we check that is also sufficient by applying again the crucial (2.4.1).

This chapter is organised as follows: We start by a motivation and some examples in Section 2.1, and then in Section 2.2 we give some preliminary results. In Section 2.3, we state and prove the generation of semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. In Section 2.4, we extrapolate the semigroup to all L^p-spaces, $p \in (1, \infty)$. We then show in Section 2.5 that the domain of the L^p-realization A_p of $\mathcal{A}$ coincides with the maximal domain $D_{p,\max}(\mathcal{A})$. In Section 2.6 we extend the semigroup to $L^1(\mathbb{R}^d, \mathbb{R}^m)$ and in Section 2.7 we collect some properties of the semigroup, especially positivity and analyticity. Last, in Section 2.8 we apply our results to Schrödinger operators with complex potentials.

2.1. Motivation

Throughout this chapter we assume the following hypotheses

Hypotheses 2.1. (a) $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be *Lipschitz continuous* such that $q_{ij} = q_{ji}$, for all $i, j \in \{1, \dots, d\}$ and satisfy

$$\eta_1|\xi|^2 \le \langle Q(x)\xi, \xi\rangle \le \eta_2|\xi|^2 \tag{2.1.1}$$

(b) $V : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ be such that $v_{ij} \in L^\infty_{loc}(\mathbb{R}^d)$ and

$$\langle V(x)\xi, \xi\rangle \ge 0, \tag{2.1.2}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^m$.

We now present an example which shows that without an assumption on V as in (2.1.2) we cannot expect generation of a semigroup in general.

Example 2.2. We consider the situation where $d = 1$ and $m = 2$. Let $\mathscr{A}$ be the vector-valued operator defined on smooth functions $\zeta : \mathbb{R} \to \mathbb{R}^2$ by $\mathscr{A}\zeta = \zeta'' + V\zeta$, where

$$V(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

Obviously, the quadratic form $\xi \mapsto \langle V(x)\xi, \xi\rangle$ takes for $x \neq 0$ arbitrary values in $\mathbb{R}$ so that V does not satisfy the semiboundedness assumption (2.1.2). Fix $p \in (1, \infty)$. We are going to prove that no realization in $L^p(\mathbb{R}; \mathbb{R}^2)$ of the operator $\mathscr{A}$ generates a semigroup. For this purpose, it suffices to prove that, for every $\lambda > 0$ and properly chosen $f \in L^p(\mathbb{R}; \mathbb{R}^2)$, the resolvent equation $\lambda u - \mathscr{A}u = f$ does not admit any solution in the maximal domain $D_{p,\max}(\mathscr{A}) = \{u \in L^p(\mathbb{R}; \mathbb{R}^2) : \mathscr{A}u \in L^p(\mathbb{R}; \mathbb{R}^2)\}$. The resolvent equation can be rewritten as a system as follows:

$$\begin{cases} \lambda u_1(x) - u_1''(x) - xu_2(x) = f_1(x), & x \in \mathbb{R}, \\ \lambda u_2(x) - u_2''(x) = f_2(x), & x \in \mathbb{R}. \end{cases}$$

For simplicity, we will only consider functions f_1, f_2 which are supported in $[1,\infty)$. Solving the second equation in $L^p(\mathbb{R})$, we find that the unique solution u_2 is given by

$$u_2(x) = \frac{1}{2\sqrt{\lambda}} \int_x^\infty e^{\sqrt{\lambda}(x-t)} f_2(t)\,dt + \frac{1}{2\sqrt{\lambda}} \int_1^x e^{-\sqrt{\lambda}(x-t)} f_2(t)\,dt + ce^{-\sqrt{\lambda}x}$$

for $x \geq 1$ and $u_2(x) = 0$ for $x < 1$. The constant c is chosen such that $u_2(1) = 0$ so that u_2 is a continuous function.

From now on, we pick $f_2(t) = t^{-1}$ for $t \geq 1$ and $f_2(t) = 0$ for $t < 1$. One has $xu_2(x)$ converges to λ^{-1} as $x \to \infty$. In particular, there exists $x_0 > 1$ such that $xu_2(x) \geq (2\lambda)^{-1}$ for all $x \geq x_0$. Inserting this into the first equation and choosing $f_1 \equiv 0$, we obtain the differential inequality $u_1'' \leq \lambda u_1 - (2\lambda)^{-1}$.

Integrating this inequality, we obtain first

$$u_1'(x) \leq c_{1,\lambda} + \lambda \int_{x_0}^x u_1(t)\,dt - \frac{x}{2\lambda}, \qquad x \geq x_0,$$

and then

$$u_1(x) \leq c_{2,\lambda} + c_{1,\lambda}x + \lambda \int_{x_0}^x \int_{x_0}^t u_1(s)\,ds\,dt - \frac{x^2}{4\lambda}, \qquad x \geq x_0, \tag{2.1.3}$$

for certain constants $c_{1,\lambda}, c_{2,\lambda}$. Suppose now that our resolvent equation has a solution $(u_1, u_2) \in L^p(\mathbb{R}; \mathbb{R}^2)$. As $u_1 \in L^p(\mathbb{R})$, we can use Hölder's inequality to estimate

$$\left| \int_{x_0}^x \int_{x_0}^t u_1(s)\,ds\,dt \right| \leq \|u_1\|_p \int_{x_0}^x t^{1-\frac{1}{p}}\,dt = c_3 x^{2-\frac{1}{p}} + c_4$$

for all $x \geq x_0$. Inserting this into (2.1.3) and letting $x \to \infty$ we obtain that $u_1(x)$ diverges to $-\infty$ for $x \to \infty$, which contradicts the condition $u_1 \in L^p(\mathbb{R})$.

2.2. Preliminaries

To simplify notation, we write for $\xi, \eta \in \mathbb{R}^d$

$$\langle \xi, \eta \rangle_Q := \sum_{i,j=1}^d q_{ij}\xi_i\eta_j \quad \text{and} \quad |\xi|_Q := \sqrt{\langle \xi, \xi \rangle_Q}.$$

We define the operator $\Delta_Q : W^{1,1}_{loc}(\mathbb{R}^d) \to \mathscr{D}(\mathbb{R}^d)$ by setting

$$\langle \Delta_Q u, \varphi \rangle = -\int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle_Q\,dx. \tag{2.2.1}$$

for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$. As usual, we will say that $\Delta_Q u \in L^1_{loc}(\mathbb{R}^d)$, if there is a function $f \in L^1_{loc}(\mathbb{R}^d)$ such that

$$\langle \Delta_Q u, \varphi \rangle = \int_{\mathbb{R}^d} f\varphi\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. In this case we will identify $\Delta_Q u$ and the function f.

We first state a lemma which gives a generalisation of the Stampacchia theorem of weak derivation of the function asbolute value, see [**28**, Lemma 7.6], to vector–valued functions

Lemma 2.3. *Let $1 < p < \infty$ and $u = (u_1(x), \ldots, u_m(x)) \in W^{1,p}(\mathbb{R}^d, \mathbb{C}^m)$. Then, $|u| \in W^{1,p}(\mathbb{R}^d)$ and*

$$\nabla|u| = \frac{1}{|u|}\sum_{j=1}^{m} \mathrm{Re}\,(\bar{u}_j \nabla u_j)\chi_{\{u\neq 0\}}. \tag{2.2.2}$$

Moreover,

$$|\nabla|u||_Q^2 \leq \sum_{j=1}^{d} |\nabla u_j|_Q^2. \tag{2.2.3}$$

Proof. (i) Let $\varepsilon > 0$ and define $a_\varepsilon(u) = (\sum_{j=1}^{m} u_j^2 + \varepsilon^2)^{\frac{1}{2}} - \varepsilon$. Then, $a_\varepsilon(u) \in W^{1,p}(\mathbb{R}^d)$ and

$$\nabla a_\varepsilon(u) = \frac{\sum_{j=1}^{m} \mathrm{Re}\,(\bar{u}_i \nabla u_j)}{(\sum_{j=1}^{m} u_j^2 + \varepsilon^2)^{\frac{1}{2}}}.$$

We have the following pointwise convergence: $a_\varepsilon(u) \underset{\varepsilon\to 0}{\longrightarrow} |u|$ and

$$\nabla a_\varepsilon(u) \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{|u|}\sum_{j=1}^{m} \mathrm{Re}\,(\bar{u}_j \nabla u_j)\chi_{\{u\neq 0\}}.$$

Now, since $a_\varepsilon(u) = \dfrac{|u|^2}{(\sum_{j=1}^{m} u_j^2 + \varepsilon^2)^{\frac{1}{2}} + \varepsilon} \leq |u|$ and by Young inequality $|\nabla a_\varepsilon(u)| \leq |\nabla u|$, thus the dominated convergence theorem yields $|u| \in W^{1,p}(\mathbb{R}^d)$ and (2.2.2).
(ii) One knows that

$$\mathrm{Re}\,(\bar{u}_j \nabla u_j) = \frac{1}{2}\nabla|u_j|^2 \text{ and } |u|\nabla|u| = \frac{1}{2}\nabla|u|^2 = \frac{1}{2}\sum_{j=1}^{m} \nabla|u_j|^2.$$

Hence, by the use of Cauchy-Schwartz inequality,

$$\begin{aligned}|u||\nabla|u||_Q &= \frac{1}{2}\left|\sum_{j=1}^{m}\nabla|u_j|^2\right|_Q \\ &\leq \frac{1}{2}\sum_{j=1}^{m}|\nabla|u_j|^2|_Q \\ &\leq \sum_{j=1}^{m}|\mathrm{Re}\,(u_j\nabla u_j)|_Q \\ &\leq \sum_{j=1}^{m}|u_j\nabla u_j|_Q \\ &\leq |u|\left(\sum_{j=1}^{m}|\nabla u_j|_Q^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, $|\nabla|u||_Q^2 \leq \sum_{j=1}^{d}|\nabla u_j|_Q^2$. □

Proposition 2.4. *Let $u = (u_1, \ldots, u_m) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ be such that $\Delta_Q u_j \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for $j = 1, \ldots, d$. Then*

$$\Delta_Q|u| = \mathbb{1}_{\{u\neq 0\}}\frac{1}{|u|}\Big(\sum_{j=1}^{m}u_j\Delta_Q u_j + \sum_{j=1}^{m}|\nabla u_j|_Q^2 - |\nabla|u||_Q^2\Big). \tag{2.2.4}$$

Moreover, we have

$$\Delta_Q|u| \geq \mathbb{1}_{\{u\neq 0\}}\frac{1}{|u|}\sum_{j=1}^{m}u_j\Delta_Q u_j \tag{2.2.5}$$

in the sense of distributions.

Proof. As in the proof of Lemma 2.3 we set $a_\varepsilon(u) = \left(|u|^2 + \varepsilon^2\right)^{\frac{1}{2}} - \varepsilon$. It was seen there that $a_\varepsilon(u) \to |u|$ in $W^{1,p}(\mathbb{R}^d)$. From this it easily follows that $\Delta_Q a_\varepsilon(u) \to \Delta_Q|u|$ in $\mathscr{D}(\mathbb{R}^d)$.

Recalling from the proof of Lemma 2.3 that $\nabla a_\varepsilon(u) = \frac{1}{u_\varepsilon+\varepsilon}\sum_{j=1}^m u_j\nabla u_j$, we see that for a function $\varphi\in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}
\langle \Delta_Q a_\varepsilon(u),\varphi\rangle &= -\int_{\mathbb{R}^d}\langle Q\nabla a_\varepsilon(u),\nabla\varphi\rangle\,dx = -\sum_{j=1}^d\int_{\mathbb{R}^d}\frac{u_j}{a_\varepsilon(u)+\varepsilon}\langle Q\nabla u_j,\nabla\varphi\rangle\,dx\\
&= -\sum_{j=1}^d\int_{\mathbb{R}^d}\big\langle Q\nabla u_j,\nabla\big((a_\varepsilon(u)+\varepsilon)^{-1}u_j\varphi\big)\big\rangle\,dx\\
&\quad+\sum_{j=1}^d\int_{\mathbb{R}^d}\big\langle Q\nabla u_j,\nabla\big((a_\varepsilon(u)+\varepsilon)^{-1}u_j\big)\big\rangle\varphi\,dx\\
&= \sum_{j=1}^d\int_{\mathbb{R}^d}\frac{u_j}{a_\varepsilon(u)+\varepsilon}\Delta_Q u_j\varphi\,dx+\sum_{j=1}^d\int_{\mathbb{R}^d}\frac{1}{a_\varepsilon(u)+\varepsilon}\langle Q\nabla u_j\nabla u_j\rangle\varphi\,dx\\
&\quad-\sum_{j=1}^d\int_{\mathbb{R}^d}\frac{u_j}{(a_\varepsilon(u)+\varepsilon)^2}\langle Q\nabla u_j,\nabla a_\varepsilon(u)\rangle\varphi\,dx\\
&= \sum_{j=1}^d\int_{\mathbb{R}^d}\frac{u_j}{a_\varepsilon(u)+\varepsilon}\Delta_Q u_j\varphi\,dx+\sum_{j=1}^d\int_{\mathbb{R}^d}\frac{1}{a_\varepsilon(u)+\varepsilon}\langle Q\nabla u_j\nabla u_j\rangle\varphi\,dx\\
&\quad-\int_{\mathbb{R}^d}\frac{1}{a_\varepsilon(u)+\varepsilon}\langle Q\nabla a_\varepsilon(u),\nabla a_\varepsilon(u)\rangle\varphi\,dx.
\end{aligned}$$

We can now let $\varepsilon\to 0$. Recall, that $a_\varepsilon(u)\to|u|$ in $L^p_{loc}(\mathbb{R}^d)$ and $\nabla a_\varepsilon(u)\to\nabla|u|$ in $L^2_{loc}(\mathbb{R}^d,\mathbb{R}^m)$. Note, that $\frac{u_j}{a_\varepsilon(u)+\varepsilon}$ is uniformly bounded by 1, so that we can use dominated convergence for the first integral above. For the other two integrals, we use monotone convergence, using that Q is strictly elliptic and observing that $a_\varepsilon(u)^{-1}\uparrow|u|^{-1}$. Note that in all integrals it sufficed to integrate over the set $\{u\neq 0\}$. For the first and third integral, this is obvious, as there are functions u_j appearing, which vanish on $\{u=0\}$. However, by Stampacchia's lemma we have $\nabla u_j = 0$ on $\{u=0\}$ as well, taking care of the second integral. We obtain

$$\langle\Delta_Q|u|,\varphi\rangle = \int_{\{u\neq 0\}}\sum_{j=1}^m\Big(\frac{u_j}{|u|}\Delta_Q u_j+\frac{1}{u}|\nabla u_j|_Q^2\Big)\varphi\,dx-\int_{\{u\neq 0\}}\frac{1}{|u|}\big|\nabla|u|\big|_Q^2\varphi\,dx.$$

It thus follow 2.2.4. Taking into the account (2.2.3), one obtains (2.2.5). □

2.3. Generation of semigroup in $L^2(\mathbb{R}^d,\mathbb{R}^m)$

Define A to be the realization on $L^2(\mathbb{R}^d,\mathbb{R}^m)$ of $\mathcal{A}$ with domain

$$D(A) = \{u\in H^1(\mathbb{R}^d,\mathbb{R}^m) : \mathcal{A}u\in L^2(\mathbb{R}^d,\mathbb{R}^m)\}.$$

Our aim is to prove that A generates a strongly continuous semigroup in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. To do so, we use standard arguments of functional analysis and operator theory to establish that $-A$ is maximal accretive and then conclude by Lumer–Philips theorem.

We introduce the following two operators: $(L_0, D(L_0))$ and $(L, D(L))$ given by

$$L_0 u = \mathcal{A}u, \qquad \forall u \in D(L_0) := C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$$

and

$$Lu = \mathcal{A}u, \qquad \forall u \in D(L) := \{u \in H^1(\mathbb{R}^d, \mathbb{R}^m) : \mathcal{A}u \in H^{-1}(\mathbb{R}^d, \mathbb{R}^m)\}.$$

L_0 and L are realizations of $\mathcal{A}$ acting from $H^1(\mathbb{R}^d, \mathbb{R}^m)$ into $H^{-1}(\mathbb{R}^d, \mathbb{R}^m)$. We define $\bar{\mathcal{A}}u = \Delta_Q - V^*$ to be the formal adjoint of $\mathcal{A}$, where V^* is the conjugate matrix of V. Similarly, we define $\bar{A}$, $\bar{L}$ and $\bar{L}_0$ with potential V^* instead of V.

Once $-L$ is showed to be monotone then follows that $-A$ is an accretive operator. On the other hand, if $-L$ is maximal i.e., $(1-L)D(L) = H^{-1}(\mathbb{R}^d, \mathbb{R}^m)$ then so is $-A$. Indeed, let $f \in L^2(\mathbb{R}^d, \mathbb{R}^m) \subset H^{-1}(\mathbb{R}^d, \mathbb{R}^m)$ and $u \in D(L)$ such that $u - \mathcal{A}u = f$. Thus $\mathcal{A}u = u - f \in L^2(\mathbb{R}^d, \mathbb{R}^m)$, then $u \in D(A)$ and $(1-A)u = f$.

In the following proposition we collect some properties of L_0.

Proposition 2.5. *(1) $\bar{L}$ is the adjoint of L_0: $\bar{L} = L_0^*$ and $L = \bar{L}_0^*$. Consequently $\bar{L}$ and L are closed.*
(2) L_0 is closable and its closure is equal to L_0^{**}.

PROOF. (1) Let $f \in D(\bar{L})$ and $g \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Then

$$\begin{aligned}(\bar{L}f, g) &= \int_{\mathbb{R}^d} \langle f(x), \operatorname{div}(Q\nabla g)(x)\rangle dx - \int_{\mathbb{R}^d} \langle V^*(x)f(x), g(x)\rangle dx \\ &= \int_{\mathbb{R}^d} \langle f(x), \operatorname{div}(Q\nabla g)(x)\rangle dx - \int_{\mathbb{R}^d} \langle f(x), Vg(x)\rangle dx \\ &= (f, L_0 g)\end{aligned}$$

Thus $\bar{L} = L_0^*$ is closed. In a similar way one shows that $L = \bar{L}_0^*$ and thus L is also closed.
(2) One has $L_0^* = \bar{L}$ and $\bar{L}$ is densely defined, hence L_0^{**} exists and equal to the closure of L_0. □

Now, we give the main theorem of this section

Theorem 2.6. *$-L$ is maximal monotone.*

PROOF. *Step 1:* We first show that $-L_0^{**}$ is maximal monotone. It is easy to see that $-L_0$ is monotone. Indeed,

$$(-L_0\varphi, \varphi) = \int_{\mathbb{R}^d} |\nabla\varphi(x)|^2_{Q(x)} dx + \int_{\mathbb{R}^d} \langle V(x)\varphi(x), \varphi(x)\rangle dx \geq 0,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Then

$$\|(1 - L_0)u\|_{H^{-1}} \geq \|u\|_{H^1}, \qquad \forall u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m).$$

Thus $-L_0^{**}$ is also monotone as it is the closure of L_0. In particular, $rg(1 - L_0^{**})$ is a closed subset of H^{-1}. It thus suffices to show that $(1 - L_0^{**})C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ is dense in $H^{-1}(\mathbb{R}^d, \mathbb{R}^m)$ and deduce that $-L_0^{**}$ is maximal. For that purpose, let $u \in H^1(\mathbb{R}^d, \mathbb{R}^m)$ such that $((1 - L_0)\varphi, u) = 0$, for all $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Thus

$$u - \Delta_Q u + V^* u = 0. \tag{2.3.1}$$

The above equality is in the distribution sense. Therefore

$$\Delta_Q u_j = u_j + \sum_{l=1}^m v_{lj} u_l,$$

for every $j \in \{1, \dots, m\}$. Applying (2.2.4), one obtains

$$\begin{aligned}
\Delta_Q |u| &\geq \frac{1}{|u|} \sum_{j=1}^m u_j \Delta_Q u_j \chi_{\{u \neq 0\}} \\
&\geq \frac{\chi_{\{u \neq 0\}}}{|u|} (\sum_{j=1}^m u_j^2 + \sum_{j,l=1}^m v_{lj} u_l u_j) \\
&\geq \frac{\chi_{\{u \neq 0\}}}{|u|} |u|^2 = |u|
\end{aligned}$$

Thus $\Delta|u| \geq |u|$ in the sense of distributions. Now, let $(\phi_n)_n \subset C_c^\infty(\mathbb{R}^d)$ be such that $\phi_n \geq 0$ and $\phi_n \to |u|$ in $H^1(\mathbb{R}^d)$. One has

$$0 \leq (\Delta_Q |u|, \phi_n) - (|u|, \phi_n) = -\int_{\mathbb{R}^d} \langle \nabla |u|, \nabla \phi_n \rangle_Q dx + \int_{\mathbb{R}^d} |u| \phi_n \, dx$$

Letting n tends to ∞, one obtains $-\|\nabla |u|\|_2^2 - \|u\|_2^2 \geq 0$. Therefore $u = 0$.
Step 2: Now we prove that $L = L_0^{**}$. One knows that L is a closed extension of L_0. Hence $L_0^{**} \subset L$. In order to get the converse, it suffices to show that $\rho(L) \cap \rho(L_0^{**}) \neq \emptyset$. Since L_0^{**} is maximal monotone then, $1 \in \rho(L_0^{**})$. On the other hand, $(1 - L)D(L) \supset (1 - L_0^{**})D(L_0^{**}) = H^{-1}(\mathbb{R}^d, \mathbb{R}^m)$. Hence, $1 - L$ is surjective. For the injectivity, one has $\mathit{ker}(1 - L) = rg(1 - L^*)^\perp$, where $\mathit{ker}(1 - L)$ is the null space of $1 - L$ and $= rg(1 + L^*)^\perp$ the orthogonal space of $rg(1 + L^*)$. Since $L^* = \bar{L}_0^{**}$ is maximal and this follows from step 1 applied for V^* instead of V. Thus $\mathit{ker}(1 - L) = 0$ and $1 \in \rho(L)$, which ends the proof. □

The following corollary states the result of generation of semigroup

Corollary 2.7. *The operator A generates a contractive strongly continuous semigroup $\{T(t) : t \geq 0\}$ on $L^2(\mathbb{R}^d, \mathbb{R}^m)$.*

PROOF. As it is explained on the beginning of this section, since $-L$ is maximal then so is $-A$. $-A$ is also accretive. In fact, let $u \in D(A)$, then $\mathcal{A}u \in L^2$. Thus

$$\langle -Au, u\rangle = \int_{\mathbb{R}^d} \langle -\mathcal{A}u(x), u(x)\rangle dx = (-Lu, u) \geq 0.$$

The claim follow by application of Lumer-Philips theorem. ☐

2.4. Extension of the semigroup to $L^p(\mathbb{R}^d, \mathbb{R}^m)$

In this section we construct an extrapolation of the semigroup $\{T(t)\}$ into the spaces $L^p(\mathbb{R}^d, \mathbb{R}^m)$, $1 \leq p < \infty$. For this purpose, we first show that $\{T(t)\}$ is given by the Trotter-Kato product formula:

$$T(t)f = \lim_{n\to\infty} \left[e^{\frac{t}{n}\Delta_Q} e^{-\frac{t}{n}V}\right]^n f, \tag{2.4.1}$$

for all $t > 0$ and $f \in L^2(\mathbb{R}^d, \mathbb{R}^m)$, where $\{e^{t\Delta_Q}\}$ is the semigroup generated, in $L^2(\mathbb{R}^d, \mathbb{R}^m)$, by Δ_q. In order to apply (2.4.1), we assume that $v_{ij} \in L^\infty_{loc}(\mathbb{R}^d)$ and use the following statement

Proposition 2.8. *Assume that Hypotheses 2.1 hold. Then, $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ is a core for A.*

PROOF. Since $-A$ is maximal accretive, it suffices to show that $(1-A)C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. Let $u \in L^2(\mathbb{R}^d, \mathbb{R}^m)$ such that $\langle (1-A)\varphi, u\rangle = 0$, for all $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Thus $u - \Delta_Q u + V^* u = 0$ in the distribution sense. Hence

$$\Delta_Q u_j = u_j + \sum_{l=1}^m v_{lj} u_l.$$

In particular, $\Delta_Q u_j = \mathrm{div}(Q\nabla u_j) \in L^2_{loc}(\mathbb{R}^d)$, for for each $j \in \{1, \dots, m\}$, equivalently Δu_j also belongs to $L^2_{loc}(\mathbb{R}^d)$. Then, by local elliptic regularity, cf [**1**, Theorem 7.1], $u_j \in H^2_{loc}(\mathbb{R}^d)$. Therefore, $|u| = \lim_{\varepsilon\to 0}(|u|^2 + \varepsilon^2)^{\frac{1}{2}}$ belongs to $H^2_{loc}(\mathbb{R}^d)$. In particular, (2.2.4) still hold true i.e.,

$$\Delta_Q |u| \geq \frac{\chi_{\{u\neq 0\}}}{|u|} \sum_{j=1}^m u_j \Delta_Q u_j$$

pointwisely. Consequently,

$$\Delta_Q |u| \geq \frac{\chi_{\{u\neq 0\}}}{|u|}(|u|^2 + \langle Vu, u\rangle) \geq |u|.$$

Now, let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $\chi_{B(1)} \leq \zeta \leq \chi_{B(2)}$ and define $\zeta_n(x) = \zeta(x/n)$ for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. We multiply both two sides of the inequality $\Delta_Q|u| \geq |u|$

by $\zeta_n|u|$ and integrating by part, we obtain

$$\begin{aligned}
0 &\geq \int_{\mathbb{R}^d} |u(x)|^2\zeta_n(x)dx - \int_{\mathbb{R}^d} \Delta_Q|u|(x)\zeta_n(x)|u(x)|dx \\
&\geq \int_{\mathbb{R}^d} |u(x)|^2\zeta_n(x)dx + \int_{\mathbb{R}^d} \langle \nabla(\zeta_n|u|)(x), Q(x)\nabla|u|(x)\rangle dx \\
&\geq \int_{\mathbb{R}^d} |u(x)|^2\zeta_n(x)dx + \int_{\mathbb{R}^d} |\nabla|u|(x)|^2_{Q(x)}\zeta_n(x)dx + \int_{\mathbb{R}^d} \langle \nabla\zeta_n(x), \nabla|u|(x)\rangle_{Q(x)}|u|dx \\
&\geq \int_{\mathbb{R}^d} |u(x)|^2\zeta_n(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} \langle Q(x)\nabla\zeta_n(x), \nabla|u|^2(x)\rangle dx \\
&= \int_{\mathbb{R}^d} |u(x)|^2\zeta_n(x)dx - \frac{1}{2}\int_{\mathbb{R}^d} \Delta_Q\zeta_n(x)|u(x)|^2dx.
\end{aligned}$$

Straightforward computations yields

$$\Delta_Q\zeta_n := \operatorname{div}(Q\nabla\zeta_n) = \frac{1}{n}\sum_{i,j=1}^m \partial_i q_{ij}\partial_j\zeta(\cdot/n) + \frac{1}{n^2}\sum_{i,j=1}^m q_{ij}\partial_{ij}\zeta(\cdot/n).$$

In particular, one can deduce that $\|\Delta_Q\zeta_n\|_\infty \to 0$ as n tends to ∞. Hence, letting n tends to ∞ in the above last inequality, one obtains $\|u\|_2 \leq 0$, and thus $u = 0$. □

Since $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m) \subset D(\Delta_Q) \cap D(V)$, where $D(\Delta_Q) = H^2(\mathbb{R}^d, \mathbb{R}^m)$ and $D(V) = \{u \in L^p(\mathbb{R}^d, \mathbb{R}^m) : Vu \in L^2(\mathbb{R}^d, \mathbb{R}^m)\}$, therefore one can apply [**23**, Corollary 5.8] and conclude that

Proposition 2.9. *Assume that Hypotheses 2.1 hold. Then the semigroup* $\{T(t) : t \geq 0\}$ *is given by the Trotter–Kato product formula* (2.4.1).

Now, we are able to extend $\{T(t) : t \geq 0\}$ to $L^p(\mathbb{R}^d, \mathbb{R}^m)$

Theorem 2.10. *Let* $1 < p < \infty$ *and assume Hypotheses 2.1. Then* $\{T(t) : t \geq 0\}$ *can be extrapolated to a* C_0*–semigroup* $\{T_p(t) : t \geq 0\}$ *on* $L^p(\mathbb{R}^d, \mathbb{R}^m)$*. Moreover, if we denote by* $(A_p, D(A_p))$ *its generator. Then,* $A_p u = \mathcal{A}u$*, for all* $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$.

Proof. Let $1 < p < \infty$ and $f \in L^2(\mathbb{R}^d, \mathbb{R}^m) \cap L^p(\mathbb{R}^d, \mathbb{R}^m)$. The assumption (2.1.2) yields $|e^{-tV(x)}f(x)| \leq |f(x)|$ for all $x \in \mathbb{R}^d$ and $t \geq 0$. So, $\|e^{-tV}f\|_p \leq \|f\|_p$, for all $t \geq 0$.

On the other hand, it is well–known that $\{e^{t\Delta_Q}\}_{t\geq 0}$ is a contractive C_0-semigroup on $L^p(\mathbb{R}^d, \mathbb{R}^m)$. Consequently, for every $t > 0$, both $e^{t\Delta_Q}$ and e^{-tV} leave invariant the set

$$\mathcal{B}_p := \{f \in L^2(\mathbb{R}^d, \mathbb{R}^m) \cap L^p(\mathbb{R}^d, \mathbb{R}^m) : \|f\|_p \leq 1\}.$$

By Fatou's lemma we conclude that $\mathcal{B}_p$ is a closed subset of $L^2(\mathbb{R}^d,\mathbb{R}^m)$. So, applying (2.4.1), it follows that $T(t)\mathcal{B}_p\subset\mathcal{B}_p$, for every $t\geq 0$. Thus,

$$\|T(t)f\|_p=\|\lim_{n\to\infty}[e^{t/n\Delta_q}e^{-t/nV}]^n f\|\leq\|f\|_p.$$

Therefore, we can extend $\{T(t):t\geq 0\}$ to a semigroup of bounded linear operators in $L^p(\mathbb{R}^d,\mathbb{R}^m)$. Moreover, let $p_\varepsilon\in(1,\infty)$ given by $p_\varepsilon=p+\varepsilon$ if $p>2$ and $p_\varepsilon=p-\varepsilon$ if $p<2$, for suitable $\varepsilon>0$. Thus, by Hölder's inequality,

$$\|T(t)f-f\|_p\leq\|T(t)f-f\|_2^{1-\theta}\|T(t)f-f\|_{p_\varepsilon}^{\theta}\leq 2^\theta\|T(t)f-f\|_2^{1-\theta}\|f\|_{p_\varepsilon}^{\theta},$$

for some $\theta\in(0,1)$ and every $f\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$. Therefore $\lim_{t\to 0}T(t)f=f$ in $L^p(\mathbb{R}^d,\mathbb{R}^m)$ for all $C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$. Thus, by density, the strong continuity of the semigroup $\{T_p(t):t\geq 0\}$ follows. Now, fix $t>0$ and $f\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$. Thus $f\in D(A)$ and

$$T(t)f-f=\int_0^t AT(s)f\,ds=\int_0^t T(s)\mathcal{A}f\,ds, \tag{2.4.2}$$

where the integral is computed in $L^2(\mathbb{R}^d,\mathbb{R}^m)$. However, $\mathcal{A}f$ is of compact support, thus $\mathcal{A}f\in L^p(\mathbb{R}^d,\mathbb{R}^m)$ and $t\mapsto T_p(t)\mathcal{A}f$ is a continuous map from $[0,\infty)$ into $L^p(\mathbb{R}^d,\mathbb{R}^m)$. Hence, (2.4.2) holds true in $L^p(\mathbb{R}^d,\mathbb{R}^m)$, i.e.

$$T_p(t)f-f=\int_0^t T_p(s)\mathcal{A}f\,ds.$$

This implies that $t\mapsto T_p(t)f$ is differentiable in $[0,\infty)$. Hence, $f\in D(A_p)$ and $A_pf=\mathcal{A}f$, which ends the proof. □

2.5. Maximal domain of A_p

In this section we characterize the domain $D(A_p)$ in terms of its L^p-maximal domain. To this purpose we first show that the space of test functions is a core for A_p.

Proposition 2.11. *Let $1<p<\infty$. Assume Hypotheses 2.1. Then,*

(i) the set of test functions $C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$ is a core for A_p,
(ii) the semigroup $\{T_p(t):t\geq 0\}$ is given by the Trotter-Kato product formula

Proof. (i) Fix $1<p<\infty$. Since $-A_p$ is m-accretive and the coefficients of $\mathcal{A}$ are real, it suffices to show that $(1-A_p)C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$ is dense in $L^p(\mathbb{R}^d,\mathbb{R}^m)$. Let $u\in L^{p'}(\mathbb{R}^d,\mathbb{R}^m)$, where $p'=\frac{p}{p-1}$ is the conjugate of p, be such that $\langle(1-\mathcal{A})\varphi,u\rangle_{p,p'}=0$ for all $\varphi\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$. So,

$$u-\Delta_Q u+V^*u=0 \tag{2.5.1}$$

in the sense of distributions. In particular,

$$\Delta_Q u_j = u_j + \sum_{l=1}^{m} v_{lj} u_l \ \in L^{p'}_{loc}(\mathbb{R}^d)$$

for all $j \in \{1, \ldots, m\}$. By local elliptic regularity, $u_j \in W^{2,p'}_{loc}(\mathbb{R}^d)$ for all $j \in \{1, \ldots, m\}$. Then, (2.5.1) holds true pointwisely in $\mathbb{R}^d$.

Consider $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that $\chi_{B(1)} \le \zeta \le \chi_{B(2)}$ and define $\zeta_n(\cdot) = \zeta(\cdot/n)$ for $n \in \mathbb{N}$. For $p' < 2$ we multiply in (2.5.1) by $\zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} u \in L^p(\mathbb{R}^d, \mathbb{R}^m)$ for $\varepsilon > 0$, $n \in \mathbb{N}$, and integrating by part, we obtain

$$\begin{aligned}
0 \;=\;& \int_{\mathbb{R}^d} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|^2\,dx + \sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla u_j, \nabla\left(\zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} u_j\right)\rangle_Q\,dx \\
&+ \int_{\mathbb{R}^d} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} \langle V^* u, u\rangle\,dx \\
\ge\;& \int_{\mathbb{R}^d} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|^2\,dx + \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j|^2_{Q(x)} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}}\,dx \\
&+ \sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla u_j, \nabla\zeta_n\rangle_Q (|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} u_j\,dx \\
&+(p'-2)\sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla u_j, \nabla|u|\rangle_Q u_j |u| \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-4}{2}}\,dx \\
\ge\;& \int_{\mathbb{R}^d} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|^2\,dx + \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j|^2_{Q(x)} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}}\,dx \\
&+ \int_{\mathbb{R}^d} \langle \nabla|u|, \nabla\zeta_n\rangle_Q (|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|\,dx \\
&+(p'-2)\int_{\mathbb{R}^d} |\nabla|u||^2_Q \zeta_n |u|^2 (|u|^2+\varepsilon^2)^{\frac{p'-4}{2}}\,dx.
\end{aligned}$$

It follows now from (2.2.3) that

$$\begin{aligned}
0 \;\ge\;& \int_{\mathbb{R}^d} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|^2\,dx + \int_{\mathbb{R}^d} \langle \nabla|u|, \nabla\zeta_n\rangle_Q (|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|\,dx \\
&+(p'-1)\int_{\mathbb{R}^d} |\nabla|u||^2_Q \zeta_n |u|^2 (|u|^2+\varepsilon^2)^{\frac{p'-4}{2}}\,dx \\
\ge\;& \int_{\mathbb{R}^d} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|^2\,dx + \frac{1}{p'}\int_{\mathbb{R}^d} \langle \nabla((|u|^2+\varepsilon^2)^{\frac{p'}{2}})), \nabla\zeta_n\rangle_Q\,dx \\
=\;& \int_{\mathbb{R}^d} \zeta_n(|u|^2+\varepsilon^2)^{\frac{p'-2}{2}} |u|^2\,dx - \frac{1}{p'}\int_{\mathbb{R}^d} \Delta_Q\zeta_n(|u|^2+\varepsilon^2)^{\frac{p'}{2}})\,dx.
\end{aligned}$$

By letting ε goes to 0, one obtains

$$\int_{\mathbb{R}^d} \zeta_n |u|^{p'}\, dx - \frac{1}{p'} \int_{\mathbb{R}^d} \Delta_Q \zeta_n |u|^{p'}\, dx \leq 0.$$

Hence, as in the proof of Proposition 2.8, we conclude, by letting n tends to ∞, that

$$\int_{\mathbb{R}^d} |u|^{p'}\, dx \leq 0.$$

Therefore, $u = 0$.
In the case when $p' > 2$, one multiplies in (2.5.1) by $\zeta_n |u|^{p'-2} u$ and argues in a similar way.
(ii) This is an immediate consequence of (i) and [**23**, Corollary III-5.8]. □

In the next result we show that the domain $D(A_p)$ is equal to the L^p-maximal domain of $\mathcal{A}$.

Proposition 2.12. *Let $1 < p < \infty$. Assume Hypotheses 2.1. Then*

$$D(A_p) = \{u \in L^p(\mathbb{R}^d, \mathbb{R}^m) \cap W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{R}^m) : \mathcal{A}u \in L^p(\mathbb{R}^d, \mathbb{R}^m)\} := D_{p,\max}(\mathcal{A}).$$

Proof. Let us show first that $D(A_p) \subseteq D_{p,\max}(\mathcal{A})$. Take $u \in D(A_p)$. Since $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ is a core for A_p, it follows that there exists $(u_n)_n \subset C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ such that $u_n \to u$ and $\mathcal{A}u_n \to A_p u$ in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, and in particular in $L^p_{loc}(\mathbb{R}^d, \mathbb{R}^m)$. As $V \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^m)$, we deduce that $Vu_n \to Vu$ in $L^p_{loc}(\mathbb{R}^d, \mathbb{R}^m)$. Consequently,

$$\Delta_Q u = A_p u + Vu = \lim_{n\to\infty} \mathcal{A}u_n + Vu_n \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^m).$$

So, by local elliptic regularity, we obtain $u \in W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$. Hence, $\mathcal{A}u = A_p u$ belongs to $L^p(\mathbb{R}^d, \mathbb{R}^m)$, which shows that $u \in D_{p,\max}(\mathcal{A})$.

In order to prove the other inclusion it suffices to show that $\lambda - \mathcal{A}$ is injective on $D_{p,\max}(\mathcal{A})$, for some $\lambda > 0$. To this purpose, let $u \in D_{p,\max}(\mathcal{A})$ such that $(\lambda - \mathcal{A})u = 0$. Assume that $p \geq 2$. Multiplying by $\zeta_n |u|^{p-2} u$ and integrating (by part) over $\mathbb{R}^d$ one obtains

$$\begin{aligned}
0 &= \lambda\int_{\mathbb{R}^d}\zeta_n(x)|u(x)|^p dx + \int_{\mathbb{R}^d}\sum_{j=1}^m\langle Q\nabla u_j, \nabla(|u|^{p-2}u_j\zeta_n)\rangle dx \\
&\quad + \int_{\mathbb{R}^d}\langle V(x)u(x), u(x)\rangle |u(x)|^{p-2}\zeta_n(x)dx \\
&\geq \lambda\int_{\mathbb{R}^d}\zeta_n(x)|u(x)|^p dx + \int_{\mathbb{R}^d}|u(x)|^{p-2}\zeta_n(x)\sum_{j=1}^m\langle Q(x)\nabla u_j(x), \nabla u_j(x)\rangle dx \\
&\quad + \int_{\mathbb{R}^d}\sum_{j=1}^m |u(x)|^{p-2}u_j(x)\langle Q(x)\nabla u_j(x), \nabla\zeta_n(x)\rangle dx \\
&\quad +(p-2)\int_{\mathbb{R}^d}|u(x)|^{p-2}\zeta_n(x)\langle Q(x)\nabla|u|(x), \nabla|u|(x)\rangle dx \\
&\geq \lambda\int_{\mathbb{R}^d}\zeta_n(x)|u(x)|^p dx + \int_{\mathbb{R}^d}|u(x)|^{p-1}\langle Q(x)\nabla|u|(x), \nabla\zeta_n(x)\rangle dx \\
&\geq \lambda\int_{\mathbb{R}^d}\zeta_n(x)|u(x)|^p dx + \frac{1}{p}\int_{\mathbb{R}^d}\langle Q(x)\nabla\zeta_n(x), \nabla|u|^p(x)\rangle dx \\
&\geq \lambda\int_{\mathbb{R}^d}\zeta_n(x)|u(x)|^p dx - \frac{1}{p}\int_{\mathbb{R}^d}\Delta_Q\zeta_n(x)|u(x)|^p dx.
\end{aligned}$$

So, as in the proof of the above proposition, we conclude that $u = 0$.

The case $p < 2$ can be obtained similarly, by multiplying the equation $(\lambda - \mathcal{A})u = 0$ by $\zeta_n(|u|^2+\varepsilon)^{\frac{p-2}{2}}u$, $\varepsilon > 0$, instead of $\zeta_n|u|^{p-2}u$.

□

2.6. Semigroup in $L^1(\mathbb{R}^d, \mathbb{R}^m)$

We next address the case where $p = 1$. We can easily extend the semigroups $\{T_p(t)\}$ to a consistent contraction semigroup on $L^1(\mathbb{R}^d;\mathbb{R}^m)$. Note, however, that we no longer have knowledge of the domain of the generator.

Theorem 2.13. *There exists a realization A_1 of $\mathcal{A}$ in $L^1(\mathbb{R}^d;\mathbb{R}^m)$ that generates a strongly continuous semigroup of contractions $\{S_1(t) :\geq 0\}$, this semigroup is consistent with all $\{T_p(t) : t \geq 0\}$ for $p \in (1,\infty)$.*

Proof. Consider $f \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$. Then, $f \in L^p(\mathbb{R}^d;\mathbb{R}^m)$ for every $p \in (1,\infty)$ and by consistency of the semigroups, we have $T_2(t)f = T_p(t)f \in L^p(\mathbb{R}^d;\mathbb{R}^m)$. Since $\{T_p(t) : t \geq 0\}$ is a contraction, we obtain

$$\int_{\mathbb{R}^d}|T_2(t)f|^p\,dx \leq \|f\|_p^p \leq \|f\|_1\|f\|_\infty^{p-1}.$$

for every $t \geq 0$. Applying the well–known Fatou's lemma to let $p \to 1^+$, we infer that $T_2(t)f \in L^1(\mathbb{R}^d;\mathbb{C}^m)$ and $\|T_2(t)f\|_1 \leq \|f\|_1$. Thus, by density of $C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$ in $L^1(\mathbb{R}^d;\mathbb{R}^m)$, $T_2(t)$ can be uniquely extended to a linear bounded operator $T_1(t)$ on $L^1(\mathbb{R}^d;\mathbb{R}^m)$, for all $t \geq 0$. The semigroup law for $\{T_1(t) : t \geq 0\}$ follows by uniqueness of the extension.

Now, let us prove that the semigroup $\{T_1(t)\}$ is strongly continuous. To that end, since $C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$ is contained in $D(A_p)$, we can write

$$\begin{aligned}\|T_1(t)f - f\|_p =& \left\| \int_0^t T(s)\mathcal{A}f\, ds \right\|_p \leq \int_0^t \|T(s)\mathcal{A}f\|_p ds\\ \leq& \|\mathcal{A}f\|_p t\\ \leq& \|\mathcal{A}f\|^{\frac{1}{p}}_{L^1(\mathbb{R}^d;\mathbb{R}^m)} \|\mathcal{A}f\|_\infty^{1-\frac{1}{p}} t\end{aligned}$$

for all $t > 0$ and $f \in C_c^\infty(\mathbb{R};\mathbb{R}^m)$. Using the same arguments as above, we can easily show that

$$\|T_1(t)f - f\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} \leq \|\mathcal{A}f\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)} t, \qquad t > 0.$$

Hence, $T_1(t)f$ converges to f in $L^1(\mathbb{R}^d;\mathbb{R}^m)$ as $t \to 0^+$. Since each operator $T_1(t)$ is a contraction, we can extend the previous convergence to any $f \in L^1(\mathbb{R}^d;\mathbb{R}^m)$ by density arguments.

Note that once $\{T_1(t) : t \geq 0\}$ is a contractive consistent semigroup. We can conclude directly by applying [**67**, Proposition 4] that $\{T_1(t) : t \geq 0\}$ is strongly continuous.

Noting that a function $f \in L^1(\mathbb{R}^d;\mathbb{R}^m) \cap L^2(\mathbb{R}^d;\mathbb{R}^m)$ can be approximated, simultaneously in L^1 and in L^2, by a sequence of test functions, we see that $\{T_1(t) : t \geq 0\}$ and $\{T_2(t) : t \geq 0\}$ are consistent. Similarly, we see that $\{T_1(t) : t \geq 0\}$ and $\{T_p(t) : t \geq 0\}$ are consistent for every $p \in (1,\infty)$.

To complete the proof we shall show that $A_p = A$ on $C_c^\infty(\mathbb{R}^d,\mathbb{R}^m)$. For this purpose fix $t > 0$ and $f \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$. One has $f \in D(A)$ and

$$T(t)f - f = \int_0^t AT(s)f\, ds = \int_0^t T(s)\mathcal{A}f\, ds, \tag{2.6.1}$$

where the integral is computed in $L^2(\mathbb{R}^d,\mathbb{R}^m)$. However, $\mathcal{A}f$ is of compact support, thus $\mathcal{A}f \in L^p(\mathbb{R}^d,\mathbb{R}^m)$ and $t \mapsto T_p(t)\mathcal{A}f$ is a continuous map from $[0,\infty)$ into $L^p(\mathbb{R}^d,\mathbb{R}^m)$. Hence, (2.6.1) holds true in $L^p(\mathbb{R}^d,\mathbb{R}^m)$, that is

$$T_p(t)f - f = \int_0^t T_p(s)\mathcal{A}f\, ds.$$

This yields $t \mapsto T_p(t)f$ is differentiable in $[0,\infty)$, then $f \in D(A_p)$ and $A_p f = \mathcal{A}f$, which ends the proof. □

Remark 2.14. From Theorem 2.13 we do not obtain explicitly the domain $D(A_1)$ of A_1 and we cannot even say that the subspace of test function is a core for it. However, as the semigroups $\{T_p(t) : t \geq 0\}$ are consistent for $1 \leq p < \infty$, so are the resolvent $(\lambda - A_p)^{-1}$ for every $\lambda > 0$. Indeed, if $f \in L^p(\mathbb{R}^d; \mathbb{R}^m) \cap L^1(\mathbb{R}^d; \mathbb{R}^m)$ for some $p \in (1, \infty)$, then for $g \in L^\infty(\mathbb{R}^d; \mathbb{R}^m)$ with compact support we have

$$\begin{aligned}\langle (\lambda - A_p)^{-1} f, g\rangle_{p,p'} &= \int_0^\infty e^{-\lambda t} \langle T_p(t) f, g\rangle_{p,p'}\, dt \\ &= \int_0^\infty e^{-\lambda t} \langle T_1(t) f, g\rangle_{1,\infty}\, dt = \langle (\lambda - A_1)^{-1} f, g\rangle_{1,\infty}.\end{aligned}$$

As g was arbitrary, it follows that $(\lambda - A_p)^{-1} f = (\lambda - A_1)^{-1} f$.

2.7. Further properties of the semigroup

2.7.1. Positivity. We start by characterizing positivity of the semigroup.

Proposition 2.15. *Let $1 < p < \infty$. The semigroup $\{T_p(t)\}$ is positive if and only if the off–diagonal entries of V satisfy $v_{kl}(x) \leq 0$ for almost every $x \in \mathbb{R}^d$ whenever $k \neq l$.*

Proof. Assume that $\{T_p(t)\}$ is positive. Let us denote the canonical basis of $\mathbb{R}^m$ by $(e_k)_{1 \leq k \leq m}$. One has, for $0 \leq \varphi \in C_c^\infty(\mathbb{R}^d)$ the nonnegative function φe_k belongs to $D(A_p)$ as well as to the dual space of $L^p(\mathbb{R}^d; \mathbb{R}^m)$, for each $k \in \{1, \ldots, m\}$. By application of TheoremA.31, it follows that for $k \neq l$ we have

$$\begin{aligned}0 \leq \langle A_p \varphi e_l, \varphi e_k\rangle &= \int_{\mathbb{R}^d} \operatorname{div}(Q\nabla\varphi)\varphi \langle e_l, e_k\rangle\, dx - \int_{\mathbb{R}^d} \varphi^2 \langle V e_l, e_k\rangle\, dx \\ &= -\int_{\mathbb{R}^d} v_{kl} \varphi^2\, dx.\end{aligned}$$

As φ is arbitrary, this implies that $v_{kl} \leq 0$ as claimed.

To prove the converse, assume that $v_{kl}(x) \leq 0$ for $k \neq l$ and almost every $x \in \mathbb{R}^d$. This is precisely the positive minimum principle for the matrix $-V(x)$. For matrices positive minimum principle is not only necessary, but also sufficient to generate positive semigroup, see [**10**, Theorem 7.1]. Hence, we see that the multiplication semigroup e^{-tV_p} is positive. As the semigroup $\{e^{t\Delta_Q}\}$ is positive, see [**54**, Corollary 4.3], the positivity of $\{T_p(t) : t \geq 0\}$ follows by the Trotter–Kato product formula. ☐

2.7.2. Analyticity. The semigroup $\{T_p(t) : t \geq 0\}$ generated by A_p is not, in general, analytic. This depends upon the potential matrix V. Indeed, if V is symmetric, the semigroup $\{T_p(t) : t \geq 0\}$ is analytic in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, $1 < p < \infty$, see Chapter 1. However, if V is antisymmetric it may happen that the semigroup is not analytic as the following example shows

Example 2.16. Consider the operator $\mathscr{A}$ defined on smooth functions $\zeta : \mathbb{R} \to \mathbb{C}^2$ by $\mathscr{A}\zeta = \zeta'' - V\zeta$, where

$$V(x) = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

By Corollary 3.7, the realization A_p of $\mathscr{L}$, with domain $W^{2,p}(\mathbb{R};\mathbb{C}^2) \cap D(V_p)$ generates a strongly continuous semigroup on $L^p(\mathbb{R};\mathbb{C}^2)$ for $p \in (1,\infty)$.
We diagonalize the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and so we obtain that A_p is similar to the operator

$$\widetilde{A}_p := P^{-1}A_pP = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} - x\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. Hence the semigroup $\{T_p(t) : t \geq 0\}$ generated by A_p is analytic if and only if both the two semigroups generated by $\Delta \pm ix$ are analytic on $L^p(\mathbb{R})$.

To see that the semigroup generated by $B := \Delta - ix$ is not analytic on $L^p(\mathbb{R})$ we introduce the transformation

$$\mathcal{U}_\sigma f(x) = f(x-\sigma), \qquad x \in \mathbb{R},\ \ f \in L^p(\mathbb{R}),$$

for arbitrary fixed $\sigma \in \mathbb{R}$. So, we have

$$\mathcal{U}_{-\sigma}B\mathcal{U}_\sigma = B - i\sigma I.$$

Hence,

$$\mathcal{U}_{-\sigma}(\mu - i\sigma - B)^{-1}\mathcal{U}_\sigma = (\mu - B)^{-1}$$

and thus,

$$\|(\mu - i\sigma - B)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}))} = \|(\mu - B)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}))}$$

for arbitrary $\sigma \in \mathbb{R}$ and every $\mu > 0$. Therefore, by [**23**, Theorem II.4.6] the semigroup generated by B is not analytic.

If one looks to the spectrum of V on obtains the following: Whenever V is symmetric obviously, $\sigma(V)$ belongs to the positive real axis $[0,\infty)$ (we suppose that (2.1.2) is satisfied). However, if V is antisymmetric, as in the above example, the spectrum of V is the imaginary axis (in Example2.16, $\sigma(V(x)) = \pm ix$ and $x \in \mathbb{R}$); thus, in this case one cannot find a sector of angle less than $\pi/2$ that includes the spectrum of V.

Our idea to construct an analytic semigroup is then to impose on the spectrum of $V(x)$ to be included in a sector of angle less than $\pi/2$, uniformly in x

Proposition 2.17. *Assume that, there exists a constant $C > 0$ such that*

$$\mathrm{Re}\,\langle V(x)\xi, \xi\rangle \geq C\,|\mathrm{Im}\,\langle V(x)\xi, \xi\rangle|\,, \tag{2.7.1}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^m$. Thus, the semigroup $\{T_p(t) : t \geq 0\}$ generated by A_p is analytic, for every $1 < p < \infty$. That is it admits a holomorphic extension to $\Sigma_{\pi/2-\arctan(C^{-1})}$.

Proof. Let us show that

$$\mathrm{Re}\,\langle -A_p u, |u|^{p-2}u\rangle \geq C_p|\mathrm{Im}\,\langle -A_p u, |u|^{p-2}u\rangle|,$$

for all $u \in D(A_p)$ and suitable $C_p > 0$. It suffices to prove it for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$, since the set of test functions is a core for A_p.

Let $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$. According to [**54**, Theorem 3.9], one has

$$\mathrm{Re}\,\langle -\Delta_Q u, |u|^{p-2}u\rangle \geq \frac{|p-2|}{2\sqrt{p-1}}|\mathrm{Im}\,\langle \Delta_Q u, |u|^{p-2}u\rangle|,$$

Set $C_p = \min(C, \frac{|p-2|}{2\sqrt{p-1}})$. Thus

$$\begin{aligned}
\mathrm{Re}\,\langle -A_p u, |u|^{p-2}u\rangle &= \mathrm{Re}\,\langle -\Delta_Q u, |u|^{p-2}u\rangle + \mathrm{Re}\,\langle Vu, |u|^{p-2}u\rangle \\
&\geq \frac{|p-2|}{2\sqrt{p-1}}|\mathrm{Im}\,\langle -\Delta_Q u, |u|^{p-2}u\rangle| + \int_{\mathbb{R}^d} \mathrm{Re}\,\langle V(x)u(x), u(x)\rangle |u(x)|^{p-2}dx \\
&\geq \frac{|p-2|}{2\sqrt{p-1}}|\mathrm{Im}\,\langle -\Delta_Q u, |u|^{p-2}u\rangle| + C\int_{\mathbb{R}^d} |\mathrm{Im}\,\langle V(x)u(x), u(x)\rangle||u(x)|^{p-2}dx \\
&\geq C_p|\mathrm{Im}\,\langle A_p u, |u|^{p-2}u\rangle|.
\end{aligned}$$

□

2.8. Application to Schrödinger operators with complex potentials

Let us consider the matrix potential

$$V(x) := \begin{pmatrix} v(x) & -w(x) \\ w(x) & v(x) \end{pmatrix} = w(x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + v(x)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $w \in L^\infty_{loc}(\mathbb{R}^d)$ and $0 \leq v \in L^\infty_{loc}(\mathbb{R}^d)$. So, Hypotheses 2.1 are satisfied and therefore, we deduce, by Theorem 2.10 and Proposition 2.12, that L_p, the L^p-realization of the operator

$$L = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} - V \text{ with domain } \{u \in L^p(\mathbb{R}^d, \mathbb{C}^2) \cap W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}^2) : Lu \in L^p(\mathbb{R}^d, \mathbb{C}^2)\},$$

generates a C_0-semigroup on $L^p(\mathbb{R}^d, \mathbb{R}^2)$. Moreover $C_c^\infty(\mathbb{R}^d, \mathbb{R}^2)$ is a core for L_p. Now, we diagonalize the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and so we obtain that L_p is similar to the operator

$$P^{-1}L_pP = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} - \begin{pmatrix} v+iw & 0 \\ 0 & v-iw \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. Thus the following Schrödinger operators with complex potentials $\Delta - v \pm iw$ with domain

$$\{f \in L^p(\mathbb{R}^d, \mathbb{C}) \cap W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}) : \Delta f \pm iwf - vf \in L^p(\mathbb{R}^d, \mathbb{C})\}$$

generates C_0-semigroups on $L^p(\mathbb{R}^d, \mathbb{C})$ and $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ is a core. In general the semigroups are not analytic, see 2.16. However, if we assume that there is a positive constant C such

$$|w(x)| \leq Cv(x), \qquad \text{for a.e. } x \in \mathbb{R}^d,$$

then

$$\begin{aligned} |\mathrm{Im}\, \langle V(x)\xi, \xi\rangle| &= |v(x)(\xi_1\overline{\xi_2} - \overline{\xi_1}\xi_2)| \\ &\leq 2|v(x)||\xi_1\xi_2| \\ &\leq Cw(x)|\xi|^2 = C\mathrm{Re}\, \langle V(x)\xi, \xi\rangle. \end{aligned}$$

Applying Proposition 2.17 we deduce that $\Delta \pm iw - v$ with domain

$$\{f \in L^p(\mathbb{R}^d, \mathbb{C}) \cap W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}) : \Delta f \pm ivf - wf \in L^p(\mathbb{R}^d, \mathbb{C})\}$$

generates an analytic C_0-semigroups on $L^p(\mathbb{R}^d, \mathbb{C})$.

CHAPTER 3

Domain characterization of matrix Schrödinger operators

In the previous Chapter 2, we associated a C_0-semigroup in L^p-spaces, $1 \leq p < \infty$, to a realization A_p of $\mathcal{A}$. We then showed that, in reflexive L^p-spaces, $1 < p < \infty$, A_p is the closure of $\mathcal{A}$ defined on the space of test functions and thus the domain of A_p is the *maximal domain* $D_{p,\max}(\mathcal{A})$.

Since $\mathcal{A}$ is the sum of a diffusion term Δ_Q and potential term $-V$. We, now ask if we may coincide $D_{p,\max}(\mathcal{A})$ with the so-called *natural domain* of $\mathcal{A}$, which will be the intersection between $W^{2,p}(\mathbb{R}^d, \mathbb{R}^m)$ which represents the domain of the diffusion part Δ_Q and $D(V_p)$ the (maximal) domain of the operator multiplication by V.

Such a coincidence between the maximal and natural domains leads to a maximal regularity for solutions to the parabolic system $\partial_t u = \mathcal{A}u$, when the semigroup is analytic. Indeed, in such a situation the domain $D(A_p)$ will be continuously embedded into $W^{2,p}(\mathbb{R}^d, \mathbb{R}^m)$, which is, in turn, embedded in Hölder spaces for large enough p's. Moreover, it holds a *maximal inequality* of type (3.2.2), such an inequality is very useful when looking for conditions of compactness of the resolvent of A.

The aim of this chapter is to investigate on this coincidence between the maximal and natural domains of $\mathcal{A}$. Such a coincidence is not always satisfied for elliptic operators as shows in [**34**, Example 2.2], where there was an unbounded drift term, in addition to the diffusion and potential terms. However, for Schrödinger operators, in the scalar case, Shen shows, in [**60**], that the coincidence of domains holds for (scalar) potentials belonging to the so-called *reverse Hölder class* B_q, for $2q > d$. This class contains, in particular, all polynomial radial potentials $|x|^r$, $r > 0$. Otherwise, Okazawa in [**52**], proves the same for nonnegative potentials v satisfying $|\nabla v| \leq Cv$, for some $C > 0$, which is equivalent to the fact that $\log(v)$ is Lipschitz continuous. Thus, v may grow as $\exp(|x|)$ as well as $|x|^r$, $r \geq 1$.

To get similar results for matrix Schrödinger operator, we propose to follow, firstly, the strategy from [**34**] in using a noncommutative Dore–Venni theorem due to Monniaux and Prüss [**50**], thereby obtaining sectoriality and, in particular, closedness, of $\mathcal{A}$ endowed with the natural domain $W^{2,p} \cap D(V_p)$. Our approach allows Lipschitz potential terms, and for some particular potential, entries growing like $|x|^r$ for some $r \in [1, 2)$. Secondly, we use a perturbation theorem due to

Okazawa [**52, 53**] to include more potentials in the diagonal part of the matrix potential.

We also obtain a sufficient condition for the compactness of the resolvent of A and apply our results to scalar complex–valued potentials.

This chapter is organized as follows. In Section 3.1 we fix our assumptions, present some examples satisfying this assumptions and recall some preliminary results that will be used subsequently. Section 3.2 contains the actual main theorem obtained by Dore–Venni's theorem and in Section 3.3, we add a diagonal perturbation to the potential matrix considered in Section 3.2 and obtain the same results. We then, establish compactness of the resolvent of A_p in Section 3.4 and end the chapter by a section where we consider Schrödinger operators with scalar complex-valued potential and we show that they are similar to some special matrix Schrödinger operators. Finally, we apply the results of this chapter for an example of matrix Schrödinger operator with polynomial entries.

The material of this chapter is mainly based on the analysis of results of [**41**, Section 2 and 3] and [**47**, Section 3].

3.1. Hypotheses, remarks and preliminaries

Throughout, we make the following assumptions.

Hypotheses 3.1. Let $d, m \in \mathbb{N}$.

(a) Let $Q = (q_{i,j}) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a symmetric matrix-valued function with Lipschitz continuous entries such that (2.1.1) holds. For $p \in (1,\infty)$ we define the operator D_p on $L^p(\mathbb{R}^d;\mathbb{C}^m)$ by $D(D_p) = W^{2,p}(\mathbb{R}^d;\mathbb{C}^m)$ and

$$D_p u := \big[\operatorname{div}(Q\nabla u_k) - u_k\big]_{k=1,\dots,m} = \big[\Delta_Q u_k) - u_k\big]_{k=1,\dots,m}.$$

(b) Let $V : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ be a matrix–valued function with entries in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ such that $V(x)$ is injective and satisfy (2.1.2).

Moreover, assume that there exists a constant $\alpha \in [0, \frac{1}{2})$ such that, for every $j = 1,\dots,d$, either the matrix–valued function $x \mapsto D_jV(x)(V(x))^{-\alpha}$ or $x \mapsto (V(x))^{-\alpha}D_jV(x)$ is uniformly bounded in $\mathbb{R}^d$. That is

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d} |\partial_j V(x)(V(x))^{-\alpha}| < \infty \qquad \text{or} \\ &\sup_{x\in\mathbb{R}^d} |(V(x))^{-\alpha}\partial_j V(x)| < \infty, \end{aligned} \tag{3.1.1}$$

for all $j \in \{1,\dots,m\}$.

For $p \in (1,\infty)$ we define the operator V_p on $L^p(\mathbb{R}^d;\mathbb{C}^m)$ by setting $D(V_p) = \{f \in L^p(\mathbb{R}^d;\mathbb{C}^m) : Vf \in L^p(\mathbb{R}^d;\mathbb{C}^m)\}$ and $V_pf := Vf$. Here Vf is to be understood as matrix–vector product.

Remark 3.2. Without loss of generality we assume that both D_p and V_p are injective operators. Indeed, if for example V were not injective but still satisfies (2.1.2), one could rescale it and consider $\tilde{V} = V + I_m$, which becomes injective. The injectivity assumption is required to compute imaginary powers.

Once V satisfies Hypotheses 3.1, then do also its conjugate matrix V^*. In fact, it is easy to see that V satisfy (2.1.2) if and only if V^* do. Now, let us look to (3.1.1). We now that

$$\partial_j V^*(V^*)^{-\alpha} = (V^{-\alpha}\partial_j V)^*.$$

Hence, $\partial_j V^*(V^*)^{-\alpha}$ is uniformly bounded if and only if $V^{-\alpha}\partial_j V$ so is. We thus can say that actually (3.1.1) is a symmetric property with respect to the adjointness.

It is easy to see that potentials V of Lipschitz entries are allowed by Hypotheses 3.1. Actually, (3.1.1) with $\alpha = 0$ holds if and only if V is Lipschitz continuous. We are next going to illustrate that Hypotheses 3.1 allow for potentials V whose entries grow more than linearly at infinity.

Example 3.3. We again consider the situation where $d = 1$ and $m = 2$. Choosing $r \in [1, 2)$, we set

$$V(x) := \begin{pmatrix} 0 & 1+|x|^r \\ -(1+|x|^r) & 0 \end{pmatrix} = (1+|x|^r)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad x \in \mathbb{R}^d.$$

As $V(x)$ is antisymmetric, we find $\langle V(x)\xi, \xi\rangle = 0$ for all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$. Note that $V(x)$ is symmetric for all $x \in \mathbb{R}$. One has

$$(V(x))^{-\alpha} = (1+|x|^r)^{-\alpha}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-\alpha}, \qquad x \in \mathbb{R},$$

so that

$$D_x V(x) \cdot (V(x))^{-\alpha} = r|x|^{r-2}(1+|x|^r)^{-\alpha} x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-\alpha}$$

for all $x \in \mathbb{R}$. Now, since $r < 2$, thus $\frac{r-1}{r} < \frac{1}{2}$. Hence, one can pick $\alpha \in (\frac{r-1}{r}, \frac{1}{2})$ so that $r - 1 - \alpha r < 0$ and thus the function $x \mapsto D_x V(x) \cdot (-V(x))^{-\alpha}$ is indeed bounded.

Actually, we can consider any constant square matrix V_0 which is invertible and satisfy (2.1.2). Taking $V(x) := (1+|x|^r)V_0$, for some $r \in [1, 2)$, with this choice of V, hypotheses 3.1 are satisfied.

We now collect some properties of the operators D_p and V_p.

Proposition 3.4. *Let $1 < p < \infty$.*

(a) The operator $-D_p$ is invertible, sectorial and admits bounded imaginary powers. Its power angle is 0. Consequently, for every $\vartheta > 0$ there exists a constant c such that for $s \in \mathbb{R}$ and $\lambda \in S_{\pi-\vartheta}$ we have

$$\|(\lambda - D_p)^{-1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \leq \frac{c}{1+|\lambda|}, \qquad \|(-D_p)^{is}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \leq ce^{\vartheta|s|}.$$

(b) The operator V_p is invertible and admits bounded imaginary powers. Its power angle is at most $\frac{\pi}{2}$. Consequently, for every $\vartheta > \frac{\pi}{2}$ there exists a constant c such that for $s \in \mathbb{R}$ and $\lambda \in S_{\pi-\vartheta}$ we have

$$\|(\lambda + V_p)^{-1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \leq \frac{c}{1+|\lambda|}, \qquad \|(V_p)^{is}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \leq ce^{\vartheta|s|}.$$

Proof. (a) We conclude by the fact that for every $\varphi \in (0, \frac{\pi}{2})$ the operator $-A_p$ has a bounded H^∞-calculus on $S_{\pi-\varphi}$, see Proposition C.13 ([**21**, Theorem 6.1]) and the boundedness of imaginary follows immediately. For more details see Appendix C.

(b) The claim follow by Theorem (D.4) and Theorem D.5 of Appendix D. □

3.2. The generation result

In this section we are going to prove that the sum $-(D_p - V_p)$, defined on the domain $D(D_p) \cap D(V_p)$ is closed and quasi–sectorial. To that end, we make use of a non–commutative version of the Dore–Venni Theorem, see Theorem C.26, elaborated by Monniaux and Prüss in [**50**]. This theorem is valid in arbitrary $\mathcal{UMD}$ Banach spaces, see Definition C.20. The spaces $X = L^p(\mathbb{R}^d;\mathbb{C}^m)$, $1 < p < \infty$ are a $\mathcal{UMD}$ Banach space. For more information we refer the reader to [**14**].

Crucial to apply Theorem C.26 is the commutator estimate. To formulate it, we use the following notation. Given a (sufficiently differentiable) matrix–valued function $M : \mathbb{R}^d \to \mathbb{R}^{m\times m}$, we write $\nabla^k M$ for the matrix whose k-column is the gradient of the k-th row of M. Thus, if $M = (m_{ij})$, then

$$\nabla^k M = \begin{pmatrix} \partial_1 m_{k1} & \dots & \partial_1 m_{km} \\ \vdots & \ddots & \vdots \\ \partial_l m_{k1} & \dots & \partial_l m_{km} \end{pmatrix}.$$

Lemma 3.5. *Fix $p \in (1,\infty)$, let D_p be defined as in Hypotheses 3.1 and $M = (m_{ij}) : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ be a matrix valued function with entries in $W^{2,\infty}(\mathbb{R}^d)$; the induced multiplication operator on $L^p(\mathbb{R}^d;\mathbb{C}^m)$ is denoted by M_p. Then, for every $f \in W^{2,p}(\mathbb{R}^d,\mathbb{C}^m)$ the k-th entry of $(D_pM_p - M_pD_p)f$ is given by*

$$\operatorname{div}(Q(\nabla^k M)f) + \operatorname{tr}\left[Q(\nabla^k M)Df\right].$$

Proof. Making use of the definition of the operators and the product rule, we find that the k–th entry of $(D_pM_p - M_pD_p)f$ is

$$\begin{aligned}
&\sum_{i,j=1}^{d} \partial_i\big(q_{ij}\partial_j(Mf)_k\big) - \sum_{l=1}^{m} m_{kl}\,\mathrm{div}(Q\nabla f_l)\\
&= \sum_{i,j=1}^{d} \partial_i\Big(q_{ij}\partial_j \sum_{l=1}^{m} m_{kl}f_l\Big) - \sum_{l=1}^{m} m_{kl}\,\mathrm{div}(Q\nabla f_l)\\
&= \sum_{i,j=1}^{d} \partial_i\Big(q_{ij} \sum_{l=1}^{m} \big[(\partial_j m_{kl})f_l + m_{kl}(\partial_j f_l)\big]\Big) - \sum_{l=1}^{m} m_{kl}\,\mathrm{div}(Q\nabla f_l)\\
&= \mathrm{div}(Q(\nabla^k M)f) + \sum_{l=1}^{m}\sum_{i,j=1}^{d} (\partial_i m_{kl})q_{ij}(\partial_j f_l)\\
&= \mathrm{div}(Q(\nabla^k M)f) + \mathrm{tr}\,\big[Q(\nabla^k M)\nabla f\big]
\end{aligned}$$

for all $f \in W^{2,p}(\mathbb{R}^d;\mathbb{C}^m)$. □

Theorem 3.6. *Assume Hypotheses 3.1 and fix $p \in (1,\infty)$. Then, the operator $-(D_p - V_p)$, defined on the domain $D(D_p) \cap D(V_p)$, is closed, densely defined and sectorial.*

Proof. We will apply the noncommutative version of Dore–Venni theorem, see Theorem C.26. According to Proposition 3.4 we can pick $\theta_D, \theta_V \in (0,\pi)$ with $\theta_D + \theta_V < \pi$ such that

$$\|(\lambda - D_p)^{-1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \le \frac{c}{1+|\lambda|} \quad \text{and} \quad \|(-D_p)^{is}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \le ce^{\theta_D|s|}$$

for $\lambda \in S_{\pi-\theta_D}$ and

$$\|(\lambda + V_p)^{-1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \le \frac{c}{|\lambda|} \quad \text{and} \quad \|V_p^{is}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \le ce^{\theta_V|s|}$$

for $\lambda \in S_{\pi-\theta_V}$.

Fixing $f \in L^p(\mathbb{R}^d,\mathbb{C}^m)$, $\lambda \in S_{\pi-\theta_D}$ and $\mu \in S_{\pi-\theta_V}$. We now proceed to estimate

$$C(\lambda,\mu)f := (-D_p)(\lambda - D_p)^{-1}\big[(-D_p)^{-1}(\mu+V_p)^{-1} - (\mu+V_p)^{-1}(-D_p)^{-1}\big]f.$$

and then get the commutator condition of TheoremC.26. In order to apply Lemma 3.5 for the commutator $C(\lambda,\mu)$ we need to approximate the potential V with smoother potentials. Let $(\rho_n)_{n\in\mathbb{N}}$ be a mollifier sequence and let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \le \zeta(x) \le 1$ for all $x \in \mathbb{R}^d$ and $\zeta(x) = 1$ for $|x| \le 1$ whereas $\zeta(x) = 0$ for $|x| \ge 2$. For a locally integrable function φ, we set

$$(K_n\varphi)(x) := \zeta\Big(\frac{x}{n}\Big)\int_{\mathbb{R}^d} \rho_n(y)\varphi(x-y)\,dy, \qquad x \in \mathbb{R}^d,\ n \in \mathbb{N}.$$

Clearly, $K_n\varphi$ belongs to $C_c^\infty(\mathbb{R}^d)$. It is well-known that $K_n\varphi$ converges locally uniformly to φ as $n \to \infty$, for every continuous function φ. Moreover, if φ belongs to $W^{j,p}(\mathbb{R}^d;\mathbb{R}^m)$ for some $j \in \mathbb{N}$, then we also get convergence in $W^{j,p}(\mathbb{R}^d;\mathbb{R}^m)$.

We now set $V^{(n)} := (K_n v_{ij})$ for $n \in \mathbb{N}$. Note that $\langle V^{(n)}\xi, \xi\rangle \geq 0$ on $\mathbb{R}^d$ for every $\xi \in \mathbb{R}^m$. Consequently, the induced multiplication operator $V_p^{(n)}$ on $L^p(\mathbb{R}^d;\mathbb{C}^m)$ is m-accretive, whence for $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > 0$ we have $\mu \in \rho(V_p^{(n)})$ and $\|(\mu + V_p^{(n)})^{-1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \leq (\operatorname{Re}\mu)^{-1}$, see Proposition D.1. In particular, for fixed μ the resolvent operators $(\mu + V_p^{(n)})^{-1}$ are uniformly bounded. We claim that $(\mu + V_p^{(n)})^{-1}$ converges strongly to $(\mu + V_p)^{-1}$. Indeed, for $g \in C_c(\mathbb{R}^d;\mathbb{C}^m)$ we have

$$(\mu + V_p^{(n)})^{-1}g - (\mu + V_p)^{-1}g = (\mu + V_p^{(n)})^{-1}(V - V^{(n)})(\mu + V_p)^{-1}g.$$

Since $(\mu + V)^{-1}g$ has compact support and $V^{(n)}$ converges to V locally uniformly on $\mathbb{R}^d$, in particular $V^{(n)}$ converges uniformly in $\operatorname{supp}((\mu + V)^{-1}g)$ to V, it follows that $(V - V^{(n)})(\mu + V_p)^{-1}g \to 0$ uniformly and thus in $L^p(\mathbb{R}^d;\mathbb{C}^m)$. Using the uniform boundedness of the resolvents, the claim follows.

Thus, setting

$$\begin{aligned} &C_{n,m}(\lambda,\mu)f \\ :=&D_p(\lambda - D_p)^{-1}\big[(-D_p)^{-1}(\mu + V_p^{(n)})^{-1} - (\mu + V_p^{(n)})^{-1}(-D_p)^{-1}\big]D_pK_m((-D_p)^{-1}f)\end{aligned}$$

we see that, letting first n and then m tend to ∞, $C_{n,m}(\lambda,\mu)f$ converges to $C(\lambda,\mu)f$ in $L^p(\mathbb{R}^d;\mathbb{C}^m)$. Noting that $(\mu + V^{(n)})^{-1}$ has C^∞-entries which, together with its derivatives, are bounded, it follows that we can rewrite $C_{n,m}(\lambda,\mu)f$ as

$$C_{n,m}(\lambda,\mu)f = (\lambda - D_p)^{-1}\big[D_p(\mu + V_p^{(n)})^{-1} - (\mu + V_p^{(n)})^{-1}D_p\big]K_m((-D_p)^{-1}f).$$

We can now apply Lemma 3.5. Noting that

$$D_j(\mu + V^{(n)})^{-1} = -(\mu + V^{(n)})^{-1}(D_jV^{(n)})(\mu + V^{(n)})^{-1}$$

we find that

$$\begin{aligned} &C_{n,m}(\lambda,\mu)f \\ =&(\lambda - D_p)^{-1}\operatorname{div}\big(-Q(\mu + V^{(n)})^{-1}\nabla V^{(n)}(\mu + V^{(n)})^{-1}K_m((-D_p)^{-1}f)\big) \\ &+ (\lambda - D_p)^{-1}(\mu + V^{(n)})^{-1}\operatorname{tr}[-Q(\nabla V^{(n)})(\mu + V^{(n)})^{-1}\nabla K_m((-D_p)^{-1}f)] \\ =&(-D_p)^{\frac{1}{2}}(\lambda - D_p)^{-1}(-D_p)^{-\frac{1}{2}}\operatorname{div}\big(-Q(\mu + V^{(n)})^{-1}\nabla V^{(n)}(\mu + V^{(n)})^{-1}K_m((-D_p)^{-1}f)\big) \\ &+ (\lambda - D_p)^{-1}(\mu + V^{(n)})^{-1}\operatorname{tr}[-Q(\nabla V^{(n)})(\mu + V^{(n)})^{-1}\nabla K_m((-D_p)^{-1}f)]. \end{aligned} \tag{3.2.1}$$

Here,

$$\operatorname{div}\big(-Q(\mu + V^{(n)})^{-1}\nabla V^{(n)}(\mu + V^{(n)})^{-1}K_m((-D_p)^{-1}f)\big)$$

should be interpreted as the vector, whose k-th component is

$$\operatorname{div}\big(- Q(\mu + V^{(n)})^{-1}\nabla^k V^{(n)}(\mu + V^{(n)})^{-1}K_m((-D_p)^{-1}f)\big).$$

The interpretation of the trace term in (3.2.1) is similar.

To be able to pass to the limit as $n \to \infty$, we have to take care of the summand involving the divergence. To that end, pick $q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$. Recall that by the results of [**5**] (the square root problem) the operator $(-D_q)^{-\frac{1}{2}}$ is bounded from $L^q(\mathbb{R}^d;\mathbb{C}^m)$ to $W^{1,q}(\mathbb{R}^d;\mathbb{C}^m)$. Therefore, for $j = 1, \ldots, d$, the operator $\big(\partial_j(-D_q)^{-\frac{1}{2}}\big)^*$ defines a bounded operator on $L^p(\mathbb{R}^d;\mathbb{C}^m)$. Consequently, we can extend $(-D_p)^{-\frac{1}{2}}\operatorname{div}$ to a bounded operator S on $L^p(\mathbb{R}^d;\mathbb{C}^m)$. Since the function $K_m((-D_p)^{-1}f$ is compactly supported on $\mathbb{R}^d$ and $\partial_j V^{(n)}$ converges to $\partial_j V$ in $L^p_{\text{loc}}(\mathbb{R}^d;\mathbb{C}^m)$ and $\partial_j V^{(n)}$ is locally uniformly bounded for $j = 1, \ldots, d$, we can affirm that

$$E_n := Q(\mu + V^{(n)})^{-1}\nabla V^{(n)}(\mu + V^{(n)})^{-1}K_m((-D_p)^{-1}f)$$

converges to

$$E := Q(\mu + V)^{-1}\nabla V(\mu + V)^{-1}K_m((-D_p)^{-1}f),$$

in $L^p(\mathbb{R}^d;\mathbb{C}^m)$ as $n \to \infty$. Indeed, set $\varphi := K_m((-D_p)^{-1}f) \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$. For each $j \in \{1, \ldots, m\}$, one has

$$\begin{aligned}
Q^{-1}(E_n - E) &= (\mu + V^{(n)})^{-1}\partial_j V^{(n)}(\mu + V^{(n)})^{-1}\varphi - (\mu + V)^{-1}\partial_j V(\mu + V)^{-1}\varphi \\
&= (\mu + V^{(n)})^{-1}\{\partial_j V^{(n)}(\mu + V^{(n)})^{-1} - \partial_j V(\mu + V)^{-1}\}\varphi \\
&\quad + \big\{(\mu + V^{(n)})^{-1} - (\mu + V)^{-1}\big\}\,\partial_j V(\mu + V)^{-1}\varphi \\
&= (I) + (II).
\end{aligned}$$

Since $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ then so is $(\mu + V)^{-1}\varphi$, thus $\partial_j V(\mu + V)^{-1}\varphi \in L^p(\mathbb{R}^d, \mathbb{C}^m)$. Hence, the second term (II), of the right hand side of the above equality, tends to 0 as n tends to ∞, since the resolvent of $-V^{(n)}$ converges to the resolvent of $-V$ as seen before. Now, rewriting (I) as

$$(I) = (\mu+V^{(n)})^{-1}\{\partial_j V^{(n)}((\mu+V^{(n)})^{-1} - (\mu+V)^{-1}) + (\partial_j V^{(n)} - \partial_j V)(\mu+V)^{-1}\}\varphi,$$

one gets

$$\begin{aligned}
\|(I)\|_p \le{}& \frac{1}{|\mu|}\sup_{x\in\mathcal{K}}|\partial_j V(x)|\left\|((\mu + V^{(n)})^{-1} - (\mu + V)^{-1})\varphi\right\|_p \\
&+ \frac{1}{|\mu|}\left\|(\partial_j V^{(n)} - \partial_j V)(\mu + V)^{-1})\varphi\right\|_p,
\end{aligned}$$

where $\mathcal{K}$ is the support of φ. The first term of the left hand side of the above inequality vanishes at $n \to \infty$, since the resolvent of $-V^{(n)}$ converges to the resolvent of $-V$. For the second term, one has $\partial_j V \in L^p_{loc}(\mathbb{R}^d, \mathbb{C}^m)$, then $\partial_j V^{(n)}$ converges to $\partial_j V$ locally in $L^p(\mathbb{R}^d, \mathbb{C}^m)$. Thus, as $(\mu + V)^{-1}\varphi$ has a compact support, $(\partial_j V^{(n)} - \partial_j V)(\mu + V)^{-1})\varphi$ converges to 0 in $L^p(\mathbb{R}^d, \mathbb{C}^m)$.

Similarly, we can show that

$$(\mu + V^{(n)})^{-1} \operatorname{tr}[Q\nabla V^{(n)}(\mu + V^{(n)})^{-1}\nabla K_m((-D_p)^{-1}f)]$$

converges to

$$(\mu + V)^{-1} \operatorname{tr}[Q\nabla V(\mu + V)^{-1}\nabla K_m((-D_p)^{-1}f)].$$

Hence, letting $n \to \infty$ in $C_{n,m}(\lambda,\mu)$ and denoting by $C_m(\lambda,\mu)f$ the limit, we thus find that

$$\begin{aligned} C_m(\lambda,\mu)f =&(-D_p)^{\frac{1}{2}}(\lambda - D_p)^{-1}S\big(-Q(\mu+V)^{-1}\nabla V(\mu+V)^{-1}(K_m(-D_p)^{-1}f)\big) \\ &+ (\lambda - D_p)^{-1}(\mu+V)^{-1}\operatorname{tr}[-Q\nabla V(\mu+V)^{-1}\nabla K_m((-D_p)^{-1}f)] \\ =:& \, T_1 + T_2. \end{aligned}$$

We are now very close to provide the crucial commutator estimate of Theorem C.26. Let us start with the term T_1. According to PropositionC.13 $-D_p$ admits bounded H^∞-calculus, consequently $(-D_p)^{\frac{1}{2}}(\lambda - D_p)^{-1}$ defines a bounded operator on $L^p(\mathbb{R}^d;\mathbb{C}^m)$ and

$$\|(-D_p)^{\frac{1}{2}}(\lambda - D_p)^{-1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \le \frac{C}{|\lambda|^{\frac{1}{2}}}$$

for a suitable constant C. As noted above, S defines a bounded linear operator, as does multiplication with the bounded matrix–valued function Q. On the other hand, $\|(\mu+V)^{-1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \le C|\mu|^{-1}$. To estimate the rest of the term T_1, we write

$$\nabla V(\mu+V)^{-1}K_m((-D_p)^{-1}f) = (\nabla V)\cdot V^{-\alpha}V^{\alpha}(\mu+V)^{-1}K_m((-D_p)^{-1}f),$$

or

$$(\mu+V)^{-1}\nabla V = (\mu+V)^{-1}V^{\alpha}V^{-\alpha}\nabla V$$

depends on, in Hypotheses 3.1, whether $(\nabla V)\cdot V^{-\alpha}$ or $V^{-\alpha}\nabla V$ is bounded; the constant α is the one appearing in Hypotheses 3.1. In both cases, using the boundedness of the H^∞-calculus of $-V_p$, TheoremD.5, we get

$$\|V^{\alpha}(\mu+V)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^d;\mathbb{C}^m))} \le \frac{C}{|\mu|^{1-\alpha}}.$$

Taking into the account that $(-D_p)^{-1}$ is a bounded operator, it follow that, overall,

$$\|T_1\|_p \le \frac{M_1}{|\lambda|^{\frac{1}{2}}|\mu|^{2-\alpha}}\|f\|_p.$$

The term T_2 can be estimated similarly, so that, altogether, we have an estimate

$$\|C(\lambda,\mu)f\|_p \le \frac{M}{|\lambda|^{\frac{1}{2}}|\mu|^{2-\alpha}}\|f\|_p.$$

Finally, the assumptions of TheoremC.26 are satisfied which yields the claim. □

Corollary 3.7. *Let A_p the realization of $\mathcal{A}$ defined on Chapter2 and $\{T_p(t); t \geq 0\}$ its associated semigroup. One has*

$$D(A_p) = W^{1,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p) = D_{p,\max}(\mathcal{A}).$$

In particular, there exists a constant $K > 0$ such that

$$\|u\|_{2,p} + \|Vu\|_p \leq K(\|u\|_p + \|\Delta_Q u - Vu\|_p), \tag{3.2.2}$$

for all $u \in W^{1,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p)$. Here $\|\cdot\|_{2,p}$ is the norm of $W^{2,p}(\mathbb{R}^d, \mathbb{R}^m)$. The same results hold true for $\mathcal{A}^*$ the adjoint of $\mathcal{A}$.

Proof. In view of Theorem 3.6, the operator $\mathcal{A}$ with domain $W^{1,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p)$ is m-dissipative. Thus, generates a strongly continuous semigroup. Also $(A_p, D_{p,\max}(\mathcal{A}))$ generates a strongly continuous semigroup according to Chapter2. Moreover, both the two operators coincide on $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ which is a core for A_p. It thus follows that the two domains coincide. □

Remark 3.8. Let θ_A be the spectral angle of $-A_p$. If $\theta_A < \frac{\pi}{2}$, then the semigroup $\{T_p(t)\}_{t \geq 0}$ (in the sequel simply denoted by $\{T(t)\}$ to ease the notation) is analytic, as is well-known, see [**23**, II Theorem 4.6]. It is a consequence of [**50**, Corollary 2], that in the noncommutative version of the Dore–Venni theorem the spectral angle of the sum is at most the maximum of the power angle of the summands. Thus, if the power angle of $-V_p$ is strictly less than $\frac{\pi}{2}$, which is for example the case when $V(x)$ is symmetric with $\langle V(x)\xi, \xi \rangle \leq -|\xi|^2$ for all $x \in \mathbb{R}^d$, then $\{T_p(t) : t \geq 0\}$ is analytic.

3.3. Scalar multiplication perturbation of the system

3.3.1. Hypotheses. The condition (3.1.1) allows Lipschitz entries for V or at most, as Example 2.4 of [**41**] shows, potentials like

$$V(x) = \pm \begin{pmatrix} 0 & 1 + |x|^r \\ -(1 + |x|^r) & 0 \end{pmatrix}$$

with $r \in [1, 2)$. Now, we want to establish the same results as in [**41**] for potentials of type

$$\tilde{V}(x) = \begin{pmatrix} |x|^\delta & 1 + |x|^r \\ -(1 + |x|^r) & |x|^\delta \end{pmatrix},$$

for $\delta \geq 1$. To do so we split such potentials $\tilde{V}$ into $\tilde{V} = V + vI_m$, where V is a potential satisfying Hypotheses 3.1 and v is a scalar nonnegative potential (nonnegative real valued function) satisfying

$$|\nabla v(x)| \leq C\, v(x),$$

for some $C > 0$ and all $x \in \mathbb{R}^d$. Such condition is satisfied by radial polynomial potentials and potentials of the form $|x|^r \log(1 + |x|)$. The I_m is nothing but the matrix identity of $\mathbb{R}^m$. Defining the operator $\tilde{\mathcal{A}} = \Delta_Q - \tilde{V}$. $\tilde{\mathcal{A}}$ can be seen as $\tilde{\mathcal{A}} = \mathcal{A} - v$. We consider $\tilde{A}_p$ the realization on $L^p(\mathbb{R}^d, \mathbb{R}^m)$ of $\tilde{\mathcal{A}}$ with domain $D(\tilde{A}_p) = D(A_p) \cap D(v)$ and we use Okazawa's perturbation theorem, see TheoremA.20, to show that $\tilde{A}_p$ generates a contractive strongly continuous semigroup.

3.3.2. Generation of semigroup. Now, consider $0 < v \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ and define on $L^p(\mathbb{R}^d, \mathbb{C}^m)$ the multiplication operator with its maximal domain

$$B_p u = vu,\ u \in D(B_p) = \{u \in L^p(\mathbb{R}^d, \mathbb{C}^m) : vu \in L^p(\mathbb{R}^d, \mathbb{C}^m)\}. \tag{3.3.1}$$

It is easy to see that B_p is m-accretive on $L^p(\mathbb{R}^d, \mathbb{C}^m)$. Define, for $\varepsilon > 0$ and $u \in L^p(\mathbb{R}^d, \mathbb{C}^m)$, $v_\varepsilon := v(1 + \varepsilon v)^{-1}$ and $B_{p,\varepsilon} u := v_\varepsilon u$. $B_{p,\varepsilon}$ is actually the so-called *Hille-Yosida* approximation of B_p.

Let us proceed to establish (A.2.5) of Theorem A.20 for the operators $-A_p$ and B_p. For that purpose fix $1 < p < \infty$, $\varepsilon > 0$ and $u \in C^\infty_c(\mathbb{R}^d, \mathbb{C}^m)$. Define $R := v_\varepsilon^{p-1}$. We first approximate the left hand side of (A.2.5) as follows

$$\mathrm{Re}\, \langle -A_p u, |B_{p,\varepsilon} u|^{p-2} B_{p,\varepsilon} u\rangle_{p,p'} = \lim_{\delta \to 0} P_\delta, \tag{3.3.2}$$

where

$$P_\delta := \mathrm{Re}\, \langle -A_p u, R u_\delta^{p-2} u\rangle_{p,p'},$$

and

$$u_\delta = \begin{cases} (|u|^2 + \delta)^{\frac{1}{2}}, & \text{if} \quad 1 < p < 2, \\ |u|, & \text{if} \quad p \geq 2. \end{cases}$$

Taking into the account that $u_\delta^{p-2} u \longrightarrow |u|^{p-2} u$, as $\delta \to 0$, in $L^{p'}(\mathbb{R}^d, \mathbb{C}^m)$, the convergence in (3.3.2) follows easily.
We now start estimating P_δ from below. The following lemma yields a first estimate

Lemma 3.9. *Let $\delta > 0$. One has, for $p \geq 2$,*

$$\begin{aligned} P_\delta \ \geq\ & (p-1) \int_{\mathbb{R}^d} |\nabla |u(x)||^2_{Q(x)} u_\delta^{p-2}(x) R(x) dx \\ +\ & \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla |u|^2(x), \nabla R(x)\rangle_{Q(x)} u_\delta^{p-2}(x) dx, \end{aligned} \tag{3.3.3}$$

and for $1 < p < 2$

$$\begin{aligned} P_\delta \ & \geq (p-1) \textstyle\sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j(x)|^2_{Q(x)} u_\delta^{p-2}(x) R(x) dx \\ & \quad + \tfrac{1}{2} \textstyle\int_{\mathbb{R}^d} \langle \nabla |u|^2(x), \nabla R(x)\rangle_{Q(x)} u_\delta^{p-2}(x) dx. \end{aligned} \tag{3.3.4}$$

Proof. Applying integration by part formula and taking into account (2.1.2) one obtains

$$
\begin{aligned}
P_\delta &= -\mathrm{Re}\,\langle A_p u, R u_\delta^{p-2} u\rangle_{p,p'} \\
&= -\mathrm{Re}\int_{\mathbb{R}^d} \langle \Delta_Q u(x), R(x)u_\delta^{p-2}(x)u(x)\rangle dx + \int_{\mathbb{R}^d} \mathrm{Re}\,\langle V(x)u(x), u(x)\rangle R(x)u_\delta^{p-2}(x)dx \\
&\geq -\sum_{j=1}^m \int_{\mathbb{R}^d} \mathrm{Re}\,\left(div(Q\nabla u_j)(x)\bar{u}_j(x)R(x)u_\delta^{p-2}(x)\right) dx \\
&= \sum_{j=1}^m \int_{\mathbb{R}^d} \mathrm{Re}\,\langle \nabla u_j(x), \nabla(R u_\delta^{p-2}u_j)(x)\rangle_{Q(x)} dx \\
&= \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j(x)|^2_{Q(x)} R(x)u_\delta^{p-2}(x)dx + \sum_{j=1}^m \int_{\mathbb{R}^d} \mathrm{Re}\,\langle \nabla u_j(x), \nabla R(x)\rangle_{Q(x)}\bar{u}_j(x)u_\delta^{p-2}(x)dx \\
&+ \frac{p-2}{2}\sum_{j=1}^m \int_{\mathbb{R}^d} \mathrm{Re}\,\langle \nabla u_j(x), \nabla |u|^2(x)\rangle_{Q(x)}\bar{u}_j(x)R(x)u_\delta^{p-4}(x)dx \\
&= \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j(x)|^2_{Q(x)} R(x)u_\delta^{p-2}(x)dx + \frac{1}{2}\sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla |u_j|^2(x), \nabla R(x)\rangle_{Q(x)} u_\delta^{p-2}(x)dx \\
&+ \frac{p-2}{4}\sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla |u_j|^2(x), \nabla |u|^2(x)\rangle_{Q(x)} R(x)u_\delta^{p-4}(x)dx \\
&= \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j(x)|^2_{Q(x)} R(x)u_\delta^{p-2}(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla |u|^2(x), \nabla R(x)\rangle_{Q(x)} u_\delta^{p-2}(x)dx \\
&+ \frac{p-2}{4}\int_{\mathbb{R}^d} \langle \nabla |u|^2(x), \nabla |u|^2(x)\rangle_{Q(x)} R(x)u_\delta^{p-4}(x)dx \\
&= \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j(x)|^2_{Q(x)} R(x)u_\delta^{p-2}(x)dx + (p-2)\int_{\mathbb{R}^d} |\nabla |u|(x)|^2_{Q(x)}|u(x)|^2 R(x)u_\delta^{p-4}(x)dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla |u|^2(x), \nabla R(x)\rangle_{Q(x)} u_\delta^{p-2}(x)dx.
\end{aligned}
$$

Taking into account (2.2.3) and that $u_\delta = |u|$ when $p \geq 2$, and $u_\delta \leq |u|$ and $p-2 < 0$ when $1 < p < 2$, one obtains (3.3.3)) and (3.3.4)). □

We next give a second inequality verified by P_δ and an inequality involving the dual product between $-A_p$ and B_p

Lemma 3.10. *Let $\delta > 0$. One has*

$$
P_\delta \geq -\frac{1}{4(p-1)}\int_{\mathbb{R}^d} u_\delta^{p-2}(x)|u(x)|^2 \frac{|\nabla R(x)|^2_{Q(x)}}{R(x)} dx, \tag{3.3.5}
$$

and hence

$$\operatorname{Re}\langle -A_p u, |B_{p,\varepsilon}u|^{p-2}B_{p,\varepsilon}u\rangle_{p,p'} \geq -\frac{1}{4(p-1)}\int_{\mathbb{R}^d}|u(x)|^p\frac{|\nabla R(x)|^2_{Q(x)}}{R(x)}dx. \tag{3.3.6}$$

Proof. Let $c = \frac{1}{2(p-1)}$. One can use, for $p \geq 2$, the inequality

$$\int_{\mathbb{R}^d} u_\delta^{p-2}(x)\left|\sqrt{R(x)}\nabla|u|(x) + \frac{c}{\sqrt{R(x)}}|u(x)|\nabla R(x)\right|^2_{Q(x)} dx \geq 0$$

which implies that

$$0 \leq \int_{\mathbb{R}^d} u_\delta^{p-2}(x)R(x)|\nabla|u|(x)|^2_{Q(x)}dx + c\int_{\mathbb{R}^d} u_\delta^{p-2}(x)\langle\nabla|u|^2(x), \nabla R(x)\rangle_{Q(x)}$$
$$+ c^2\int_{\mathbb{R}^d} u_\delta^{p-2}(x)|u(x)|^2\frac{|\nabla R(x)|^2_{Q(x)}}{R(x)}dx.$$

Multiplying by $p-1$ and using (3.3.3) one obtains (3.3.5). On the other hand, for $1 < p < 2$, one can use the inequality

$$\int_{\mathbb{R}^d} u_\delta^{p-2}(x)\sum_{j=1}^m\left|\sqrt{R(x)}\nabla u_j(x) + \frac{cu_j(x)}{\sqrt{R(x)}}\nabla R(x)\right|^2_{Q(x)} dx \geq 0,$$

arguing similarly as above and using (3.3.4) one obtains (3.3.5). The inequality (3.3.6) follows now by letting $\delta \to 0$ in (3.3.5). □

Now, we prove the main theorem of this section,

Theorem 3.11. *Assume that there exist nonnegative constants a and b such that*

$$|\nabla v_\varepsilon(x)|^2_{Q(x)} \leq a(v_\varepsilon(x))^2 + b(v_\varepsilon(x))^3, \tag{3.3.7}$$

for all $\varepsilon > 0$ and $x \in \mathbb{R}^d$. Then $-A_p + sB_p$ with domain $D(A_p) \cap D(B_p)$ is an m-accretive operator in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ for each $s > \dfrac{(p-1)b}{4}$.

Proof. We will show the following inequality

$$\operatorname{Re}\langle -A_p u, \|B_{p,\varepsilon}u\|^{2-p}|B_{p,\varepsilon}u|^{p-2}B_{p,\varepsilon}u\rangle \geq -\frac{p-1}{4}a\|B_{p,\varepsilon}u\|\|u\| - \frac{p-1}{4}b\|B_{p,\varepsilon}u\|^2, \tag{3.3.8}$$

for every $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$. Since $C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$ is a core for A_p, we conclude by Theorem A.20. Let us prove 3.3.8. Applying (3.3.6) and taking in consideration

(3.3.7), we obtain

$$\begin{aligned}\mathrm{Re}\,\langle -A_p u, \|B_{p,\varepsilon}u\|^{2-p}|B_{p,\varepsilon}u|^{p-2}B_{p,\varepsilon}u\rangle &\geq -\frac{1}{4(p-1)}\|B_{p,\varepsilon}u\|^{2-p}\int_{\mathbb{R}^d}|u(x)|^p\frac{|\nabla R(x)|^2_{Q(x)}}{R(x)}dx\\ &\geq -\frac{p-1}{4}\|B_{p,\varepsilon}u\|^{2-p}\int_{\mathbb{R}^d}|u(x)|^p(v_\varepsilon(x))^{p-3}|\nabla v_\varepsilon(x)|^2_{Q(x)}dx\\ &\geq -\frac{p-1}{4}a\|B_{p,\varepsilon}u\|^{2-p}\int_{\mathbb{R}^d}|u(x)|^p|v_\varepsilon(x)|^{p-1}dx\\ &\quad -\frac{p-1}{4}b\|B_{p,\varepsilon}u\|^{2-p}\int_{\mathbb{R}^d}|u(x)|^p|v_\varepsilon(x)|^p dx.\end{aligned}$$

Taking into account that $|B_{p,\varepsilon}(x)u| = |v_\varepsilon(x)||u(x)|$ and using Hölder's inequality, one obtains (3.3.8). By Theorem A.20, one conclude that $-(A_p - sB_p)$ is m-accretive, for every $s > \dfrac{(p-1)b}{4}$.

□

Now we state the main result of this section

Corollary 3.12. *Assume that there exists $c > 0$ such that*

$$|\nabla v(x)| \leq c\, v(x), \tag{3.3.9}$$

for all $x \in \mathbb{R}^d$. Then, $-\tilde{A}_p$ is m-accretive in $L^p(\mathbb{R}^d, \mathbb{C}^m)$.

Proof. One has $\nabla v_\varepsilon = \nabla(v(1+\varepsilon v)^{-1}) = (1+\varepsilon v)^{-2}\nabla v$, which implies $|\nabla v_\varepsilon| \leq c v_\varepsilon$. Thus (3.3.7) is verified with $a = c^2$ and $b = 0$. Thus $A_p - sB_p$, for all $s > 0$, is m-accretive. In particular, $\tilde{A}_p$ generates a contractive C_0-semigroup. □

A consequence of Corollary 3.12 is that $-\tilde{A}_p$ endowed with domain $W^{2,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p) \cap D(B_p)$ is closed, even sectorial, where B_p is the multiplication by $-v$. According to the result of Chapter 2, $\tilde{\mathcal{A}} := \Delta_Q - \tilde{V}$ is m-accretive, in particular closed, when endowed with its maximal domain

$$D_{p,\max}(\tilde{A}) = \{u \in W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{R}^m) : \tilde{\mathcal{A}}u := \Delta_Q u - \tilde{V}u \in L^p(\mathbb{R}^d, \mathbb{R}^m)\}.$$

That means $(\tilde{\mathcal{A}}, D_{p,\max}(\tilde{A}))$ is a closed extension of $\tilde{A}_p$. It thus follow the following domain coincidence

Theorem 3.13. *Assume that* (3.3.9) *holds. Then, $D(\tilde{A}_p) = D_{p,\max}(\tilde{A})$. In particular, there exists $\tilde{L} > 0$ such that*

$$\|u\|_{2,p} + \|\tilde{V}u\|_p \leq \tilde{L}(\|u\|_p + \|\Delta_Q u - \tilde{V}u\|_p), \tag{3.3.10}$$

for every $u \in D(\tilde{A}_p)$.

3.4. Compactness and Spectrum of the generator

We first give a compactness result and then we analyze the spectrum of the operator $\tilde{A}_p$. A sufficient condition of compactness is given by the following result.

Theorem 3.14. *Assume in addition to Hypotheses 3.1 that there exists a function $\kappa : \mathbb{R}^d \to [0, \infty)$ with $\lim_{|x|\to\infty} \kappa(x) = \infty$ such that*

$$|\tilde{V}(x)\xi| \geq \kappa(x)|\xi|, \tag{3.4.1}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^m$. Then, for all $p \in (1, \infty)$, the operator $\tilde{A}_p$ has compact resolvent. Consequently, its spectrum is independent of $p \in (1, \infty)$, discrete and consists of eigenvalues only.

Proof. Fix $p \in (1, \infty)$. To show that $\tilde{A}_p$ has compact resolvent, it suffices to prove that $D(\tilde{A}_p)$, endowed with the graph norm of $\tilde{A}_p$, is compactly embedded into $L^p(\mathbb{R}^d; \mathbb{R}^m)$. The inequality 3.3.10 implies that the graph norm of $\tilde{A}_p$ is equivalent to the norm $u \mapsto |||u||| := \|u\|_{2.p} + \|\tilde{V}_p u\|_p$. Now, let us show that the closed unit ball of $D(\tilde{A}_p)$ is compact (or, equivalently, totally bounded) in $L^p(\mathbb{R}^d; \mathbb{R}^m)$. Let u belong to the unit ball of $D(A_p)$ so that in particular $\|\tilde{V}_p u\|_p \leq 1$. By our additional assumption (3.4.1), we have

$$\|\tilde{V}_p u\|_p^p \geq \int_{\mathbb{R}^d} \kappa(x)^p |u(x)|^p \, dx \tag{3.4.2}$$

for every $u \in D(\tilde{A}_p)$. Given $\varepsilon > 0$ we fix $R > 0$ sufficiently large so that $\kappa \geq \varepsilon^{-1}$ outside the ball $B_R := \{x \in \mathbb{R}^d : |x| < R\}$. Then, from Equation (3.4.2), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d \setminus B_R} |u(x)|^p \, dx \leq & \varepsilon^p \int_{\mathbb{R}^d \setminus B_R} \kappa(x)^p |u(x)|^p \, dx \\ \leq & \varepsilon^p \int_{\mathbb{R}^d} \kappa(x)^p |u(x)|^p \, dx \leq \varepsilon^p \|V_p u\|_p^p \leq \varepsilon^p.\end{aligned}$$

Since $\tilde{V} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^m)$, then the set of the restriction to B_R of functions in $D(\tilde{A}_p)$ coincides with $W^{2,p}(B_R; \mathbb{R}^m)$. According to the Sobolev embedding theorem (the Rellich-Kondarov theorem), we refer to [**?**, Theorem 6.2], [**28**, Section 7.10] or [**24**, Sections 5.6 and 5.7], $W^{2,p}(B_R; \mathbb{R}^m)$ is compactly embedded into $L^p(B_R; \mathbb{R}^m)$. Thus, overall, we can find finitely many functions $g_1, \ldots, g_k \in L^p(B_R; \mathbb{R}^m)$ such that, for every u in the unit ball of $D(\tilde{A}_p)$, there exists an index $j \in \{1, \ldots, k\}$ such that

$$\int_{B_R} |u(x) - g_j(x)|^p \, dx \leq \varepsilon^p.$$

Denoting the trivial extension of g_j to $\mathbb{R}^d$ by $\tilde{g}_j$, we have

$$\int_{\mathbb{R}^d} |u(x) - \tilde{g}_j(x)|^p \, dx = \int_{B_R} |f(x) - g_j(x)|^p \, dx + \int_{\mathbb{R}^d \setminus B_R} |u(x)|^p \, dx \leq 2\varepsilon^p.$$

This shows that the unit ball of $D(\tilde{A}_p)$ is covered by the balls in $L^p(\mathbb{R}^d; \mathbb{R}^m)$ centered at $\tilde{g}_j$ of radius $2^{\frac{1}{p}}\varepsilon$. As $\varepsilon > 0$ was arbitrary, it follows that the unit ball of $D(\tilde{A}_p)$ is totally bounded in $L^p(\mathbb{R}^d; \mathbb{R}^m)$.

The fact that the spectrum consists only of eigenvalues follows from spectral properties of compact operators with help of the spectral mapping theorem for the resolvent, cf. Theorem A.12 together with [**23**, Theorem IV.1.13].

Since the resolvent operators $(\lambda - \tilde{A}_p)^{-1}$ are consistent (see Remark 2.14) and compact, the p-independence of the spectrum follows from [**17**, Corollary 1.6.2]. □

Remark 3.15. (1) As $\tilde{V} = V + vI_m$, the assumption (3.4.1) can be obtained in terms of V and v. In fact, suppose that one of the following is satisfied
- There exists $\rho : \mathbb{R}^d \to \mathbb{R}^+$ measurable such that $\lim_{|x|\to\infty} \rho(x) = \infty$ and
$$|V(x)\xi| \geq \rho(x)|\xi|, \qquad \forall x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$
- $\lim_{|x|\to\infty} v(x) = \infty$.

If we set $\kappa(x) := \min(\rho(x), v(x))$. Then, $|\tilde{V}(x)\xi| \geq \kappa(x)|\xi|$, for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^m$. Indeed,

$$|\tilde{V}(x)\xi|^2 = |V(x)\xi|^2 - 2v(x)\langle V(x)\xi, \xi\rangle + v(x)^2|\xi|^2 \geq |V(x)\xi|^2 + v(x)^2|\xi|^2.$$

(2) Using the Cauchy–Schwartz inequality, we see that the compactness assumption (3.4.1) is, in particular, satisfied if we have

$$\langle \tilde{V}(x)\xi, \xi\rangle \geq \kappa(x)|\xi|^2$$

for all $\xi \in \mathbb{R}^m$, which is exactly the one used in Chapter1, for symmetric potentials. Actually, if $\tilde{V}(x)$ is symmetric for every $x \in \mathbb{R}^d$, then the two conditions are equivalent. Indeed, the assumption in Theorem 3.14 implies that every eigenvalue $\lambda(x)$ of $\tilde{V}(x)$ satisfy $\lambda(x) \geq \kappa(x)$. This, in turn, is equivalent to the condition $\langle \tilde{V}(x)\xi, \xi\rangle \geq \kappa(x)|\xi|^2$, for all $\xi \in \mathbb{R}^m$.

However, the assumption in Theorem 3.14 is more general than this, since it is satisfied for some antisymmetric potentials as in Example 3.3, i.e.,

$$\tilde{V}(x) = V(x) := \begin{pmatrix} 0 & |x|^r \\ -|x|^r & 0 \end{pmatrix}, \qquad x \in \mathbb{R}^d,$$

where $r \in [1, 2)$. Indeed, in this case we have $|\tilde{V}(x)\xi| = (1 + |x|^r)|\xi|$, so that we can choose $\kappa(x) = 1 + |x|^r$ for all $x \in \mathbb{R}^d$. On the other hand, we have $\langle \tilde{V}(x)\xi, \xi\rangle = 0$ for all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$.

3.5. Application to complex potential

Let us consider the following *scalar* Schrödinger operator, with complex potential,

$$\Delta_Q - (v + iw),$$

where $v, w : \mathbb{R}^d \to \mathbb{R}$ be measurable real–valued functions. The corresponding matrix Schrödinger operator to the above complex operator is the following

$$\Delta_Q - \begin{pmatrix} v & -w \\ w & v \end{pmatrix}.$$

Let us denote

$$\tilde{V} := \begin{pmatrix} v & -w \\ w & v \end{pmatrix} = vI_2 + w \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} := vI_2 + V.$$

We thus conclude, by results of Chapter2, that $\Delta_Q - (v+iw)$ generates a strongly continuous semigroup of contraction in $L^p(\mathbb{R}^d, \mathbb{C})$, under the conditions v, w are locally bounded and $v \geq 0$. Furthermore, if

$$\sup_{x\in\mathbb{R}^d} \frac{|\nabla v(x)|}{v(x)} < \infty \qquad \text{and}$$

$$\sup_{x\in\mathbb{R}^d} \frac{|\nabla w(x)|}{|w(x)|^\alpha} < \infty, \tag{3.5.1}$$

for some $\alpha \in [0, 1/2)$, thus we conclude by the result of this chapter that the domain in $L^p(\mathbb{R}^d, \mathbb{C})$, $1 < p < \infty$, of $\Delta_Q - (v + iw)$ is given by

$$\{f \in W^{2,p}(\mathbb{R}^d, \mathbb{C}) : (v + iw)f \in L^p(\mathbb{R}^d, \mathbb{C})\},$$

and it holds the inequality

$$\|f\|_{2,p} + \|(v + iw)f\|_p \leq M(\|f\|_p + \|(\Delta_Q - (v + iw))f\|_p),$$

for some constant $M \geq 1$.

In addition to (3.5.1), if we assume that $\inf(v, |w|)$ blows up when $|x| \to \infty$. Thus, by Remark 3.15, $\Delta_Q - (v + iw)$ has a compact resolvent and then discrete spectrum consisting of eigenvalues only.

The equivalent to the analyticity condition (2.7.1) for complex potentials is the following: There exists $C > 0$ such that

$$v(x) \geq C\,|w(x)|, \qquad \forall x \in \mathbb{R}^d.$$

We end this chapter by an example where we take $\Delta_Q = \Delta$ be the Laplacian operator and $\tilde{V}$ with polynomial entries.

Examples 3.16. The condition (2.7.1) is satisfied for symmetric potential matrices but never for antisymmetric ones. Moreover, it has been proved in Example 2.16 that the semigroup generated by A_p with the antisymmetric potential

$V(x) = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$ is not analytic. However, we find analyticity when perturbing V by $(1+|x|^r)I_2$, for some $r \geq 1$. Indeed, consider $\tilde{V} : \mathbb{R} \to \mathbb{R}^{2\times 2}$ given by

$$\tilde{V}(x) = \begin{pmatrix} (1+|x|^r) & -x \\ x & (1+|x|^r) \end{pmatrix} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} - (1+|x|^r)I_2,$$

where $r \geq 1$. Let us show that $\tilde{V}$ verify (2.7.1). Let $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{C}^2$. One has

$$\langle \tilde{V}(x)\xi, \xi\rangle_{\mathbb{C}^2} = (1+|x|^r)(\xi_1^2 + \xi_2^2) + x(\xi_1\bar{\xi}_2 - \bar{\xi}_1\xi_2).$$

Then

$$\mathrm{Re}\,\langle \tilde{V}(x)\xi, \xi\rangle = (1+|x|^r)(\xi_1^2 + \xi_2^2)$$

and

$$\mathrm{Im}\,\langle \tilde{V}(x)\xi, \xi\rangle = x(\xi_1\bar{\xi}_2 - \bar{\xi}_1\xi_2).$$

Moreover, one has

$$\left|\mathrm{Im}\,\langle \tilde{V}(x)\xi, \xi\rangle\right| \leq 2|x||\xi_1\xi_2| \leq (1+|x|^r)(\xi_1^2 + \xi_2^2) = \mathrm{Re}\,\langle \tilde{V}(x)\xi, \xi\rangle.$$

Hence (2.7.1) holds for $\tilde{V}$.

Furthermore, conditions of Remark 3.15 are satisfied and thus we have compact resolvent.

CHAPTER 4

Ultracontractivity, kernel estimates and spectrum

After associating a semigroup to the matrix Schrödinger operator in Chapter 2 and given explicitly its domain in Chapter 3. Now, in this chapter, we want to establish, under the hypotheses of Chapter 3 (which include the ones of Chapter 2), at first the ultracontractivity property for the matrix Schrödinger semigroup. The ultracontractivity is a strong regularity property for semigroups. For analytic semigroups with L^p-domain, $1 < p < \infty$, contained in the Sobolev space $W^{2,p}$, this leads to a maximal regularity as Theorem 4.3 shows.

Another crucial consequence of ultracontractivity is that the semigroup will be given by an integral kernel. For matrix Schrödinger operators we are talking about matrix kernel:

$$T(t)f(x) = \int_{\mathbb{R}^d} K(t,x,y)f(y)dy, \qquad t > 0,\ x \in \mathbb{R}^d,$$

where the product $K(t,x,y)f(y)$ should be understood as matrix–vector product; $K(t,\cdot,\cdot)$ is the matrix kernel and $f \in L^p(\mathbb{R}^d,\mathbb{R}^m)$, for some $p \in [1,\infty)$.

The matrix kernel is bounded in space variables by the Dunford–Pettis theorem; and is firstly estimated by $Ct^{-\frac{d}{4}}$. Considering the twisted semigroup $T_{\lambda,\varphi}(t) = e^{-\lambda t}T(t)(e^{\lambda t}\cdot)$, we get a Gaussian upper estimates for all entries of the kernel $K(t,\cdot,\cdot)$:

$$|k_{ij}(t,x,y)| \leq Ct^{-\frac{d}{2}} \exp\{-\tau\frac{|x-y|^2}{4t}\}, \qquad \forall i,j \in \{1,\ldots,m\}.$$

The above estimate becomes, for $y = x$, $k_{ij}(t,x,x) \leq Ct^{-\frac{d}{2}}$. This is not an optimal bound for semigroups of Hilbert-Schmidt, for which the trace in $L^2(\mathbb{R}^d,\mathbb{R}^m)$ is finite, which corresponds with integrability of $x \mapsto k_{ii}(t,x,x)$ over $\mathbb{R}^d$. Actually, for every $t > 0$, the trace of $T(t)$ is given by

$$\mathrm{tr}(T(t)) = \int_{\mathbb{R}^d} \sum_{j=1}^{m} k_{ii}(t,x,x)dx.$$

It is then necessary to look for another way to estimate $k_{ii}(t,x,x)$. We first show that, for each $i \in \{1,\ldots,m\}$, $k_{ii}(t,\cdot,\cdot)$ is nothing but the heat kernel associated to scalar Schrödinger operator with potential the i-th diagonal component of V. Hence, we obtain upper and lower estimates for k_{ii}, $i \in \{1,\ldots,m\}$, from the literature of kernel estimates in the scalar case. We remind that for scalar

Schrödinger semigroup, kernel estimates are widely studied, we refer to [**45, 49, 48, 56, 61**]. Estimates of the off-diagonal kernels k_{ij}, $i \neq j$ are also obtained in different ways.

After establishing suitable estimates of the trace of the semigroup which will be the sum of its eigenvalues, we apply a Tauberian theorem due to Karamata to obtain asymptotic behaviour of eigenvalues of the matrix Schrodinger operator.

We consider the realization A_p in $L^p(\mathbb{R}^d, \mathbb{R}^m)$ of $\mathcal{A}$ defined in Chapter 2. We still denote by $\{T_p(t) : t \geq 0\}$ its associated semigroup. We assume that the diffusion matrix is of Lipschitz entries and the potential matrix denoted by V to be split into the sum of a 'principal part' V^{ess} that satisfies 3.1 and a diagonal part vI_m with $v \geq 0$ and satisfies (3.3.9). Likewise, as showed in Chapter 3, $D(A_p) = W^{2,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p)$ and thus the maximal inequality of type (3.2.2) is satisfied for every $1 < p < \infty$.
Actually any locally bounded potential matrix V satisfying (2.1.2) and (3.2.2) is allowed.

We divide this chapter into five sections, in the first one we established ultracontractivity and existence of the kernel, then in Section 4.2, we give a result of regularity for the Schrödinger semigroup. In Section 4.3, we prove Gaussian upper estimates for the kernel entries and then in Section 4.4, we investigate further upper kernel estimates. Finally, Section 4.5 is devoted to the asymptotic distribution of the eigenvalues.

4.1. Ultracontractivity

In this subsection we will establish ultracontractivity property of the semigroup $\{T_p(t) : t \geq 0\}$. As a consequence, the semigroup is given by an integral matrix kernel. Since for $1 \leq p < \infty$ the semigroups $\{T_p(t) : t \geq 0\}$ are consistent, we drop the index p and merely write $\{T(t) : t \geq 0\}$ for our semigroup. In what follows, we denote by $\{T_0(t) : t \geq 0\}$ the scalar semigroup on $L^p(\mathbb{R}^d)$ generated by the scalar operator $\Delta_Q = \mathrm{div}(Q\nabla \cdot)$, defined on $W^{2,p}(\mathbb{R}^d)$. We start by the following technical lemma which gives a pointwise domination of $\{T(t) : t \geq 0\}$.

Lemma 4.1. *We have the following semigroup domination*

$$|T(t)f|^2 \leq T_0(t)|f|^2, \qquad t > 0, \tag{4.1.1}$$

for all $f \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$.

Proof. Let $f \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$ be given. Let us also fix $p \in (1, \infty)$. We set $u(t, \cdot) = T(t)f$, for $t \geq 0$. One has $f \in D(A_q)$ which is continuously embedded into $W^{2,q}(\mathbb{R}^d; \mathbb{R}^m)$, according to (3.2.2). Thus u belongs to $C([0,\infty); W^{2,q}(\mathbb{R}^d; \mathbb{R}^m)) \cap C^1([0,\infty); L^q(\mathbb{R}^d; \mathbb{R}^m))$ for every $q \in [1, \infty)$. It thus follows that the scalar function

$|u|^2$ belongs to $C([0,\infty);W^{2,p}(\mathbb{R}^d))$. Since u solves the system of coupled partial differential equations $\partial_t u = (\mathrm{div}(Q\nabla u_k) - u_k) - Vu$, we get

$$\begin{aligned}\frac{1}{2}\partial_t|u|^2 = \langle \partial_t u, u\rangle &= \sum_{k=1}^m \mathrm{div}(Q\nabla u_k)u_k - |u|^2 - \langle Vu, u\rangle \\ &\le \sum_{k=1}^m\sum_{i,j=1}^d \partial_i(q_{ij}\partial_j u_k)u_k - 2|u|^2 \\ &= \sum_{k=1}^m\sum_{i,j=1}^d \partial_i(q_{ij}u_k\partial_j u_k) - \sum_{k=1}^m\sum_{i,j=1}^d q_{ij}\partial_j u_k\partial_i u_k - 2|u|^2 \\ &\le \frac{1}{2}\sum_{i,j=1}^d \partial_i(q_{ij}\partial_j|u|^2) - 2|u|^2 \\ &\le \frac{1}{2}\Delta_Q|u|^2.\end{aligned}$$

Thus, the function $v := \partial_t|u|^2 - \Delta_Q|u|^2$ belongs to $C([0,\infty);L^p(\mathbb{R}^d))$ and is nonpositive. Fix $t>0$ and set $w(s,\cdot) = T_0(t-s)|u|^2(s,\cdot)$ for every $s\in[0,t]$. As is immediately seen,

$$\begin{aligned}\partial_s w(s,\cdot) =& -T_0(t-s)\Delta_Q|u|^2(s,\cdot) + T_0(t-s)\partial_s|u|^2(s,\cdot) \\ =& T_0(t-s)(\partial_s|u|^2(s,\cdot) - \Delta_Q|u|^2(s,\cdot)) \\ =& T_0(t-s)v(s,\cdot) \le 0,\end{aligned}$$

since the semigroup $\{T_0(t)\}$ preserves positivity (see, [**54**, Corollary 4.3]). Hence, $w(t,\cdot)\le w(0,\cdot)$, which is exactly (4.1.1). □

We can now establish ultracontractivity of the semigroup.

Theorem 4.2. *There there exists $M>0$ such that*

$$\|T(t)f\|_\infty \le Mt^{-\frac{d}{2}}\|f\|_1, \qquad f\in L^1(\mathbb{R}^d;\mathbb{R}^m). \tag{4.1.2}$$

Consequently, for every $t>0$, there exists a matrix kernel $K(t,\cdot,\cdot)\in L^\infty(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R}^{m\times m})$ such that

$$(T(t)f)(x) = \int_{\mathbb{R}^d} K(t,x,y)f(y)dy, \qquad x\in\mathbb{R}^d,\ \ f\in L^p(\mathbb{R}^d;\mathbb{R}^m). \tag{4.1.3}$$

Moreover, for $t>0$, $T(t)$ is positive if, and only if, $k_{ij}(t,x,y)\ge 0$ for almost every $x,y\in\mathbb{R}^d$.

Proof. Let us first prove Estimate (4.1.2). We fix $f\in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$ and show that

$$\|T(t)f\|_\infty \le Mt^{-\frac{d}{4}}\|f\|_2, \qquad t>0. \tag{4.1.4}$$

Throughout the proof M is a constant, independent of f and t, which may vary from line to line. Using (4.1.1) and the ultracontractivity of the semigroup $\{T_0(t) : t \geq 0\}$ we get

$$\|T(t)f\|_\infty^2 \leq \|T_0(t)|f|^2\|_\infty \leq Mt^{-\frac{d}{2}}\||f|^2\|_1 = Mt^{-\frac{d}{2}}\|f\|_2^2$$

for $t > 0$. Taking square roots, this shows (4.1.4). Next, we prove the L^1–L^2 estimate

$$\|T(t)f\|_2 \leq Mt^{-\frac{d}{4}}\|f\|_1, \qquad t > 0. \tag{4.1.5}$$

To that end, note that the adjoint V^* also satisfies the same hypotheses as V. Hence, (4.1.1) and then (4.1.4) hold true also for $\{T^*(t) : t \geq 0\}$. Consequently,

$$\begin{aligned}\|T(t)f\|_2 &= \sup_{\|\varphi\|_2=1} \langle T(t)f, \varphi\rangle_{2,2} = \sup_{\|\varphi\|_2=1} \langle f, T^*(t)\varphi\rangle_{2,2} \\ &\leq \sup_{\|\varphi\|_2=1} \|T^*(t)\varphi\|_\infty \|f\|_1 \leq Mt^{-\frac{d}{4}}\|f\|_1\end{aligned}$$

and (4.1.5) thus follows. By the semigroup law, Estimates (4.1.4) and (4.1.5) we obtain

$$\|T(t)f\|_\infty = \|T(t/2)T(t/2)f\|_\infty \leq Mt^{-\frac{d}{4}}\|T(t/2)f\|_2 \leq Mt^{-\frac{d}{2}}\|f\|_1$$

for every $t > 0$. Finally, by density of $C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$ in $L^1(\mathbb{R}^d;\mathbb{R}^m)$, we can easily complete the proof of (4.1.2).

We next establish the existence of matrix kernel. We fix $t > 0$, $f = (f_1, \ldots, f_m)$ and denote the canonical basis of $\mathbb{R}^m$ by $\{e_i\}_{1\leq i\leq m}$. Then, we have

$$T(t)f = \sum_{j=1}^m T(t)(f_j e_j) = \sum_{i,j=1}^m \langle T(t)(f_j e_j), e_i\rangle e_i.$$

For $i, j \in \{1, \ldots, m\}$ and $u \in L^1(\mathbb{R}^d)$, let $T_{i,j}(t)u = \langle T(t)(ue_j), e_i\rangle$. Using (4.1.2), we obtain

$$\|T_{i,j}(t)u\|_\infty = \|\langle T(t)(ue_j), e_i\rangle\|_\infty \leq \|T(t)(ue_j)\|_\infty \leq Mt^{-\frac{d}{2}}\|ue_j\|_1 = Mt^{-\frac{d}{2}}\|u\|_1.$$

Thus, $T_{i,j}(t)$ maps $L^1(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$ continuously. We thus conclude by the Dunford–Pettis theorem, see [**4**, Theorem 1.3], the existence of a kernel $k_{i,j}(t,\cdot,\cdot) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$(T_{i,j}(t)u)(x) = \int_{\mathbb{R}^d} k_{ij}(t,x,y)u(y)dy,$$

for all $x \in \mathbb{R}^d$. Setting $K(t,\cdot,\cdot) := (k_{ij}(t,\cdot,\cdot))_{i,j=1}^m$ the matrix whose entries be $k_{i,j}$, $1 \leq i,j \leq m$, we conclude that $K(t,\cdot,\cdot) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^{m\times m})$ and

$$T(t)f = \int_{\mathbb{R}^d} \sum_{i,j=1}^m k_{ij}(t,x,y)f_j(y)e_i dy = \int_{\mathbb{R}^d} K(t,x,y)f(y)dy,$$

for all $f \in L^p(\mathbb{R}^d;\mathbb{R}^m)$.

For the positivity, one can deduce from (4.1.3) that if all entries of the kernel matrix are nonnegative, $T(t)$ is then positive. Conversely, let $t > 0$, $i, j \in \{1, \ldots, m\}$ $\mathcal{B}$ be any bounded measurable set of $\mathbb{R}^d$. Then

$$\begin{aligned} T(t) \geq 0 &\Longrightarrow \langle T(t)(\chi_{\mathcal{B}} e_i), e_j \rangle \geq 0 \\ &\Longrightarrow \int_{\mathcal{B}} k_{ij}(t,x,y)dy \geq 0. \end{aligned}$$

As $\mathcal{B}$ was arbitrary chosen, it thus follow $k_{ij}(t,x,y) \geq 0$ for almost every $x, y \in \mathbb{R}^d$. ☐

4.2. Maximal regularity

The ultracontracitivity property is a crucial result, since it implies that the solution $u(t,\cdot) = T(t)f$, for an initial datum $f \in L^1(\mathbb{R}^d,\mathbb{R}^m)$, belongs to all L^p-spaces, in particular, to $L^\infty(\mathbb{R}^d,\mathbb{R}^m)$, that is the solution is bounded. When the semigroup $\{T(t) : t \geq 0\}$ is analytic and since the domain of A_p is (continuously) embedded in $W^{2,p}(\mathbb{R}^d,\mathbb{R}^m)$, one gets more regularity for u thanks to the Sobolev imbedding. The result is formulated as follow

Theorem 4.3. *Assume that $\tilde{V}$ satisfies* (2.7.1). *Let $p \in (1,\infty)$ and $t > 0$ Then, for all $f \in L^p(\mathbb{R}^d;\mathbb{R}^m)$, the function $u(t,\cdot) := T(t)f$ belongs to $C_b^{1+\alpha}(\mathbb{R}^d;\mathbb{R}^m)$ for every $\alpha \in (0,1)$.*

Proof. Fix $t > 0$, $\alpha \in (0,1)$. Let us prove the claim for $p \in (1,\infty)$. Let k be the smallest integer such that $p^{-1} - 2kd^{-1} \leq 0$. By repeatedly applying of the Sobolev embedding theorem, see [**24**, Section 5.6], we get the assertion. Indeed, since $T(t/(k+1))f \in D(A_p) \hookrightarrow W^{2,p}(\mathbb{R}^d,\mathbb{R}^m)$ and $W^{2,p}(\mathbb{R}^d,\mathbb{R}^m) \hookrightarrow L^{p_1}(\mathbb{R}^d,\mathbb{R}^m)$, where $p_1^{-1} = p^{-1} - 2d^{-1}$, we conclude that $T(t/(k+1))f \in L^{p_1}(\mathbb{R}^d,\mathbb{R}^m)$. It thus follows that

$$T(2t/(k+1))f = T(t/(k+1))T(t/(k+1))f \in W^{2,p_1}(\mathbb{R}^d,\mathbb{R}^m).$$

Arguing as above, gives $T(2t/(k+1))f \in L^{p_2}(\mathbb{R}^d,\mathbb{R}^m)$ where $p_2^{-1} = p^{-1} - 4d^{-1}$. Iterating this argument we find that $T(kt/(k+1))f \in W^{2,p_k}(\mathbb{R}^d,\mathbb{R}^m)$, where $p_k^{-1} = p^{-1} - 2kd^{-1}$, so that $T(kt/(k+1))f \in L^q(\mathbb{R}^d,\mathbb{R}^m)$ for any $q \in [p,\infty)$. Hence, $T(t)f \in W^{2,q}(\mathbb{R}^d;\mathbb{R}^m)$ and choosing $q \geq (1-\alpha)^{-1}d$ we conclude that $T(t)f \in C^{1+\alpha}(\mathbb{R}^d;\mathbb{R}^m)$.

For the case $p = 1$, the ultracontractivity of $\{T(t) : t \geq 0\}$ implies, for $f \in L^1(\mathbb{R}^d,\mathbb{R}^m)$, $g = T(t/2)f \in L^p(\mathbb{R}^d;\mathbb{R}^m)$ for all $p > 1$. Thus splitting $T(t)f = T(t/2)T(t/2)f = T(t/2)g$. Applying the machinery of above for g instead of f and $T(t/2)$ instead of $T(t)f$, the claim follows. ☐

4.3. Gaussian Kernel estimates

In this section we give a Gaussian upper bound estimate for $\{T(t) : t \geq 0\}_{t\geq 0}$. For the proof, we follow the strategy of [**34**, section 4].

Theorem 4.4. *There exist nonnegative constants C_1 and C_2 such that*

$$|k_{ij}(t,x,y)| \leq C_1 t^{-\frac{d}{2}} \exp\{-C_2 \frac{|x-y|^2}{4t}\}, \tag{4.3.1}$$

for all $i, j \in \{1, \dots, m\}$ and $x, y \in \mathbb{R}^d$.

Proof. Let $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{J} := \{\psi \in C_b^\infty(\mathbb{R}^d) : \|\nabla\psi\|_\infty \leq 1\}$. Define the *twisted* semigroup $\{T_{\lambda,\varphi}(t) : t \geq 0\}$ by

$$T_{\lambda,\varphi}(t)f := e^{-\lambda\varphi} T(t)(e^{\lambda\varphi} f),$$

for all $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^d, \mathbb{R}^m)$. $\{T_{\lambda,\varphi}(t) : t \geq 0\}$ has the following kernel representation

$$T_{\lambda,\varphi}(t)f = \int_{\mathbb{R}^d} e^{-\lambda(\varphi(x)-\varphi(y))} K(t,x,y) f(y) dy. \tag{4.3.2}$$

Obviuosely $\{T_{\lambda,\varphi}(t) : t \geq 0\}$ is a strongly continuous semigroup in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. Let us denote by $A_{\lambda,\varphi}$ its generator. A straightforward calculation yields

$$A_{\lambda,\varphi} f = \Delta_Q f + 2\lambda\langle Q\nabla\varphi, \nabla f\rangle + (-V + \lambda\Delta_Q\varphi + \lambda^2|\nabla\varphi|_Q^2)f.$$

and

$$a_{\lambda,\varphi}(f) := \langle -A_{\lambda,\varphi} f, f\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^m)} = -\langle (A_{\lambda,\varphi}+V)f, f\rangle + \langle Vf, f\rangle \geq -\langle B_{\lambda,\varphi} f, f\rangle := b_{\lambda,\varphi}(f).$$

for every $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Where the operator $B_{\lambda,\varphi}$ is defined on $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ by

$$B_{\lambda,\varphi} f := \Delta_Q f + 2\lambda\langle Q\nabla\varphi, \nabla f\rangle + (\lambda\Delta_Q\varphi + \lambda^2|\nabla\varphi|_Q^2)f$$

Note that $\langle \nabla\varphi, \nabla f\rangle$ is the vector valued function whose components are $\langle \nabla\varphi, \nabla f_j\rangle$. By integrating by parts,

$$\begin{aligned} b_{\lambda,\varphi}(f) = &\int_{\mathbb{R}^d} \langle Q(x)\nabla f(x), \nabla f(x)\rangle dx - 2\lambda \sum_{i=0}^m \int_{\mathbb{R}^d} \langle Q(x)\nabla\varphi(x), \nabla f_i(x)\rangle f_i(x) dx \\ &- \int_{\mathbb{R}^d} \{\lambda div(Q\nabla\varphi)(x) + \lambda^2|\nabla\varphi(x)|_{Q(x)}^2\}|f(x)|^2 dx \\ &= \int_{\mathbb{R}^d} \langle Q(x)\nabla f(x), \nabla f(x)\rangle dx - \lambda^2 \int_{\mathbb{R}^d} \langle Q(x)\nabla\varphi(x), \nabla\varphi(x)\rangle |f(x)|^2 dx \\ &\geq \eta_1\|\nabla f\|_2^2 - \eta_2\lambda^2\|f\|_2^2. \end{aligned}$$

If we set $\omega = \eta_2\lambda^2$ then

$$a_{\lambda,\varphi}(f) - \omega||f||_2^2 \geq b_{\lambda,\varphi}(f) - \omega||f||_2^2 \geq \eta_1\|\nabla f\|_2^2.$$

Now, consider $\gamma(t) = \|e^{-\omega t}T_{\lambda,\varphi}(t)\|_2^{-\frac{4}{d}}$ for all $t \geq 0$. One has

$$\begin{aligned}\gamma'(t) &= \frac{d}{dt}(\|e^{-\omega t}T_{\lambda,\varphi}(t)f\|_2^2)^{-\frac{2}{d}}\\ &= -\frac{2}{d}\|e^{-\omega t}T_{\lambda,\varphi}(t)f\|_2^{-\frac{4}{d}-2}\langle (A_{\lambda,\varphi}-\omega)f, f\rangle\\ &\geq \frac{2\eta_1}{d}\|e^{-\omega t}T_{\lambda,\varphi}(t)f\|_2^{-\frac{4}{d}-2}\|\nabla(e^{-\omega t}T_{\lambda,\varphi}(t)f)\|_2^2.\end{aligned}$$

Applying Nash's inequality, see [**17**, Theorem 2.4.6], one obtains

$$\gamma'(t) \geq \frac{2\eta_1}{d\,C}\|e^{-\omega t}T_{\lambda,\varphi}(t)f\|_1^{-\frac{4}{d}}.$$

Now, remarking that $B_{\lambda,\varphi} - \omega$ is actually an elliptic operator with bounded coefficients which does not represent any coupling and since its associated form is accretive, we conclude by [**54**, Chapter 4] that its associated semigroup say $\{e^{-\omega t}e^{tB_{\lambda,\varphi}} : t \geq 0\}$ is contractive in all L^p-spaces, in particular in $L^1(\mathbb{R}^d, \mathbb{R}^m)$. Moreover, one has $A_{\lambda,\varphi} = B_{\lambda,\varphi} - V$ and $|e^{-tV}| \leq 1$. Applying Trotter-Kato product formula we deduce that $\{e^{-\omega t}T_{\lambda,\varphi}(t) : t \geq 0\}$ is L^p-contractive, for every $p > 1$ and Fatou's lemma yields

$$\|T_{\lambda,\varphi}(t)f\|_1 \leq e^{\omega t}\|f\|_1.$$

It thus follows

$$\gamma(t) \geq \int_0^t \gamma'(s)ds \geq \frac{2\eta_1}{d\,C}t\|f\|_1^{-\frac{4}{d}}.$$

Therefore,

$$\|T_{\lambda,\varphi}(t)f\|_2 \leq \frac{2\eta_1}{d\,C}e^{\omega t}t^{-\frac{d}{4}}\|f\|_1. \tag{4.3.3}$$

Since V^* verifies the same hypotheses as V, one can reproduce the same for

$$T^*_{\lambda,\varphi}(t) = e^{\lambda\varphi}T^*(t)(e^{-\lambda\varphi}\cdot)$$

and obtain

$$\|T^*_{\lambda,\varphi}(t)f\|_2 \leq \frac{2\eta_1}{d\,C}e^{\omega t}t^{-\frac{d}{4}}\|f\|_1.$$

On the other hand, one has

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\langle T_{\lambda,\varphi}(t)f(x), g(x)\rangle dx\right| &= |\langle T_{\lambda,\varphi}(t)f, g\rangle_{L^2}|\\ &= \left|\langle f, T^*_{\lambda,\varphi}(t)g\rangle_{L^2}\right|\\ &\leq \|T^*_{\lambda,\varphi}(t)g\|_2\|f\|_2\|g\|_1\\ &\leq \frac{2\eta_1}{d\,C}e^{\omega t}t^{-\frac{d}{4}}\|f\|_2\|g\|_1,\end{aligned}$$

for every $g \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Therefore, $g \mapsto \int_{\mathbb{R}^d} \langle S_{\lambda,\varphi}(t)f(x), g(x)\rangle dx$ can be extended to a bounded linear form over $L^1(\mathbb{R}^d, \mathbb{R}^m)$ which has $L^\infty(\mathbb{R}^d, \mathbb{R}^m)$ as a dual space. Thus, $T_{\lambda,\varphi}(t)f \in L^\infty(\mathbb{R}^d, \mathbb{R}^m)$ and

$$\|T_{\lambda,\varphi}(t)f\|_\infty \leq \frac{2\eta_1}{d\,C} e^{\omega t} t^{-\frac{d}{4}} \|f\|_2. \tag{4.3.4}$$

Combining (4.3.3) and (4.3.4) one obtains $T_{\lambda,\varphi}(t)f \in L^\infty(\mathbb{R}^d, \mathbb{R}^m)$ for every $f \in L^1(\mathbb{R}^d, \mathbb{R}^m)$ and

$$\begin{aligned} \|T_{\lambda,\varphi}(t)f\|_\infty = \|T_{\lambda,\varphi}(t/2)T_{\lambda,\varphi}(t/2)f\|_\infty &\leq \frac{2\eta_1}{d\,C} e^{\omega t/2}(t/2)^{-\frac{d}{4}} \|T_{\lambda,\varphi}(t/2)f\|_2 \\ &\leq C_1 e^{\omega t} t^{-\frac{d}{2}} \|f\|_1, \end{aligned}$$

with $C_1 = 2^d(\frac{2\eta_1}{d\,C})^2$. It thus follow that the twisted semigroup is ultracontractive with $\|\cdot\|_{1\to\infty}$–norm less than or equal to $C_1 e^{\omega t} t^{-\frac{d}{2}}$. Taking into account (4.3.2) one gets

$$|k_{ij}(t,x,y)| \leq C_1 t^{-\frac{d}{2}} \exp\{\eta_2 \lambda^2 t + \lambda(\varphi(x) - \varphi(y))\}.$$

Choosing $\lambda = \dfrac{\varphi(y) - \varphi(x)}{2\eta_2 t}$, we get

$$|k_{ij}(t,x,y)| \leq C_1 t^{-\frac{d}{2}} \exp\{-\frac{|\varphi(x) - \varphi(y)|^2}{4\eta_2 t}\}.$$

If we define the distance δ on $\mathbb{R}^d$ by

$$\delta(x,y) := \sup\{\psi(x) - \psi(y) : \psi \in \mathcal{J}\}, \qquad x, y \in \mathbb{R}^d.$$

It is well-known that δ is equivalent to the euclidean distance in $\mathbb{R}^d$. Finally, there exists $C_2 > 0$ such that

$$|k_{ij}(t,x,y)| \leq C_1 t^{-\frac{d}{2}} \exp\{-C_2 \frac{|x-y|^2}{4t}\}.$$

□

4.4. Further kernel estimates

We assume that $V = (v_{ij})_{1\leq i,j\leq m} + vI_m$, where $(v_{ij})_{1\leq i,j\leq m}$ satisfies Hypotheses 3.1 and v to be as in Section 3.3. We denote by $d_{ii} := v_{ii} + v$, for each $i \in \{1, \ldots, m\}$.

Let us denote by $\{T^*(t) : t \geq 0\}$ the adjoint semigroup of $\{T(t) : t \geq 0\}$ in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. We start by given the matrix kernel associated to $\{T^*(t) : t \geq 0\}$

Proposition 4.5. *Let $g \in L^2(\mathbb{R}^d, \mathbb{R}^m)$ and $t > 0$. Then,*

$$T(t)^*(t)g(x) = \int_{\mathbb{R}^d} K^*(t,z,x)g(z)dz, \qquad \forall x \in \mathbb{R}^d. \tag{4.4.1}$$

In particular, if V is a symmetric, then $K(t,y,x) = K^*(t,x,y)$ and $k_{ij}(t,y,x) = k_{ji}(t,x,y)$, for every $x, y \in \mathbb{R}^d$ and $i, j \in \{1, \ldots, m\}$.

PROOF. Let $f, g \in L^2(\mathbb{R}^d, \mathbb{R}^m)$. By Fubini's theorem one obtains

$$\begin{aligned}\langle T(t)f, g\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^m)} &= \int_{\mathbb{R}^d} \langle T(t)f(x), g(x)\rangle dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \langle K(t,x,y)f(y), g(x)\rangle dydx \\ &= \int_{\mathbb{R}^d} \langle f(y), \int_{\mathbb{R}^d} K^*(t,x,y)g(x)dx\rangle dy\end{aligned}$$

Hence, $T^*(t)g(x) = \int_{\mathbb{R}^d} K^*(t,z,x)g(z)dz$. Moreover, if V is symmetric, thus $\{T(t) : t \geq 0\}$ will be a self-adjoint semigroup in $L^2(\mathbb{R}^d, \mathbb{R}^m)$. Hence, by uniqueness of the kernel $K(t,y,x) = K^*(t,x,y)$, for almost every $x, y \in \mathbb{R}^d$. □

Proposition 4.6. *For each $i \in \{1,\ldots,m\}$, $k_{ii}(t,\cdot,\cdot)$, the i–th diagonal entry of the matrix kernel $K(t,\cdot,\cdot)$, represents the heat kernel associated to the scalar Schrödinger operator $A_{ii} := \Delta_Q - d_{ii}$. Moreover, if we denote by $k_v(t,\cdot,\cdot)$ the kernel of $\Delta_Q - v$, one gets*

$$0 \leq k_{ii}(t,x,y) \leq k_v(t,x,y), \tag{4.4.2}$$

for evey $t > 0$ and almost every $x, y \in \mathbb{R}^d$.

PROOF. Let $i \in \{1,\ldots,m\}$ and $t > 0$. Define $T_{ii}(t)f = \langle T(t)(fe_i), e_i\rangle$, for all $f \in L^2(\mathbb{R}^d)$. One can check easily that $\{T_{ii}(t) : t \geq 0\}$ is a strongly continuous semigroup in $L^2(\mathbb{R}^d)$. Moreover, one has

$$T_{ii}(t)f(x) = \int_{\mathbb{R}^d} k_{ii}(t,x,y)f(y)dy,$$

for all $f \in L^2(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$. Now, let us compute the infinitesimal generator of $\{T_{ii}(t) : t \geq 0\}$. Let $f \in L^2(\mathbb{R}^d)$ and $t > 0$. One has

$$\begin{aligned}\frac{1}{t}\{T_{ii}(t)f - f\} &= \frac{1}{t}\{\langle T(t)(fe_i), e_i\rangle - f\} \\ &= \langle \frac{T(t)(fe_i) - fe_i}{t}, e_i\rangle.\end{aligned}$$

Hence, $f \in D(A_{ii})$ if, and only if, $fe_i \in D(A_2)$ and

$$A_{ii}f = \langle A(fe_i), e_i\rangle = \Delta_Q f - (v_{ii} + v)f.$$

In particular, $k_{ii}(\cdot,\cdot,\cdot)$ is the kernel of A_{ii}. According to Proposition D.6, one has $v_{ii} \geq 0$, thus $-d_{ii} \leq -v$, then, by application of Trotter–Kato product formula, we get the following pointwise semigroup comparison

$$|T_{ii}(t)f| \leq |T_v(t)f|, \qquad \forall f \in L^2(\mathbb{R}^d).$$

This implies immediately (4.4.2). □

Here we give another proposition, holding in the symmetric case, which yields the domination of the off-diagonal entries of the matrix kernel by the diagonal one

Proposition 4.7. *Assume that V is symmetric. Let $t > 0$ and $i \neq j \in \{1, \dots, m\}$. One has*

$$|k_{ij}(t,x,y) + k_{ij}(t,y,x)| \leq 2\sqrt{k_{ii}(t,x,y)k_{jj}(t,x,y)} \leq k_{ii}(t,x,y) + k_{jj}(t,x,y), \tag{4.4.3}$$

for almost every $x, y \in \mathbb{R}^d$. In particular,

$$|k_{ij}(t,x,x)| \leq \sqrt{k_{ii}(t,x,x)k_{jj}(t,x,x)} \leq \frac{1}{2}\{k_{ii}(t,x,x) + k_{jj}(t,x,x)\}, \tag{4.4.4}$$

for almost every $x \in \mathbb{R}^d$.

Proof. let $\xi \in \mathbb{R}^d$, $t > 0$ and $\mathcal{B}$ any bounded subset of $\mathbb{R}^d$. Consider $f = \chi_{\mathcal{B}}\xi \in L^2(\mathbb{R}^d, \mathbb{R}^m)$. Since V is symmetric, then $T(t)$ is self-adjoint. Hence, $\langle T(t)f, f\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^m)} = \|T(t/2)f\|_2^2 \geq 0$. Thus,

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^d} \langle T(t)f(x), f(x)\rangle dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \langle K(t,x,y)\xi, \xi\rangle \chi_{\mathcal{B}}(x)\chi_{\mathcal{B}}(y)dxdy \\ &= \int_{\mathcal{B}\times\mathcal{B}} \langle K(t,x,y)\xi,\xi\rangle dxdy. \end{aligned}$$

Thanks to the arbitrariness of $\mathcal{B}$, one obtains $\langle K(t,x,y)\xi,\xi\rangle \geq 0$, for almost every $x, y \in \mathbb{R}^d$. Taking into the account Proposition D.6, one obtains

$$|k_{ij}(t,x,y) + k_{ji}(t,x,y)| \leq 2\sqrt{k_{ii}(t,x,y)}\sqrt{k_{jj}(t,x,y)},$$

for almost every $x, y \in \mathbb{R}^d$. Now, (4.4.3) and (4.4.4) follow taking into the account Proposition 4.5. □

As a consequence of Proposition 4.6, one obtains the following kernel estimates

Proposition 4.8. *Assume that $v(x) = |x|^\alpha$, for some $\alpha > 2$. Then, for all $i \in \{1, \dots, m\}$, one has*

$$k_{ii}(t,x,y) \leq Ce^{-\gamma t}e^{ct^{-b}}\phi(|x|)\phi(|y|), \tag{4.4.5}$$

for large x and y in $\mathbb{R}^d$ and all $t > 0$. Where γ, c and C are some positive constants and

$$\phi(R) = \frac{\exp\left(-\frac{2\sqrt{\theta}}{2+\alpha}R^{1+\frac{\alpha}{2}}\right)}{R^{\frac{\alpha}{4}+\frac{d-1}{2}}},$$

for every $R \geq 0$. Here $b > \frac{\alpha+2}{\alpha-2}$ and $\theta > 0$ is properly chosen. Moreover, if $v_{lk} \geq 0$ for all $k \neq l \in \{1, \dots, m\}$, then (4.4.5) holds true also for k_{ij} for all $i, j \in \{1, \dots, m\}$.

Proof. Estimate (4.4.2) together with [**56**, Theorem 2.7] yield (4.4.5). Note that the constant θ is such that $\theta\eta_2 < 1$, where η_2 is the constant appearing in (2.1.1). Moreover, if $v_{lk} \geq 0$ for all $k \neq l \in \{1, \dots, m\}$, thus by Theorem 4.2,

$k_{ij}(t,x,y) \geq 0$, for every $t>0$ and $x,y \in \mathbb{R}^d$. Due to (4.4.3), it thus follow the estimate (4.4.5) for k_{ij} as well. $\square$

4.5. Asymptotic distribution of eigenvalues of A

In this section, we assume the following:

Hypotheses 4.9. V is symmetric and $v(x) = |x|^\alpha$, for some $\alpha \geq 1$ and all $x \in \mathbb{R}^d$. Moreover, suppose that $v_{ii}(x) = o(v(x))$, when $|x|$ goes to infinity, for all $i \in \{1,\dots,m\}$.

Let $\{\lambda_n : n \in \mathbb{N}\}$ be the eigenvalues of $-A$ and $\{\Psi_n : n \in \mathbb{N}\}$ the orthonormal basis of $L^2(\mathbb{R}^d,\mathbb{R}^m)$ constituted from the eigenvectors of $-A$: $A\Psi_n = -\lambda_n\Psi_n$. In the following proposition we compute the trace of $\{T(t) : t \geq 0\}$ in two different ways

Proposition 4.10. *For all $i,j \in \{1,\dots,m\}$, one has*

$$k_{ij}(t,x,y) = \sum_{n\in\mathbb{N}} e^{-\lambda_n t}\Psi_n^{(i)}(x)\Psi_n^{(j)}(y), \tag{4.5.1}$$

for all $x,y \in \mathbb{R}^d$ and $t>0$. Here $\Psi_n^{(i)}(x)$ is the i*-th component of the vector $\Psi_n(x)$. In particular,*

$$\int_{\mathbb{R}^d}\sum_{i=1}^m k_{ii}(t,x,x)dx = \sum_{n\in\mathbb{N}} e^{-\lambda_n t}, \qquad \forall t>0. \tag{4.5.2}$$

Proof. Let $f \in L^2(\mathbb{R}^d,\mathbb{R}^m)$. Thus $f = \sum_{n\in\mathbb{N}}\langle f,\Psi_n\rangle_{L^2}\Psi_n$. Then, by linearity and continuity of $T(t)$, one gets

$$T(t)f = \sum_{n\in\mathbb{N}}\langle f,\Psi_n\rangle_{L^2}T(t)\Psi_n = \sum_{n\in\mathbb{N}}\langle f,\Psi_n\rangle_{L^2}e^{-\lambda_n t}\Psi_n.$$

for every $t>0$. Hence,

$$\langle T(t)f(x),e_i\rangle = \sum_{\mathbb{N}} e^{-\lambda_n t}\int_{\mathbb{R}^d}\sum_{j=1}^m f_j(y)\Psi_n^{(j)}(y)\Psi_n^{(i)}(x)\,dy.$$

for each $i \in \{1,\dots,m\}$. Therefore, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} k_{ij}(t,x,y)\varphi(y)dy &=\leq T(t)(\varphi e_j)(x),e_i\rangle \\ &= \int_{\mathbb{R}^d}\sum_{\mathbb{N}} e^{-\lambda_n t}\Psi_n^{(j)}(y)\Psi_n^{(i)}(x)\varphi(y)\,dy,\end{aligned}$$

for all $t > 0$, $x \in \mathbb{R}^d$ and $i, j \in \{1, \dots, m\}$. From which we deduce (4.5.1). Moreover,

$$\sum_{i=1}^{m} \int_{\mathbb{R}^d} k_{ii}(t,x,x)\,dx = \int_{\mathbb{R}^d} \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \Psi_n^{(i)}(x)^2\,dx = \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \int_{\mathbb{R}^d} |\Psi(x)|^2\,dx = \sum_{n\in\mathbb{N}} e^{-\lambda_n t}.$$

□

Let us now introduce the measure γ defined over $\mathbb{R}^+$ by $\gamma(X) = |\{n : \lambda_n \in X\}|$. Define, for $\lambda > 0$, $\mathcal{N}(\lambda) = \gamma[0, \lambda]$ which corresponds to the number of eigenvalues λ_n that are less or equal than λ. Let us denote by $\hat{\gamma}$ the Laplace transform of γ:

$$\hat{\gamma}(t) := \int_{\mathbb{R}} e^{-tx} d\gamma(x) = \sum_{n\in\mathbb{N}} e^{-\lambda_n t},$$

for all $t > 0$. According to (4.5.2), one has

$$\hat{\gamma}(t) = \sum_{n\in\mathbb{N}} e^{-\lambda_n t} = \int_{\mathbb{R}^d} \sum_{i=1}^{m} k_{ii}(t,x,x)dx.$$

We are looking for the asymptotic behavior ($\lambda \to \infty$) of $\mathcal{N}(\lambda)$. This is related to the behavior near 0 of $\hat{\mu}$ by the famous Tauberian (Karamata's) theorem, see [**62**, Theorem 10.3]. One has the following

Theorem 4.11. *Assume that Hypotheses 4.9 hold and $Q = I_d$. Then,*

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}(\lambda)}{\lambda^{d(\frac{1}{2}+\frac{1}{\alpha})}} = \frac{1}{\alpha} \frac{d\,m\,\omega_d}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(d/\alpha)}{\Gamma(d(\frac{1}{2}+\frac{1}{\alpha})+1)}. \tag{4.5.3}$$

Proof. Taking into account Proposition 4.6 and by [**49**, Example 2.6], for each $\tau < 1$, one hass

$$k_{ii}(t,x,x) \le \left(\frac{e^{-t\tau^\alpha |x|^\alpha}}{(4\pi t)^{\frac{d}{2}}} + \frac{\tau^d \omega_d\, C(d)}{(1-\tau)^d t^{\frac{d}{2}}} \exp\left(-\frac{(1-\tau)^2}{4t} |x|^2 \right) \right),$$

for each $i \in \{1, \dots, m\}$ and every $t > 0$ and $x \in \mathbb{R}^d$. Where $C(d)$ is a constant depending only on d and $\omega_d = |S_d|$ the volume of the unit sphere of $\mathbb{R}^d$.
Integrating over $\mathbb{R}^d$ and using suitable change of variables, one gets

$$\int_{\mathbb{R}^d} k_{ii}(t,x,x)dx \le \frac{1}{(4\pi)^{\frac{d}{2}} t^{\frac{d}{2}+\frac{d}{\alpha}}} \int_{\mathbb{R}^d} e^{-\tau^\alpha |x|^\alpha} dx + \frac{\tau^d \omega_d\, C(d)}{(1-\tau)^d \pi^{\frac{d}{2}}} e^{C_\varepsilon t} \int_{\mathbb{R}^d} \exp(-(1-\tau)^2 |x|^2) dx.$$

Hence, by letting τ tends to 1,

$$\limsup_{t\to 0} t^{\frac{d}{2}+\frac{d}{\alpha}} \sum_{i=1}^{m} \int_{\mathbb{R}^d} k_{ii}(t,x,x)dx \le \frac{m}{(4\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-|x|^\alpha} dx = \frac{m\,\omega_d}{(4\pi)^{\frac{d}{2}}\,\alpha}\Gamma(\frac{d}{\alpha}). \tag{4.5.4}$$

On the other hand, according to Hypotheses 4.9 one has $d_{ii}(x) = |x|^\alpha + o(|x|^\alpha)$. Hence, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$d_{ii}(x) \le (1+\varepsilon)|x|^\alpha + C_\varepsilon$$

for all $x \in \mathbb{R}^d$. Therefore,

$$k_{ii}(t,x,x) \ge e^{-C_\varepsilon t} k_{\varepsilon,\alpha}(t,x,x),$$

where $k_{\varepsilon,\alpha}$ is the kernel associated to $\Delta - (1-\varepsilon)|x|^\alpha$. Now, arguing as in [**17**, Lemma 4.5.9], one obtains

$$k_{\varepsilon,\alpha}(t,x,x) \ge \exp((1-\varepsilon)(1+|x|)^\alpha t) k_\Delta(t,x,x),$$

where k_Δ is the heat kernel associated to the Dirichlet Laplacian on the ball $B(x,1)$, of center x and radius 1, of $\mathbb{R}^d$. The *Kac's principle* yields

$$k_\Delta(t,x,x) \ge \frac{1}{(4\pi t)^{\frac{d}{2}}}(1-e^{-\frac{1}{4t}}) = \gamma(t),$$

for every $0 < t < \frac{1}{2d}$. Thus,

$$k_{ii}(t,x,x) \ge e^{-C_\varepsilon t}\gamma(t)\exp((1-\varepsilon)(1+|x|)^\alpha t),$$

for every $0 < t < \frac{1}{2d}$, $x \in \mathbb{R}^d$ and $i \in \{1,\dots,m\}$. Now, fix $t \in (0,\frac{1}{2d})$ and $\varepsilon > 0$. Integrating over x and summing over i, one gets

$$\begin{aligned}\int_{\mathbb{R}^d} \sum_{i=1}^{m} k_{ii}(t,x,x)dx &\ge m\,e^{-C_\varepsilon t}\gamma(t) \int_{\mathbb{R}^d} \exp((1-\varepsilon)(1+|x|)^\alpha t)dx \\ &= m\,e^{-C_\varepsilon t}\gamma(t)\,\omega_d \int_0^\infty \exp((1-\varepsilon)(1+r)^\alpha t) r^{d-1} dr.\end{aligned}$$

Using the change of variable $\rho = t^{1/\alpha}(1+r)$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} \sum_{i=1}^{m} k_{ii}(t,x,x)dx &\ge m\,e^{-C_\varepsilon t}\gamma(t)\frac{1}{t^{\frac{d}{\alpha}}}\omega_d \int_{t^{\frac{1}{\alpha}}}^\infty e^{-(1-\varepsilon)\rho^\alpha}(\rho - t^{\frac{1}{\alpha}})^{d-1} d\rho \\ &\ge m\,e^{-C_\varepsilon t}\gamma(t)\frac{1}{t^{\frac{d}{\alpha}}}\omega_d \int_{t^{\frac{1}{\alpha}}}^\infty e^{-(1-\varepsilon)\rho^\alpha}\rho^{d-1} d\rho \\ &= \frac{m\,\omega_d}{(4\pi)^{\frac{d}{2}}} e^{-C_\varepsilon t}\frac{1}{t^{\frac{d}{\alpha}+\frac{d}{2}}}(1-e^{-\frac{1}{4t}}) \int_{t^{\frac{1}{\alpha}}}^\infty \rho^{d-1} e^{-(1-\varepsilon)\rho^\alpha} d\rho.\end{aligned}$$

Hence, for every $\varepsilon > 0$,

$$\liminf_{t\to 0} t^{d(\frac{1}{\alpha}+\frac{1}{2})} \int_{\mathbb{R}^d} \sum_{i=1}^{m} k_{ii}(t,x,x)dx \geq \frac{m\,\omega_d}{(4\pi)^{\frac{d}{2}}} \int_0^{\infty} \rho^{d-1} e^{-(1-\varepsilon)\rho^{\alpha}} d\rho.$$

Thus, by letting ε tends to 0, one gets

$$\liminf_{t\to 0} t^{d(\frac{1}{\alpha}+\frac{1}{2})} \int_{\mathbb{R}^d} \sum_{i=1}^{m} k_{ii}(t,x,x)dx \geq \frac{m\,\omega_d}{(4\pi)^{\frac{d}{2}}\,\alpha}\Gamma(\frac{d}{\alpha}),$$

which leads together with (4.5.4) and (4.5.2) to

$$\lim_{t\to 0} t^{d(\frac{1}{\alpha}+\frac{1}{2})} \sum_{n}^{\infty} e^{-\lambda_n t} = \frac{m\,\omega_d}{(4\pi)^{\frac{d}{2}}\,\alpha}\Gamma(\frac{d}{\alpha}).$$

Now, the claim follows applying [**62**, Theorem 10.3]. □

APPENDIX A

Semigroup theory

This appendix will cover the theory of semigroups and homogeneous linear evolution equation. All the result of this appendix are given without proofs, for more details we refer to [**23**], the book from which are taken most of content of this appendix. We start first by some notions of operator theory

A.1. Operator theory

Throughout X is a Banach space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. X is endowed by a norm $\|\cdot\|_X$ which will be denoted easily $\|\cdot\|$. Let A be a linear operator acting on a subspace $D(A)$ of X, called domain of A. We define $\|\cdot\|_A : D(A) \to \mathbb{R}^+$ given by

$$\|x\|_A := \|x\| + \|Ax\|.$$

The above norm is called graph norm of A. We now state some definitions

Definition A.1. Let $A : D(A) \subset X \to X$ be a linear operator. We adopt the following terminology

- A is called *densely defined* if $D(A)$ is dense in X.
- A is called *bounded* if A is densely defined and there exist $L \geq 0$ such that
$$\|Ax\| \leq L\|x\|, \qquad ,\forall x \in X.$$
In this case $D(A) = X$ and the graph norm $\|\cdot\|_A$ is equivalent to the space norm $\|\cdot\|$ and
$$\|A\| := \inf\{L \geq 0 : \|Ax\| \leq L\|x\|, \quad \forall x \in X\}$$
is called norm of the operator A.
- A is called *closed* if $D(A)$ endowed with $\|\cdot\|_A$ is a Banach space.

We state the graph theorem

Theorem A.2. [**12**, Théorème II.7] *Let $(A, D(A))$ be such that $D(A) = X$. Then, A is closed if, and only if, A is bounded.*

We now define the resolvent set of an operator A by

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A \quad \textit{is invertible and} \quad (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

For every $\lambda \in \rho(A)$, we denote $R(\lambda, A) := (\lambda - A)^{-1}$. The application $R(\cdot, A) : \rho(A) \to \mathcal{L}(X)$ is called resolvent of A. The spectrum of A is $\sigma(A) := \mathbb{C}\backslash\rho(A)$. The punctual spectrum $\sigma_p(A)$ constitutes of eigenvalues of A.

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda - A \quad \textit{is not injective}\}.$$

In other words, $\lambda \in \sigma_p(A)$ if, and only if, there exists $x \neq 0$ such that $Ax = \lambda x$. Now we give some properties related to the resolvent set and spectrum of a linear operator.

Proposition A.3. *Let $A, D(A) \subset X \to X$.*

- *If A is bounded then $\sigma(A)$ is a bounded subset of $\mathbb{C}$. Indeed, $\sigma(A) \subset B(0, \|A\|) := \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$.*
- *If $\rho(A) \neq \emptyset$, then A is closed.*
- *$\rho(A)$ is an open subset of $\mathbb{C}$.*
- *One has the following resolvent equation*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \qquad \forall \lambda, \mu \in \rho(A). \tag{A.1.1}$$

- *The resolvent application $R(\cdot, A) : \rho(A) \to \mathcal{L}(X)$ is holomorphic.*

Closable operators, Core of an operator. The closedness of an operator depends upon the 'chosen' domain. Here we introduce extension and restriction of operators

Definition A.4. Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be two operators. Then, B is said to be an extension of A if $D(A) \subset D(A)$ and $Bx = Ax$, for all $x \in D(A)$. We also say that A is the restriction of B on $D(A)$.

Now, we define closure of operators and core of an operator

Definition A.5. Let $A : D(A) \subset X \to X$ be an operator. Then,

- $(A, D(A))$ is said to be *closable* if it admits a closed extension $(B, D(B))$.
- If $(A, D(A))$ is closed then, the *closure* of A denoted by $(\bar{A}, D(\bar{A}))$ is the smallest closed extension of A.
- $C \subset D(A)$ is called *core* of $(A, D(A))$ if C is dense in $(D(A), \|\cdot\|_A)$.

Accretive and m-accrtive operators. We start by a definition

Definition A.6. Let $A : D(A) \subset X \to X$ be an operator.

- A is said to be accretive if, and only if,
$$\|(\lambda + A)u\| \geq \lambda\|u\|,$$
for all $u \in D(A)$ and $\lambda > 0$.
- A is said to be m-accretive (maximal accretive) if it is accretive and there exists $\lambda_0 > 0$ such that $(\lambda_0 + A)D(A) = X$.

- A is dissipative (resp. m-dissipative) if $-A$ is accretive (resp. $-A$ is m-accretive)

In order to characterize accretive operators, we first define the so-called *duality map*

Definition A.7. Let X be a Banach space and X^* its dual space. Assume that X^* is a uniformly convex Banach space. Let, for every $x \in X$, $F(x)$ be the unique element of X^* satisfying

$$\langle x, F(x)\rangle_{X,X^*} = \|x\|_X^2 = \|F(x)\|_{X^*}^2.$$

The map $F : X \to X^*$ is called *duality map*. In the case where X is a Hilbert space, then $X^* = X$ and $F(x) = x$, for all $x \in X$.

Now we are able to give a characterization of accretive operators

Proposition A.8. *Let $A : D(A) \subset X \to X$. A is accretive if and only if*

$$\langle Ax, F(x)\rangle_{X,X^*} \geq 0, \qquad \forall x \in D(A).$$

In particular, if X is a Hilbert space then, A is accretive if and only if $\langle Ax, x\rangle \geq 0$, for all $x \in D(A)$.

Compact operators. In the infinite dimensional case $\dim(X) = \infty$, bounded (continuous) operators are not necessarily compact.

Definition A.9. Let $T : X \to Y$ be a bounded linear operator, where Y is a Banach space. T is said to be compact if, and only if, $T(B)$ is relatively compact on Y, for every bounded subset B of X.
We denote by $\mathcal{K}(X, Y)$ the set of all compact operators.

$\mathcal{K}(X, Y)$ is a subspace of $\mathcal{L}(X, Y)$ and also ideal of it. Indeed, if $T \in \mathcal{K}(X, Y)$ and $L \in \mathcal{L}(Y)$ (resp. $L \in \mathcal{L}(X)$), then $LT \in \mathcal{K}(X, Y)$ (resp. $TL \in \mathcal{K}(X, Y)$). Moreover, $\mathcal{K}(X, Y)$ is a closed subspace $\mathcal{K}(X, Y)$.
We now introduce the notion of compact resolvent

Definition A.10. Let $(A, D(A))$ be a closed operator such that $\rho(A) \neq \emptyset$. A has a compact resolvent if, and only if, $R(\lambda, A)$ is a compact operator for every\some $\lambda \in \rho(A)$.

The equivalence between compactness of the resolvent for all $\lambda \in \rho(A)$ and for only some (at least one) $\lambda \in \rho(A)$ is due to the resolvent equation (A.1.1).
Here we state a characterization of compactness of the resolvent

Proposition A.11. *Let $(A, D(A))$ be a closed operator such that $\rho(A) \neq \emptyset$. The following are equivalent*

- *A has a compact resolvent,*

- *$D(A)$ is compactly embedded in X, i.e., the embedding\injection $i : (D(A), \|\cdot\|_A) \to (X, \|\cdot\|_A$ is compact.*

We end this section with a very powerful result which yields the spectrum of compact operators

Theorem A.12. [**12**, Théorème VI.8] *Let $T \in \mathcal{L}(X)$ be a compact operator. Then, the spectrum of T is either finite or countable discrete and accumulates at 0. Moreover, the spectrum contains eigenvalues only and $0 \in \sigma(A) \backslash \sigma_p(A)$.*

A.2. Strongly continuous semigroups

We give the background about semigroups of bounded operators acting on Banach spaces. We start by the definition of a semigroup

Definition A.13. Let $\{T(t) : t \geq 0\}$ be a family of linear bounded operators acting on X, i.e., $T(t) \in \mathcal{L}(X)$, for every $t \geq 0$. $\{T(t) : t \geq 0\}$ is called semigroup if, and only if,

i) $T(0) = \mathrm{I}$, where I is the identity operator.
ii) $T(t+s) = T(t) \circ T(s)$, for every $t, s \geq 0$.

Moreover, if

$$\lim_{t \to 0} T(t)x = x, \qquad \forall x \in X.$$

Then, $\{T(t) : t \geq 0\}$ is called strongly continuous semigroup or shortly C_0-semigroup.

Strongly continuous semigroups have at most exponential growth of order 1. That is

Proposition A.14. *Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup. Then, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq M e^{\omega t}, \tag{A.2.1}$$

for every $t \geq 0$. Consider

$$\omega_0(\mathcal{T}) := \inf\{\omega \in \mathbb{R} : \sup_{t \geq 0} e^{-\omega t} \|T(t)\| < \infty\}.$$

$\omega_0(\mathcal{T})$ is called type *of the semigroup $\mathcal{T}$.*
When $\omega = 0$ in (A.2.1)*, the semiroup $\mathcal{T}$ is then called* contractive *semigroup or semigroup of contraction.*

Now, we define the generator of a semigroup

Definition A.15. Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup. The infinitesimal generator $(A, D(A))$ of $\{T(t) : t \geq 0\}$ is defined in this way

$$Ax := \lim_{t \to 0} \frac{T(t)x - x}{t}, \qquad \forall x \in X.$$

defined on the domain

$$D(A) := \{x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists in } X\}.$$

From now on $\mathcal{T} = \{T(t) : t \geq 0\}$ is a strongly continuous semigroup and $(A, D(A))$ its generator. We endow $D(A)$ with the graph norm $\| \cdot \|_A : x \mapsto \|x\| + \|Ax\|$ and let (M, ω) be such that (A.2.1) is satisfied.

Proposition A.16. *One has the following properties*

a) $D(A)$ *is dense in* X*,*
b) A *is closed,*
c) $\mathbb{C}^+_\omega := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ *and*

$$\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re} \lambda - \omega}, \qquad \forall \lambda \in \mathbb{C}^+_\omega.$$

d)

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt, \qquad \forall \lambda \in \mathbb{C}^+_\omega, f \in X. \tag{A.2.2}$$

e)

$$T(t)f = \lim_{n \to \infty} [\frac{n}{t} R(\frac{t}{n}, A)]^n f, \qquad \forall t > 0, f \in X. \tag{A.2.3}$$

Here we state further properties

Proposition A.17. *One has the following*

a) $T(t)D(A) \subseteq D(A)$*, for every* $t \geq 0$ *and*

$$AT(t)x = T(t)Ax, \qquad \forall x \in D(A).$$

b) For every $x \in X$*,* $t \mapsto T(t)x$ *is continuously differentiable and*

$$\frac{d}{dt} T(t)x = AT(t)x.$$

c) For every $x \in D(A)$ *and* $t \geq 0$*, one has*

$$T(t)x - x = \int_0^t AT(s)x \, ds.$$

d) If $f \in D(A)$. Then, $u(\cdot) = T(\cdot)f$ is the unique strong solution of the evolution equation

$$\begin{cases} u'(t) = (\mathcal{A}u)(t), & t \geq 0 \\ u(0) = f. \end{cases} \tag{A.2.4}$$

Lummer-Phillips theorem. The Lumer-phillips theorem characterizes generators of strongly continuous semigroups. C_0-semigroups appear naturally as solutions of evolution equations of the form (A.2.4). So in order to solve such an equation one needs to show that the operator say A generates strongly continuous semigroup. We now state Lumer-Phillips theorem

Theorem A.18. [**23**, Chap II, Theorem 3.8] *Let $M \geq 1$, $\omega \in \mathbb{R}$ and $A : D(A) \subset X \to X$. The following are equivalent*

- *$(A, D(A))$ generates a strongly continuous semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ satisfying*
$$\|T(t)\| \leq Me^{\omega t}, \qquad \forall t \geq 0.$$
- *$A : D(A) \subset X \to X$ is closed, densely defined such that $(\omega, \infty) \subset \rho(A)$ and*
$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \qquad \forall n \in \mathbb{N}, \lambda \in (\omega, \infty).$$

As a consequence of this theorem, one has

Corollary A.19. *Assume that $(A, D(A))$ is a densely defined m-dissipative operator. Then, A generates a contractive strongly continuous semigroup.*

We end this section by a Theorem due to N. Okazawa that yields m-dissipativity, then generation of contractive strongly-continuous semigroup, of unbounded perturbation of an m-disspipative operator

Theorem A.20. [**53**, Theorem 1.6] *Let A and B be linear m-accretive operators on X with uniformly convex X^*. Let D be a core for A. Assume that there are nonnegative constants c, a and b such that for all $u \in D$ and $\varepsilon > 0$,*

$$\operatorname{Re} \langle Au, F(B_\varepsilon u)\rangle \geq -c\|u\|^2 - a\|B_\varepsilon u\|\|u\| - b\|B_\varepsilon u\|^2, \tag{A.2.5}$$

where $B_\varepsilon := B(I + \varepsilon B)^{-1}$ denotes the Yosida approximation of B. If $t > b$ then $A + tB$ with domain $D(A) \cap D(B)$ is m-accretive and $D(A) \cap D(B)$ is a core for A. Furthermore, $A + bB$ is essentially m-accretive on $D(A) \cap D(B)$.

A.3. Analytic semigroups

In this section we introduce the notion of analytic (holomorphic) semigroups. We start by defining such semigroups

Definition A.21. Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup. $\mathcal{T}$ is said to be analytic if there exists $\delta \in (0, \pi/2]$ such that $\mathcal{T}$ admits a holomrphic extension $\{T(z) : z \in S_\delta\}$ on S_δ satisfying

- $T(z_1 + z_2) = T(z_1) \circ T(z_2)$, for all $z_1, z_2 \in S_\delta$ such that $z_1 + z_2 \in S_\delta$,
- $\lim\limits_{z \to 0, z \in S_{\delta'}} T(z)x = x$, for all $x \in X$ and all $\delta' \in [0, \delta)$.

In this case we say that $\mathcal{T}$ is analytic of angle δ.

We now characterize generators of analytic semigroups

Proposition A.22. *Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be an analytic semigroup of angle $\delta \in (0, \pi/2]$ and $(A, D(A))$ its generator. Assume furthermore that*

$$M_{\delta'} := \sup_{z \in S_{\delta'}} \|T(z)\| < \infty, \tag{A.3.1}$$

for every $\delta' \in (0, \delta)$. Then, $-A$ is a sectorial operator of angle $\pi/2 + \delta$. Conversely, assume that $-A$ is sectorial of angle $\varphi > \pi/2$. Then, $\mathcal{T} = \{T(t) : t \geq 0\}$ is analytic of angle $\varphi - \pi/2$ and satisfies (A.3.1) *for every $\delta' < \varphi - \pi/2$.*

The definition of sectorial operators is given in Appendix C. We choose to use the definition of sectorial operators used in topics of functional calculus and harmonic analysis that is why we did not give any definition here in the appendix about semigroup theory to not confuse the reader, since usually sectorial operators for a man of semigroup intends generator of holomorphic semigroup which is incompatible with one we use in this manuscript. For us, when A is sectorial, in general, neither A nor $-A$ generate analytic semigroup, see for instance SubsectionD.1 of Appendix D.
Here we state another series of equivalence for setorial operators

Theorem A.23. [**23**, Chap II, Theorem 4.6] *Let $A : D(A) \subset X \to X$ be a closed operator. The following are equivalent*

a) $-A$ is sectorial with angle of sectoriality $\varphi := \pi/2 + \delta$ with $\delta > 0$.
b) For every $\nu \in (-\delta, \delta)$, $e^{i\nu}A$ generates a bounded strongly continuous semigroup on X.
c) A generates a bounded strongly continuous semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ which satisfies $\mathcal{R}(T(t)) \subset D(A)$, for all $t > 0$ and

$$M := \sup_{t>0} t\|AT(t)\| < \infty.$$

d) A generates a bounded strongly continuous semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ and it exists $L \geq 0$ such that

$$\|R(r+is, A)\| \leq \frac{L}{|s|}, \tag{A.3.2}$$

for all $r > 0$ and $s \in \mathbb{R}\backslash\{0\}$.

Compact semigroups. Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup on a Banach space X.

Definition A.24. The semigroup $\mathcal{T}$ is said to be compact if, and only if, $T(t) \in \mathcal{K}(X)$, for every $t > 0$.

Proposition A.25. *Assume that $\mathcal{T}$ is compact, then its generator $(A, D(A))$ has a compact resolvent. The reverse implication holds true when $\mathcal{T}$ is analytic.*

As a consequence of TheoremA.12, one has

Theorem A.26. *Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup and $(A, D(A))$ its generator. Assume that, A has compact resolvent, then the spectrum of A is punctual, discrete, countable and accumulates at $-\infty$. In other words, $\sigma(A) = \sigma_p(A) = \{\lambda_n : n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} \lambda_n = -\infty$.*

A.4. Positive semigroups

In order to study positivity of semigroups in a Banach space X, one has to endow X by an order. Moreover, such an order needs to be compatible with the structure of vector space and some other properties are required for the order Banach space $(X, \leq)$. In literature such Banach spaces are called *Banach lattices*, see [**51**, Part C]. Here we give its definition

Definition A.27. Let $(X, \leq)$ be a partially ordered Banach space. $(X, \leq)$ is called a Banach lattice if, and only if,

- $f \leq g$ implies $f + h \leq g + h$, for all $h \in X$;
- $f \geq 0$ implies $\lambda f \geq 0$, for every $\lambda \geq 0$;
- For all $f, g \in X$, the supremum $\sup(f, g)$, denoted also $f \vee g$, is defined (in some sens) and belongs to X and satisfy $f \vee g$ is equal to g (resp., to f) when $f \leq g$ (resp., when $g \leq f$);
- $|f| = \sup(f, 0)$ belongs to X, for every $f \in X$;
- $|f| \leq |g|$ implies $\|f\| \leq \|g\|$, for all $f, g \in X$.

Once an order $\leq$ is defined on X, its dual X^* is endowed by the following order

$$x^*, y^* \in X^*, x^* \leq y^* \Leftrightarrow \langle f, x^* \rangle_{X,X^*} \leq \langle f, x^* \rangle_{X,X^*}, \qquad \forall 0 \leq f \in X.$$

We define positive operators on X

Definition A.28. Let $T \in \mathcal{L}(X)$. T is said to be positive if, and only if, $0 \leq f$ implies $0 \leq Tf$, for every $f \in X$.

Now we define positive semigroups.

Definition A.29. Let $(X, \leq)$ be a Banach lattice and $\mathcal{T} = \{T(t) : t \geq 0\}$ a strongly continuous semigroup on X. $\mathcal{T}$ is said to be positive if, and only if, $T(t)$ is a positive operator for every $t \geq 0$.

Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup and $(A, D(A))$ its generator. Taking into account (A.2.3) and (A.2.2), one obtains the following

Proposition A.30. *The semgroup $\mathcal{T}$ is positive if, and only if, $R(\lambda, A)$ is positive for every $\lambda > \omega_0(\mathcal{T})$*

Now, we state a very useful result which yields a necessary condition for positivity of semigroups, the so-called *positive minimum principle*

Theorem A.31. [**51**, Chap C-II, Proposition 1.7] *Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup and $(A, D(A))$ its generator. Assume that $\mathcal{T}$ is positive. Then, A satisfies the positive minimum principle, i.e.,*

$$0 \leq f \in D(A), 0 \leq f^* \in X^*, \quad \langle f, f^* \rangle_{X,X^*} = 0 \Longrightarrow \langle Af, f^* \rangle_{X,X^*} \geq 0 \tag{A.4.1}$$

APPENDIX B

Sesquilinear forms

In this appendix we give the essential background about the theory of sesquilinear forms in Hilbert spaces. All terminology and results announced in this appendix are taken from the book by Ouhabaz [**54**] where one can find more details an deep study of the topic of sesquilinear forms and application to elliptic equations.

B.1. Definition and properties of sesquilinear form

Let H be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and $\mathcal{V}$ a linear subspace of H. Let us denote by $(\cdot, \cdot)$ the inner product of H and by $\|\cdot\|$ its corresponding norm. A *sesquilinear form* is a map

$$\mathfrak{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

which satisfies, for every $\alpha \in \mathbb{C}$ and $u, v, w \in \mathcal{V}$, the folowing

$$\mathfrak{a}(\alpha u + v, w) = \alpha \mathfrak{a}(u, w) + \mathfrak{a}(v, w) \quad \text{and} \quad \mathfrak{a}(u, \alpha v + w) = \overline{\alpha} \mathfrak{a}(u, v) + \mathfrak{a}(u, w).$$

The space $\mathcal{V}$ in which $\mathfrak{a}$ is defined is called domain of $\mathfrak{a}$ and denoted $D(\mathfrak{a}) = \mathcal{V}$. The quadratic form associated to $\mathfrak{a}$ is given by $D(\mathfrak{a}) \ni u \mapsto \mathfrak{a}(u) := \mathfrak{a}(u, u)$ the associated quadratic form. If $\mathfrak{a}$ is a positive form, i.e.,

$$\text{Re}\, \mathfrak{a}(u) \geq 0 \text{ for all } u \in D(\mathfrak{a}). \tag{B.1.1}$$

we also say that $\mathfrak{a}$ is *accretive*. In this case $\|\cdot\|_{\mathfrak{a}} : D(\mathfrak{a}) \to [0, \infty)$, defined by

$$\|u\|_{\mathfrak{a}} := \sqrt{\text{Re}\, \mathfrak{a}(u) + \|u\|^2}, \qquad \forall u \in D(\mathfrak{a}),$$

is a norm on $D(\mathfrak{a})$ and it derives from the inner-product

$$(u, v)_{\mathfrak{a}} := (u, v) + \frac{1}{2}\{\mathfrak{a}(u, v) + \overline{\mathfrak{a}(v, u)}\}, \qquad u, v \in D(\mathfrak{a}).$$

Definition B.1. Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$ be a sesquilinear form. We say that

(a) $\mathfrak{a}$ is *densely defined* if

$$D(\mathfrak{a}) \text{ is dense in } H. \tag{B.1.2}$$

(b) $\mathfrak{a}$ is *continuous* if there exists a non-negative constant M such that

$$|\mathfrak{a}(u, v)| \leq M \|u\|_{\mathfrak{a}} \|v\|_{\mathfrak{a}} \text{ for all } u, v \in D(\mathfrak{a}) \tag{B.1.3}$$

where $\|u\|_{\mathfrak{a}} := \sqrt{\text{Re}\, \mathfrak{a}(u) + \|u\|^2}$.

(c) $\mathfrak{a}$ is *closed* if

$$(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}) \text{ is a complete space.} \tag{B.1.4}$$

Similarly to linear operators, the closedness of form depends upon its domain of definition. Hence, if a form is not closed one asks if it has a closed extension and how to define its closure, that shall be the smallest closed extension.

Definition B.2. Let $(\mathfrak{a}, D(\mathfrak{a}))$ be a continuous accretive sesquilinear form. $\mathfrak{a}$ is said to be *closable* if there exists a closed accretive form $\mathfrak{b} : D(\mathfrak{b}) \subseteq H \to H$ such that $D(\mathfrak{a}) \subseteq D(\mathfrak{b})$ and $\mathfrak{b}(u,v) = \mathfrak{a}(u,v)$ for all $(u,v) \in D(\mathfrak{a})$.

When, an accretive form $\mathfrak{a}$ is not closed, this means that its domain $D(\mathfrak{a})$ endowed with $\|\cdot\|_{\mathfrak{a}}$ is not a complete space. Hence, one thinks to 'extend' $D(\mathfrak{a})$ in order to get complete superspace. This is the idea behind the construction of the closure of a closable form.

Definition B.3. Let $\mathfrak{a}$ be an accretive continuous closable form. We define *closure* $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ as follows

$$D(\overline{\mathfrak{a}}) = \{u \in H | \exists (u_n)_n \subset D(\mathfrak{a}) : \lim_{n\to\infty} u_n = u, \lim_{n,m\to\infty} \mathfrak{a}(u_n - u_m) = 0\},$$

and

$$\overline{\mathfrak{a}}(u,v) := \lim_{n\to\infty} \mathfrak{a}(u_n, v_n),$$

for $u, v \in D(\overline{\mathfrak{a}})$, and $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are any Cauchy sequences in $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ and converge respectively to u and v in H.

One has the following.

Proposition B.4. *Let $\mathfrak{a}$ be an accretive continuous sesquilinear form. If $\mathfrak{a}$ is closable, then $\overline{\mathfrak{a}}$ satisfies* (B.1.1)-(B.1.4).

We define a *core* of a densely defined accretive sesquilinear form as follow

Definition B.5. Let $(\mathfrak{a}, D(\mathfrak{a}))$ be a densely defined accretive sesquilinear form and $\mathcal{V}_0 \subseteq D(\mathfrak{a})$. $\mathcal{V}_0$ is said to be a *core* for $\mathfrak{a}$ if, and only if, $\mathcal{V}_0$ is dense in $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$.

B.2. Associated operator

Now, for a given densely defined sesquilinear form $(\mathfrak{a}, D(\mathfrak{a}))$, we want to associate a linear operator $(A, D(A))$ on H. The construction of such operator, called *associated operator to the form* $\mathfrak{a}$, is the following

$$D(A) := \{u \in H | \ \exists v \in H : \mathfrak{a}(u,\phi) = (v,\phi), \ \forall \phi \in D(\mathfrak{a})\}, \quad Au := v.$$

With this construction, one has $\mathfrak{a}$ is accretive if and only if A is accretive.
The mean result of sesquilinear form theory consists on generation of analytic

semigroup for the operator associated to a form $\mathfrak{a}$. Actually, the form method is the easiest way to get generation of analytic semigroup. The result is formulated as follow

Theorem B.6. *Let* $(\mathfrak{a}, D(\mathfrak{a}))$ *be a sesquilinear form satisfying* (B.1.1)–(B.1.4). *Let* $(A, D(A))$ *be its associated operator. Then,* $-A$ *generates a contractive strongly continuous semigroup* $\mathcal{T} = \{T(t) : t \geq 0\}$ *on* H*. Moreover,* $\mathcal{T}$ *has an extension to a holomorphic semigroup on the sector* $S_{\frac{\pi}{2}-\arctan M}$*, where* M *is the constant appearing in the continuity assumption* (B.1.3).

B.3. Beurling-Deny condition and submarkovian semigroups

Throughout this section $H = L^2(\mathbb{R}^d, \mathbb{C}^m)$ is the Hilbert space of complex vector-valued functions with square integrable norm.

$$H := \{f = (f_1, \dots, f_m) : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} \sum_{j=1}^m |f_j|^2 \, dx < \infty\}.$$

The inner product of H is given by

$$(f, g) := \int_{\mathbb{R}^d} \langle f, \bar{g} \rangle \, dx = \int_{\mathbb{R}^d} \sum_{j=1}^m f_j \bar{g}_j \, dx, \qquad \forall f, g \in H.$$

Let $(\mathfrak{a}, D(\mathfrak{a}))$ be a sesquilinear form over H satisfying (B.1.1)–(B.1.4), $(A, D(A))$ its associated operator and $\mathcal{T} = \{T(t) : t \geq 0\}$ the semigroup generated by $-A$.

Consider $\mathcal{C}$ be a closed convex subset of H. Let $P : H \to \mathcal{C}$ the projection on $\mathcal{C}$. We recall that such a projection is uniquely defined and satisfies $(Pu, u - Pu) = 0$, for every $u \in H$. We aim to characterize the invariance of $\mathcal{C}$ under the semigroup $\mathcal{T}$ via the associated form.

Definition B.7. $\mathcal{C}$ is invariant under $\mathcal{T}$ if $T(t)\mathcal{C} \subset \mathcal{C}$, for every $t \geq 0$.

The 'generalized' *Beurling-Deny* characterizes this invariance as follow

Theorem B.8. [**54**, Theorem 2.2] *The following are equivalent*

a) $\mathcal{C}$ *is invariant under* $\mathcal{T}$.
b) $P(D(\mathfrak{a})) \subset D(\mathfrak{a})$ *and* $\operatorname{Re} \mathfrak{a}(Pu, u - Pu) \geq 0$*, for every* $u \in D(\mathfrak{a})$.
c) There exists a core $\mathcal{V}_0$ *of* $\mathfrak{a}$ *such that* $P(\mathcal{V}_0) \subset D(\mathfrak{a})$ *and* $\operatorname{Re} \mathfrak{a}(Pu, u - Pu) \geq 0$*, for every* $u \in \mathcal{V}_0$.

In the case where $\mathfrak{a}$ *is symmetric, the above items are equivalent to*

$$P(D(\mathfrak{a})) \subset D(\mathfrak{a}) \quad \text{and} \quad \mathfrak{a}(Pu) \leq \mathfrak{a}(u) \quad \text{for every} \quad u \in D(\mathfrak{a}).$$

As a consequence of the above theorem, one characterizes positive semigroups on $H = L^2(\mathbb{R}^d, \mathbb{C}^m)$ considering

$$\mathcal{C}^+ := \{f = (f_1, \dots, f_m) \in L^2(\mathbb{R}^d, \mathbb{R}^m) : f_j \geq 0, \quad \forall 1 \leq j \leq m\},$$

and $P^+ f = f^+ := (f_j^+)_{1 \leq j \leq m}$, for all $f \in L^2(\mathbb{R}^d, \mathbb{R}^m)$.

Corollary B.9. *$\mathcal{T}$ is a positive semigroup if and only if*

- *$f^+ \in D(\mathfrak{a})$, for every $f \in D(\mathfrak{a})$.*
- *$\mathfrak{a}(f^+, f^-) \geq 0$, for every $f \in D(\mathfrak{a})$.*

Now, we will characterize the L^∞-contractivity of the semigroup $\mathcal{T}$, that is

$$\|T(t)f\|_\infty \leq \|f\|_\infty, \qquad \forall f \in L^2(\mathbb{R}^d, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^m).$$

Consider $C_\infty := \{f = (f_1, \dots, f_m) \in L^2(\mathbb{R}^d, \mathbb{R}^m) : |f| := \sqrt{\sum_{j=1}^m f_j^2} \leq 1\}$. The projection P_∞ on C_∞ is given by

$$P_\infty f = (1 \wedge |f|) sign(f), \qquad \forall f \in L^2(\mathbb{R}^d, \mathbb{R}^m).$$

Where $sign(f) = \frac{f}{|f|} \chi_{\{f \neq 0\}}$. One has

Lemma B.10. *$\mathcal{T}$ is L^∞-contractive if, and only if, C_∞ is invariant under $\mathcal{T}$.*

This lemma together with Theorem B.8 yield

Corollary B.11. *$\mathcal{T}$ is L^∞-contractive if, and only if,*

- *$(1 \wedge |f|) sign(f) \in D(\mathfrak{a})$, for all $f \in D(\mathfrak{a})$.*
- *$\mathfrak{a}\left((1 \wedge |f|) sign(f), f - (1 \wedge |f|) sign(f)\right) \geq 0$, for all $f \in D(\mathfrak{a})$.*

In the case where the form $\mathfrak{a}$ is symmetric, the above is equivalent to

- *$f \in D(\mathfrak{a})$ implies $(1 \wedge |f|) sign(f) \in D(\mathfrak{a})$ and*
- *$\mathfrak{a}((1 \wedge |f|) sign(f) \in D(\mathfrak{a})) \leq \mathfrak{a}(f)$, for all $f \in D(\mathfrak{a})$.*

Semigroups in $L^2(\mathbb{R}^d, \mathbb{R}^m)$ which are positive and L^∞-contractive are called *submarkovian semigroups.*

APPENDIX C

Functional calculus for sectorial operators

In this appendix we collect the background about functional calculus for an unbounded 'sectorial operator'. What we mean by functional calculus for a given operator A is the definition and study of the family of operators $f(A)$, where f can be a holomorphic function. In the case where A is a bounded operator on a Banach space X, $f(A)$ is well defined for all complex functions f which are holomorphic on a neighborhood of the (finite) spectrum of A. $f(A)$ is defined by the following Cauchy formula

$$f(A) = \frac{1}{2\pi i} \int_{\gamma^+} f(z)(z - A)^{-1} dz, \tag{C.0.1}$$

where $\gamma = \partial\Omega$ is the boundary of Ω, a bounded open subset of $\mathbb{C}$ which satisfies $\sigma(A) \subset \Omega$; γ^+ indicates the positive orientation (anti-clockwise) of the path γ. As the spectrum of a unbounded operator is (in general) unbounded, the question which arise is how to extend (C.0.1) to closed unbounded A's?

Throughout this Appendix, X is a complex Banach space and A is a closed densely defined operator on X with domain $D(A)$.

All results, definitions and notation which are given in this appendix are taken from [**29**], especially chapters 2,3 and 5. Elsewhere, the reference will be cited together with the result.

C.1. Sectorial operators and natural functional calculus

We begin by defining sectorial operators; We adopt the definitions and notations of [**29**]. For $\theta \in (0, \pi)$, we define the sector

$$S_\theta = \{z \in \mathbb{C} : |\arg(z)| < \theta\}$$

and $S_0 = (0, \infty)$.

Definition C.1. $A : D(A) \subseteq X \to X$ is called sectorial of angle $\theta \in [0, \pi)$ if

(i) $\sigma(A) \subseteq \bar{S}_\theta$,

(ii) There exists $M > 0$ such that $\|\lambda R(\lambda, A)\| \leq M$, for all $\lambda \notin \bar{S}_{\theta'}$, for each $\theta' \in (\theta, \pi)$.

Remark C.2. The above definition does not mean that $-A$ generates an analytic semigroup in X, unless the angle $\theta < \frac{\pi}{2}$. This definition is incompatible with the one of [**23**, Definition II-4.1], where the sectorial operators are exactly generators of bounded holomorphic semigroups. Moreover, closed operators satisfying Hille-Yosida conditions, see Appendix A, are sectorial as shows the following proposition

Proposition C.3. *Let $A : D(A) \subset X \to X$ be a closed operator such that*

- $(-\infty, 0) \subset \rho(A)$,
- $M = \sup_{t>0} \|t(t+A)^{-1}\| < \infty$.

Then A is sectorial of angle $\theta = \pi - \arcsin(M^{-1})$.

Notation. Let A be a sectorial operator of angle $\omega \in [0, \pi)$ and fix $\varphi \in (\omega, \pi)$. We adopt the following notation: $\mathcal{O}(S_\varphi)$ (resp. $\mathcal{M}(S_\varphi)$) denotes the space of all holomorphic (resp. meromorphic) function over S_φ, $\mathcal{H}^\infty(S_\varphi)$ the set of all bounded holomorphic function on S_φ. We also introduce an intermediate space between $\mathcal{O}(S_\varphi)$ and $\mathcal{H}^\infty(S_\varphi)$, this space is called the **Dunford Riesz Class**, denoted by $\mathcal{H}_0^\infty(S_\varphi)$ and is defined by

$$\mathcal{H}_0^\infty(S_\varphi) := \{f \in \mathcal{O}(S_\varphi) : f \textit{ is regularly decaying decay at } 0 \textit{ and } \infty\}.$$

A function $f \in \mathcal{H}^\infty(S_\phi)$ is said to be *regularly decaying* at 0 if $|f(z)| = o(|z|^\alpha)$ when $z \to 0$, for some $\alpha > 0$. Similarly, we say that f is *regularly decaying* at ∞ if $|f(z)| = o(\frac{1}{|z|^\alpha})$ for $|z| \to \infty$, for some $\alpha > 0$.
Now, we define $f(A)$, for $f \in \mathcal{H}_0^\infty(S_\varphi)$ by the **Dunford-Riesz integral**

Definition C.4. Let $f \in \mathcal{H}_0^\infty(S_\varphi)$. We define $f(A) \in \mathcal{L}(X)$ by

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma_\varphi^+} f(z)(z-A)^{-1} dz, \tag{C.1.1}$$

where $\Gamma_\varphi = \partial S_\varphi$ is the boundary of S_φ and Γ_φ^+ indicates the positive orientation of Γ_φ.

Here is a characterization of $\mathcal{H}_0^\infty(S_\varphi)$

Proposition C.5. *Let $f \in \mathcal{O}(S_\varphi)$. Then, $f \in \mathcal{H}_0^\infty(S_\varphi)$ if, and only if, one the following holds true*

a) There exist $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \min(|z|^\alpha, |z|^{-\alpha})$, for all $z \in S_\varphi$.
b) There exist $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \frac{|z|^s}{1+|z|^{2s}}$, for all $z \in S_\varphi$.

Remark C.6. The estimate b) of the above proposition together with the sectoriality inequality yield the integrability of $z \mapsto f(z)(z-A)^{-1}$. Thus, the Dunford-Riesz integral (C.1.1) is meaningful. This is why $\mathcal{H}_0^\infty(S_\varphi)$ is called the **Dunford-Riesz** class.

We introduce the space $\mathcal{E}(S_\varphi)$ as the smallest subspace of $\mathcal{O}(S_\varphi)$ containing $\mathcal{H}_0^\infty(S_\varphi)$, $z \mapsto (\lambda+z)^{-1}$, $\lambda > 0$ and the constant functions. If $f \in \mathcal{E}(S_\varphi)$ such that $f(z) = g(z) + a(\lambda+z)^{-1} + bz$, where $g \in \mathcal{H}_0^\infty(S_\varphi)$, $a, b \in \mathbb{R}$ and $\lambda > 0$. Then, $f(A)$ can be defined as

$$f(A) := g(A) + a(\lambda + A)^{-1} + bA.$$

Now, we define the space of regularisable functions as the greatest space where one can define functional calculus for the operator A. This space depends on A as shows the below definition

Definition C.7. The set of regularisable meromorphic functions is defined by (C.1.2)

$$\mathcal{M}_A(S_\varphi) := \{f \in \mathcal{M}(S_\varphi) | \exists e \in \mathcal{E}(S_\varphi) : e(A) \textit{ is injective and } \quad ef \in \mathcal{E}(S_\varphi)\}.$$

If $f \in \mathcal{M}_A(S_\varphi)$ and e as above, then

$$f(A) := (e(A))^{-1}(ef)(A).$$

The function e is a kind of regularising function for f.

Remark C.8. The functional set $\mathcal{M}_A(S_\varphi)$ depends upon A. In the case where A is injective, $\mathcal{M}_A(S_\varphi)$ contains $\mathcal{H}^\infty(S_\varphi)$ and all powers $z \mapsto z^\alpha$, $\alpha \in \mathbb{R}$. Hence, one could define, for injective A, A^α and $f(A)$, for every bounded holomorphic f. The following two sections yield more details on those topics.

Composition Rule. Let us assume that A is injective. According to [**29**, Proposition 2.4.1] and since A sectorial, A^{-1} is also sectorial with the same angle. One has the following

Proposition C.9. *Let* $f \in \mathcal{M}(S_\varphi)$. *Then,*

$$f \in \mathcal{M}(S_\varphi)_{A^{-1}} \Longleftrightarrow f(z^{-1}) \in \mathcal{M}(S_\varphi)_A.$$

In this case, $f(A^{-1}) = f(z^{-1})(A)$.

We intend by *composition rule* the formula $(g \circ f)(A) = g(f(A))$. The theorem related to *composition rule* is the following

Theorem C.10. [**29**, Theorem 2.4.2]
Assume the following are satisfied

- $f \in \mathcal{M}(S_\omega)_A$ *and* $f(A)$ *is sectorial of angle* ω'.
- $g \in \mathcal{M}(S_{\omega'})_{f(A)}$.
- $f(S_\omega) \subset \bar{S}_{\omega'}$.

Then $g \circ f \in \mathcal{M}(S_\omega)_A$ *and*

$$(g \circ f)(A) = g(f(A)).$$

C.2. Bounded H^∞-functionnal calculus

In this section we define $f(A)$ for bounded holomorphic functions f and injective operators A. Throughout, assume that A is injective.

Proposition C.11. *Let $f \in \mathcal{H}^\infty(S_\varphi)$. Then, $f \in \mathcal{M}_A(S_\varphi)$.*

Proof. Let $e(z) = z(1+z^2)^{-1}$. One has $e(A)$ is injective, $e(A)^{-1} = (1 + A^2)A^{-1}$ and, by Proposition C.5, $ef \in \mathcal{H}_0^\infty(S_\varphi) \subset \mathcal{E}(S_\varphi)$. Thus, $f \in \mathcal{M}_A(S_\varphi)$ and

$$f(A) = (1 + A^2)A^{-1}(ef)(A).$$

Note that since $ef \in \mathcal{H}_0^\infty(S_\varphi)$, then $(ef)(A) = \left(z \mapsto \frac{zf(z)}{1+z^2}\right)(A)$ can be defined by the Dunford Riesz integral (C.1.1). □

Now, we define what is boundedness of H^∞-functional calculus

Definition C.12. Assume that A is injective. A admits bounded H^∞-functional calculus on S_φ if, and only if, $f(A) \in \mathcal{L}(X)$ and there exists $C_\varphi > 0$ such that

$$\|f(A)\| \le C_\varphi \|f\|_\varphi,$$

for all $f \in \mathcal{H}^\infty(S_\varphi)$. Where, $\|f\|_\varphi = \sup_{z \in S_\varphi} |f(z)|$.

Example 1: Elliptic systems. Let $p \in (1,\infty)$ and E_p the L^p-realization of the elliptic differential operator E given by

$$(Eu)(x) = \sum_{i,j=1}^{m} a_{ij}(x) D_{ij} u\,(x) + \sum_{i=1}^{m} B_j(x)\partial_j u\,(x) + C(x)u(x), \tag{C.2.1}$$

where $u = (u_1, \dots, u_m) : \mathbb{R}^d \to \mathbb{R}^m$ is smooth enough. Assume the coefficients of the differential operator E satisfy the following:

- For every $i,j \in \{1,\dots,m\}$, $a_{ij} = a_{ji}$ and there exists $0 < \eta_1 < \eta_2$ such that
$$\eta_1 |\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \le \eta_2 |\xi|^2,$$
for all $\xi \in \mathbb{R}^m$ and $x \in \mathbb{R}^d$.
- For every $i,j \in \{1,\dots,m\}$, $a_{ij} \in BUC(\mathbb{R}^d)$.
- For every $j \in \{1,\dots,m\}$, $B_j, C \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m\times m})$.

Proposition C.13. [**21**, Theorem 6.1]
Assume the above hypotheses are satisfied. Then, there exists $s \in \mathbb{R}$ such that $s - E_p$ admits a bounded H^∞-functional calculus on $S_{\pi-\varepsilon}$, for every $\varepsilon > 0$.

Remark C.14. The choice of the constant s in the above proposition is to obtain injectivity of $s - E_p$. If E is written in divergence form and $B_j = C = 0$, for all $j \in \{1,\dots,m\}$, Then $\varepsilon - E_p$ has H^∞-bounded functional calculus for every $\varepsilon > 0$.

Example 2: m-accretive operators in Hilbert spaces. Let $X = H$ be a Hilbert space and $A : D(A) \subset H \to H$ be a m-accretive operator, i.e., $\mathcal{R}(1 + A) = H$ and

$$\langle Ax, x\rangle \geq 0, \qquad \forall x \in H.$$

According to [**29**, Proposition 2.1.1], A is sectorial of angle $\frac{\pi}{2}$. Assume that A is injective. One has the following

Proposition C.15. [**29**, Corollary 7.1.8]
A admits H^∞-bounded functional calculus on $S_{\frac{\pi}{2}}$. Moreover,

$$\|f(A)\| \leq \|f\|_{\frac{\pi}{2}}, \tag{C.2.2}$$

for every $f \in \mathcal{H}^\infty(S_{\frac{\pi}{2}})$.

Remark C.16. For m-accretive operators on Hilbert spaces, it was firstly established boundedness of imaginary powers by Prüss and Sohr, see [**58**, Example 2]. Later on, in [**15**, Theorem 2.4], the authors show the equivalence between boundedness of H^∞-calculus and of imaginary powers in Hilbert spaces. Which yields the result of the above proposition.

C.3. Fractionnal powers

Here we give definition of A^α, $\alpha \in \mathbb{C}$, for injective A. To do so, we first define real powers A^α, $\alpha \in \mathbb{R}$ and imaginary powers A^{is}, $s \in \mathbb{R}$ and by properties of functional calculus one $A^\alpha = A^{\mathrm{Re}\,\alpha} A^{i\,\mathrm{Im}\,\alpha}$. Since, for every $s \in \mathbb{R}$, $z \mapsto z^{is}$ is bounded in sectors, the imaginary power A^{is} is given by Section C.2.

Assume that A is injective. Let us define A^α for $\alpha \in \mathbb{R}$.

Definition C.17. Let $\alpha \in \mathbb{R}$.

a) If $\alpha > 0$, $z \mapsto z^\alpha$ belongs to $\mathcal{M}_A(S_\varphi)$. If $n \in \mathbb{N}$ is such that $n > \alpha$, then

$$A^\alpha := (1 + A)^n \left(z \mapsto \frac{z^\alpha}{(1+z)^n}\right)(A).$$

b) If $\alpha < 0$, A^α is given by the composition rule $A^\alpha := (A^{-1})^{-\alpha}$.

One has the following properties of exponents, see [**29**, Chapter 3],

Proposition C.18. *Let $\alpha, \beta \in \mathbb{R}$. One has*

(1) $A^{\alpha+\beta} = A^\alpha A^\beta$.
(2) If $\beta\omega < \pi$. Then, $A^{\alpha\beta} = (A^\alpha)^\beta$.

Earlier definition of fractional powers was throughout the so-called Balakrishnan formula, see [**9**] or [**29**, Proposition 3.1.12],

$$A^\alpha x = \frac{sin(\alpha\pi)}{\pi} \int_0^\infty t^{\alpha-1}(t + A)^{-1} Axdt, \tag{C.3.1}$$

for every $\alpha \in \mathbb{C}$ such that $0 < \operatorname{Re}\alpha < 1$.

Another way to express A^α is the Komatsu representation, see [**39**] or [**29**, Proposition 3.2.2]. Once A is injective and $-1 < \operatorname{Re}\alpha < 1$, then

(C.3.2)
$$A^\alpha x = \frac{sin(\alpha\pi)}{\pi}\left(\frac{1}{\alpha}x + \frac{1}{1+\alpha}A^{-1}x + \int_0^1 t^{1+\alpha}(t+A)^{-1}A^{-1}x\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt\right)$$

Finally, one has the following

Proposition C.19. [**29**, Corollary 3.3.6] *Assume $-A$ generates an exponentially stable semigroup $\{T(t)\}_{t\geq 0}$. Then*

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}T(t)x\,dt,$$

for all $x \in X$ and all $\operatorname{Re}(\alpha) > 0$.

C.3.1. Boundedness of imaginary powers. Imaginary powers in $\mathcal{UMD}$ Banach spaces appear essentially in theorems dealing with maximal regularity of a perturbed problems. Dore-*Venni theorem is an example, see [**20**], [**50**]. $\mathcal{UMD}$ Banach spaces or also Banach spaces of $\mathcal{HT}$ class refer to Banach spaces with bounded *Hilbert Transform*.

Definition C.20. [**14**] Let Y be a Banach space. Y is said to be of $\mathcal{HT}$ class if, and only if,

$$H : f \to Hf(t) := \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{s\geq\varepsilon}\frac{1}{s}f(t-s)ds \tag{C.3.3}$$

extends to a bounded linear operator on $L^p(\mathbb{R}, Y)$ for some, equivalently for all, $p \in (1, \infty)$.

Assume that X is a Banach space of $\mathcal{HT}$ class.

Definition C.21. A admits bounded imaginary powers if, and only if, $D(A) \cap \mathcal{R}(A)$ is dense in X and $A^{is} \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$.
By $\mathcal{BIP}(X)$ we denote the set of all injective sectorial operators on X which have bounded imaginary powers.

One has the following property

Proposition C.22. [**29**, Corollary 3.5.7] *The following are equivalent*

- $A \in \mathcal{BIP}(X)$
- *The family $\{A^{is}\}_{s\in\mathbb{R}}$ is a C_0-group of linear bounded operators in X.*

In this case, the generator of $\{A^{is}\}_{s\in\mathbb{R}}$ is $B = i\log(A)$.

Let $A \in \mathcal{BIP}(X)$. Then, there exists $M \geq 1$ and $\theta \in \mathbb{R}$ such that

$$\|A^{is}\| \leq Me^{\theta|s|}, \qquad \forall s \in \mathbb{R}.$$

Let

$$\theta_A = \inf\{\theta \in \mathbb{R} \mid \exists M \geq 1 : \|A^{is}\| \leq Me^{\theta|s|}, \quad \forall s \in \mathbb{R}\}. \tag{C.3.4}$$

θ_A is called the power angle of A.

Remark C.23. It it easy to see that the boundedness of imaginary power is a consequence of boundedness of H^∞ functional calculus. However, in [**15**], the authors prove that in Hilbert spaces the two notions are equivalent.

Examples C.24. Every closed injective A which admits $\mathcal{H}^\infty$-bounded functional calculus has bounded imaginary powers, i.e., $\mathcal{H}^\infty(S_\varphi) \subset \mathcal{BIP}(X)$ and $\theta_A > \varphi$. Indeed, assume that A admits $\mathcal{H}^\infty$-bounded functional calculus on S_φ, for some $\theta \in [0, \pi)$. One has, for every $s \in \mathbb{R}$, $z \mapsto z^{is} \in \mathcal{H}^\infty(S_\varphi)$ and $|z^{is}| \leq e^{s\varphi}$, for all $z \in S_\varphi$. Thus

$$\|A^{is}\| \leq M\, e^{s\varphi}.$$

Consequentely,

- according to PropositionC.13, every elliptic system with bounded uniformly continuous second order coefficients and bounded lowest order coefficients has bounded imaginary powers in L^p-spaces for every $1 < p < \infty$ and the power angle is equal to 0.
- according to PropositionC.15, every accretive operator A on a Hilbert space admits bounded imaginary powers with power angle $\theta_A \leq \pi/2$.

C.3.2. Two versions of Dore–Venni theorem. Now we give a very important result which yields the closedness\sectoriality of the sum of two sectorial operators of bounded imaginary powers, the sum is defined on the natural domain: intersection of the two domains.

Let X be a $\mathcal{UMD}$ Banach space, $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ two closed operators such that $A, B \in \mathcal{BIP}(X)$. Here we recall the commutative Dore–Venni theorem.

Theorem C.25. [**20**, Theorem 2.1] *Assume that*

- $(-\infty, 0] \subset \rho(A) \cap \rho(B)$ *and there exists* $M > 0$ *such that*
$$\|(t + A)^{-1}\|, \|(t + B)^{-1}\| \leq \frac{M}{1 + t}, \qquad \forall t \geq 0.$$
- *There exist* $K \geq 1$ *and* $0 \leq \theta_A, \theta_B < \pi$ *with* $\theta_A + \theta_B < \pi$ *such that*
$$\|A^{is}\| \leq K \exp(\theta_A|s|), \quad \|B^{is}\| \leq K \exp(\theta_B|s|)$$
for all $s \in \mathbb{R}$.

- *For some\all* $\lambda \in \rho(A)$, $\mu \in \rho(B)$ *one has* $(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}$

Then, $A + B$ *defined on* $D(A + B) := D(A) \cap D(B)$ *is closed (in particular sectorial).*

The above theorem was generalized for the sum of two non commutative operators by Monniaux and Prüss as follows.

Theorem C.26. [**50**] *Assume the following hold*

- *There exist* $K \geq 1$ *and* $0 \leq \theta_A, \theta_B < \pi$ *with* $\theta_A + \theta_B < \pi$ *such that*
$$\|A^{is}\| \leq K \exp(\theta_A |s|), \quad \|B^{is}\| \leq K \exp(\theta_B |s|)$$
for all $s \in \mathbb{R}$.
- *There exists* $M \geq 0$ *such that*
$$|(\lambda + A)^{-1}| \leq \frac{M}{1 + |\lambda|}, \qquad \forall \lambda \in S_{\pi - \theta_A};$$
$$|(\mu + B)^{-1}| \leq \frac{M}{|\mu|}, \qquad \forall \mu \in S_{\pi - \theta_B}.$$
- *There exist* $c \geq 0$ *and* $0 \leq \alpha < \beta \leq 1$ *such that*
$$\left| A(\lambda + A)^{-1} \left(A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1} A^{-1} \right) \right| \leq \frac{c}{(1 + |\lambda|^{1-\alpha})|\mu|^{1-\beta}},$$
for all $\lambda \in S_{\pi - \theta_A}$ *and* $\mu \in S_{\pi - \theta_B}$.

Then, $A + B$ *with domain* $D(A) \cap D(B)$ *is closed and there exists* $\nu_0 \geq 0$ *such that* $\nu_0 + A + B$ *is sectorial (*$A + B$ *is quasi-sectorial).*

APPENDIX D

Matrix Multiplication Operator

This appendix is devoted to developing properties of the multiplication operator by a matrix in L^p-spaces. Throughout, we consider the Banach space $E = L^p(\mathbb{R}^d, \mathbb{R}^m)$, where $1 \leq p < \infty$ and $d, m \in \mathbb{N}$. Let $M : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ be a measurable matrix–valued function. Assume that M satisfies the following algebraic condition

$$\langle M(x)\xi, \xi\rangle \leq \beta|\xi|^2 \tag{D.0.5}$$

for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^m$ and some real number β. We define M_p ato be the multiplication operator by M in $L^p(\mathbb{R}^d, \mathbb{R}^m)$ with its maximal domain $D(M_p) = \{f \in L^p(\mathbb{R}^d, \mathbb{R}^m) : Mf \in L^p(\mathbb{R}^d, \mathbb{R}^m)\}$. Note that Mf should be understood as a matrix-vector multiplication: $(Mf)_i(x) = \sum_{j=1}^m M_{ij}(x)f_j(x)$, $1 \leq i \leq m$.

D.1. Semigroup associated to M_p

We prove that M_p generates a strongly continuous semigroup in the following proposition.

Proposition D.1. • *M_p generates a strongly continuous semigroup in $L^p(\mathbb{R}^d, \mathbb{R}^m)$; its associated semigroup is the family of multiplication operators: $\{e^{tM}\}_{t\geq 0}$. Moreover, one has*

$$\sup_{x\in\mathbb{R}^d} |e^{tM(x)}| \leq e^{\beta t}, \qquad \forall t \geq 0.$$

- *The half-plan $\{z \in \mathbb{C} : Re(z) \geq \beta\}$ is contained in the resolvent set $\rho(M_p)$ and*

$$\|(\lambda - M_p)^{-1}\| = \sup_{x\in\mathbb{R}^d} |(\lambda - M(x))^{-1}| \leq \frac{1}{Re(\lambda) - \beta}, \qquad \forall Re(\lambda) > \beta.$$

In particular, $\beta - M_p$ is a sectorial operator.

Proof. • Let us show first that e^{tM} is uniformly bounded on x. Fix $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^m$. Set $\phi(t) = |e^{-\beta t}e^{tM(x)}\xi|^2$, for all $t \geq 0$. One has

$$\phi'(t) = 2\langle (M - \beta)\xi, \xi\rangle \leq 0.$$

Then, ϕ is deacrising. In particular, $\phi(t) \leq \phi(0)$ which yields $|e^{tM(x)}| \leq e^{\beta t}$, for all $t \geq 0$.
It follows that

$$\int_{\mathbb{R}^d} |e^{tM(x)} f(x)|^p dx \leq \int_{\mathbb{R}^d} e^{p\beta t} |f(x)|^p dx \leq e^{p\beta t} \|f\|_p^p,$$

for every $f \in L^p(\mathbb{R}^d, \mathbb{R}^m)$. Let us define $U_p(t)f = e^{tV} f$, for all $f \in L^p(\mathbb{R}^d, \mathbb{R}^m)$. Thus, $\{U_p(t)\}_{t\geq 0}$ is a semigroup of linear bounded operator on $L^p(\mathbb{R}^d, \mathbb{R}^m)$ and

$$\|U_p(t)\| \leq e^{\beta t}.$$

To get the continuity of $\{U_p(t)\}_{t\geq 0}$, one has $\lim_{t\to 0} e^{tM} f = f$ almost everywhere, and

$$\sup_{t\in[0,1]} \|U_p(t)f\|_p \leq \sup_{t\in[0,1]} \sup_{x\in\mathbb{R}^d} |e^{tM(x)}| \|f\|_p \leq \sup_{t\in[0,1]} e^{\beta t} \|f\|_p < \infty.$$

hence, we conclude by the dominated convergence theorem that $\{U_p(t)\}_{t\geq 0}$ is a C_0-semigroup.
It remains to prove that $(M_p, D(M_p))$ is the generator of $\{U_p(t)\}_{t\geq 0}$. Let us denote by $(B_p, D(B_p))$ the generator of $\{U_p(t)\}_{t\geq 0}$ and let $f \in D(B_p)$. One has $B_p f = \lim_{t\to 0} 1/t(e^{tM} f - f)$ and the limit is taken in $L^p(\mathbb{R}^d, \mathbb{R}^m)$. The pointwise limit yields $B_p f = M f \in L^p(\mathbb{R}^d, \mathbb{R}^m)$ and thus $f \in D(M_p)$. Conversely, if $f \in D(M_p)$ then $\lim_{t\to 0} \frac{1}{t}(e^{tM} f - f) = V f$ pointwisely. However, applying the main theorem

$$|e^{tM} f - f| \leq \sup_{0\leq s\leq t} (|e^{sM}|) |Mf| \leq e^{t|\beta|} |Mf|$$

as $Mf \in L^p(\mathbb{R}^d, \mathbb{R}^m)$, we conclude by dominated convergence theorem that $f \in D(B_p)$.

- It suffices to show that the resolvent operator $(\lambda - M_p)^{-1}$ is the multiplication by the matrix-valued function $(\lambda - M)^{-1}$ and conclude by the Hille–Yosida theorem. This follows by applying the Laplace transform of $s \mapsto e^{sV}$. Indeed,

$$(\lambda - M)^{-1} = \int_0^{+\infty} e^{-\lambda t} e^{tM} dt, \qquad \forall Re(\lambda) > \beta.$$

□

What about analytic semigroup. One could ask if the semigroup $\{U_p(t)\}_{t\geq 0}$ generated by M_p is holomorphic. The answer is, in general, negative. Indeed, if one considers the antisymetric matrix

$$M(x) = a(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $a : \mathbb{R}^d \to \mathbb{R}$ is any *unbounded* real valued function: $\sup_{x\in\mathbb{R}^d} |a(x)| = \infty$. Since M is antisymmetric, then $\langle M(x)\xi, \xi\rangle = 0$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^2$. Thus (D.0.5) is satisfied. Moreover, one can easily get

$$e^{tM(x)} = \begin{pmatrix} cos(a(x)t) & sin(a(x)t) \\ -sin(a(x)t) & cos(a(x)t) \end{pmatrix},$$

therefore

$$M(x)e^{tM(x)} = \begin{pmatrix} -a(x)\, sin(a(x)t) & a(x)\, cos(a(x)t) \\ -a(x)\, cos(a(x)t) & -a(x)\, sin(a(x)t) \end{pmatrix}.$$

Since a is not bounded, then Me^{tM} is not uniformly bounded on x and thus the multiplication by Me^{tM} cannot be a bounded operator in L^p-spaces. We conclude by [**23**, Chap II, Theorem 4.6] that $\{U_p(t)\}$ is not a holomorphic semigroup in this case.
If M where symmetric, then M_2 will be self–adjoint in $L^2(\mathbb{R}^d, \mathbb{R}^m)$ and thus $\{U_p(t)\}$ will be holomorphic in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, for all $1 < p < \infty$. We, now give a sufficient condition that yields the analyticity of $\{U_p(t)\}$ on $L^p(\mathbb{R}^d, \mathbb{R}^m)$ for $1 < p < \infty$. The condition of the below proposition is that the numerical ranges of the matrices $-M(x)$, $x \in \mathbb{R}$ is included in a sector of angle less than $\frac{\pi}{2}$, independently on x. This can be seen also as the sesquilinear form associated to the symmetric part of $-M$ dominates the antisymmetric one.

Without loss of generalities we assume that β in (D.0.5) is 0. Hence $-M_p$ is m-accretive in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, $1 < p < \infty$.

Proposition D.2. *Assume that there exists a positive constant C such that*

$$\text{(D.1.1)} \qquad \mathrm{Re}\,\langle -M(x)\xi, \bar{\xi}\rangle \geq C|\mathrm{Im}\,\langle M(x)\xi, \bar{\xi}\rangle|,$$

for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^m$. Then, for all $1 < p < \infty$, $-M_p$ is sectorial of angle $\phi = \arctan(\frac{1}{C}) < \frac{\pi}{2}$. In particular $\{U_p(t)\}$ can be extended to a bounded holomorphic semigroup.

Proof. Let $u \in D(M_p)$, one has

$$\begin{aligned} \mathrm{Re}\,\langle -M_p u, |u|^{p-2}\bar{u}\rangle_p &= \mathrm{Re} \int_{\mathbb{R}^d} \langle -M(x)u(x), \bar{u}(x)\rangle |u(x)|^{p-2} dx \\ &= \int_{\mathbb{R}^d} \mathrm{Re}\,(\langle -M(x)u(x), \bar{u}(x)\rangle) |u(x)|^{p-2} dx \\ &\geq C \int_{\mathbb{R}^d} |\mathrm{Im}\,\langle M(x)u(x), \bar{u}(x)\rangle| |u(x)|^{p-2} dx \\ &\geq C|\mathrm{Im}\,\langle V_p u, |u|^{p-2}\bar{u}\rangle_p|. \end{aligned}$$

Hence $\sigma(-M_p) \subset S_\phi$ and then $\sigma(M_p) \subset S_{\pi-\phi}$, where $\phi = \arctan(\frac{1}{C})$ which ends the proof. □

D.2. Functional calculus associated to $-M_p$

We assume that $\beta = 0$ in (D.0.5). The aim of this section is to show that $f(-M_p)$ is exactly the multiplication by the matrix $f(-M)$ and show that $-M_p$ admits bounded H^∞-functional calculus. In fact, in Proposition D.1, it has been shown that the semigroup $\{e^{tM_p} : t \geq 0\}$ associated to M_p is the operator multiplication by e^{tM} and the resolvent $R(\lambda, M_p)$ coincides with the multiplication by $(\lambda - M)^{-1}$. Now, we have the following.

Lemma D.3. *Let $f \in \mathcal{H}_0^\infty(S_{\pi/2})$. Then, $f(-M_p)$ is nothing but the multiplication operator by the matrix $f(-M)$. In particular,*

$$\|f(-M_p)\|_p = \sup_{x\in\mathbb{R}^d} \|f(-M(x))\| \tag{D.2.1}$$

Proof. Let $\varphi > \pi/2$. According to (C.1.1), one has

$$f(-M_p)u = \frac{1}{2\pi i}\int_{\Gamma_\varphi^+} f(z)(z+M_p)^{-1}u\,dz, \qquad \forall u \in L^p(\mathbb{R}^d, \mathbb{R}^m).$$

Now, $z \in \Gamma_\varphi$ and $\varphi > \pi/2$ imply $|\arg(z)| < \pi/2$. Thus, $|\arg(-z)| < \pi/2$ and $\mathrm{Re}\,(-z) \geq 0$. Thus, $(z + M_p)^{-1} = -R(-z, M_p)$, which implies that $(z + M_p)^{-1}$ is the multiplication by $-(-z - M_)^{-1} = (z + M)^{-1}$. That is

$$((z + M_p)^{-1}u)(x) = (z + M(x))^{-1}u(x),$$

for every $x \in X$ and $u \in L^p(\mathbb{R}^d, \mathbb{R}^m)$. Finally, since (C.1.1) hold true for the matrix $-M(x)$, for every $x \in \mathbb{R}^d$ as it is an accretive operator in $\mathbb{R}^m$, one has $(f(-M_p)u)(x) = f(-M(x))u(x)$ and the claim is proved. □

Consequently, one obtains

Theorem D.4. *$\mathcal{M}_{-M_p}(S_{\pi/2}) = \cup_{x\in\mathbb{R}^d}\mathcal{M}_{-M(x)}(S_{\pi/2})$ and $f(-M_p)$ coincides with the mutliplication operator by the matrix $f(-M(\cdot))$, for every $f \in \mathcal{M}_{-M_p}(S_{\pi/2})$. In particular,*

$$((-M_p)^{is}u)(x) = (-M(x))^{is}u(x),$$

for every $u \in L^p(\mathbb{R}^d, \mathbb{R}^m)$ and every $x \in \mathbb{R}^d$.

Now we state the man and crucial result of this appendix

Theorem D.5. *$-M_p$ admits a bounded $\mathcal{H}^\infty$ functional calculus on $S_{\pi/2}$ and*

$$\|f(-M_p)\|_p \leq \|f\|_{S_{\pi/2}}, \tag{D.2.2}$$

for every $f \in \mathcal{H}^\infty(S_{\pi/2})$. In particular, $-M_p \in \mathcal{BIP}(L^p(\mathbb{R}^d, \mathbb{R}^m))$ and

$$\|(-M_p)^{is}\| \leq e^{\frac{\pi}{2}|s|}, \qquad \forall s \in \mathbb{R}. \tag{D.2.3}$$

PROOF. Let $f \in \mathcal{H}^\infty(S_{\pi/2})$ and fix $x \in \mathbb{R}^d$. The matrix $-M(x)$ is actually an m-accretive operator on $\mathbb{R}^m$. Thus, according to Proposition C.15, $-M(x)$ admits $\mathcal{H}^\infty$ functional calculus on $S_{\pi/2}$ and

$$|f(-M(x))| \leq \|f\|_{S_{\pi/2}}.$$

Therefore,

$$\|f(-M_p)\| = \sup_{x\in\mathbb{R}^d} |f(-M(x))| \leq \|f\|_{S_{\pi/2}} < \infty.$$

Hence, $-M_p$ admits $\mathcal{H}^\infty$–functional calculus on $S_{\pi/2}$ and (D.2.2) holds true. □

D.3. Some inequalities for positive matrices

We assume that $\beta = 0$ in (D.0.5). That is

$$\langle M(x)\xi, \xi\rangle \leq 0, \qquad \forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^m. \tag{D.3.1}$$

One has the following

Proposition D.6. *Let $M = (m_{ij})_{1\leq i,j\leq m}$ and $x \in \mathbb{R}^d$ and assume that* (D.3.1) *holds. Then,*

(D.3.2)

$$\begin{cases} m_{ii}(x) \leq 0, \quad \forall i \in \{1,\dots,m\} \\ |m_{ij}(x) + m_{ji}(x)| \leq 2\sqrt{m_{ii}(x)m_{jj}(x)} \leq -(m_{ii}(x)+m_{jj}(x)), \quad \forall i \neq j \in \{1,\dots,m\}. \end{cases}$$

In particular, if $M(x)$ is a symmetric matrix then

$$|m_{ij}(x)| \leq \sqrt{m_{ii}(x)m_{jj}(x)} \leq -\frac{1}{2}(m_{ii}(x)+m_{jj}(x)).$$

PROOF. Let $i \neq j \in \{1,\dots,m\}$ and consider $\xi = \xi_i e_i + \xi_j e_j \in \mathbb{R}^m$, where e_i and e_j are respectively the i-th and j-th component of the canonical basis of $\mathbb{R}^m$, and $\xi_i, \xi_j \in \mathbb{R}$. According to (D.3.1), one has $m_{ii}(x) = \langle M(x)e_i, e_i\rangle \leq 0$ and

$$0 \geq \langle M(x)\xi,\xi\rangle = m_{ii}(x)\xi_i^2 + (m_{ij}(x)+m_{ji}(x))\xi_i\xi_j + m_{jj}(x)\xi_j^2.$$

Now, if $m_{ii}(x) = 0$, then $(m_{ij}(x) + m_{ji}(x)\xi_i\xi_j + m_{jj}(x)\xi_j^2 \leq 0$, for every $\xi_i, \xi_j \in \mathbb{R}$, which yields $m_{ij}(x) + m_{ji}(x) = 0$. Thus, (D.3.2) is satisfied. If $m_{ii}(x) \neq 0$, one has

$$0 \leq \frac{1}{m_{ii}(x)}\langle M(x)\xi,\xi\rangle = \left(\xi_i + \frac{m_{ij}(x)+m_{ji}(x)}{2m_{ii}(x)}\xi_j\right)^2 + \xi_j^2\left(\frac{m_{jj}(x)}{m_{ii}(x)} - \frac{(m_{ij}(x)+m_{ji}(x))^2}{4m_{ii}^2(x)}\right).$$

A suitable choice of ξ_i and ξ_j implies $\frac{m_{jj}(x)}{m_{ii}(x)} - \frac{(m_{ij}(x)+m_{ji}(x))^2}{4m_{ii}^2(x)} \geq 0$ and thus (D.3.2) follows. □

List of symbols

- $\mathbb{R}^j$: Euclidean j-dimensional space, $j = d, m$.
- $\mathbb{C}^j$: Complex euclidean space of dimension $j = d, m$.
- $\langle x, y\rangle$: Inner euclidean product between the vectors $x, y \in \mathbb{C}^j$, $j = d, m$.
- $|x|$: Euclidean norm of $x \in \mathbb{R}^j$, $j = d, m$.
- $B(r)$: Centred ball of $\mathbb{R}^d$ of radius r.
- $\operatorname{supp} u$: Support of a given function u.
- $L^p(\mathbb{R}^d)$: Space of measurable $u : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} |u(x)|^p \, dx < \infty$.
- $L^p(\mathbb{R}^d, \mathbb{R}^m)$: Space of measurable $u : \mathbb{R}^d \to \mathbb{R}^m$ such that $|u| \in L^p(\mathbb{R}^d)$.
- $\|\cdot\|_p$: Norm of $L^p(\mathbb{R}^d, \mathbb{R}^m)$.
- $\|u\|_p = \left(\int_{\mathbb{R}^d} |u(x)|^p \, dx\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^d} (\sum_{j=1}^m |u_j(x)|^2)^{p/2} \, dx\right)^{\frac{1}{p}}$
- $W^{k,p}(\mathbb{R}^d)$: Sobolev space of order k over $L^p(\mathbb{R}^d)$.
- $W^{k,p}(\mathbb{R}^d, \mathbb{R}^m)$: Space of functions $u : \mathbb{R}^d \to \mathbb{R}^m$ with components u_j in $W^{k,p}(\mathbb{R}^d)$.
- $\|\cdot\|_{k,p}$: Norm of the Sobolev space $W^{k,p}(\mathbb{R}^d, \mathbb{R}^m)$.
- χ_E: Characteristic function of the set E, i.e.: $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.
- $W^{k,p}_{loc}(\mathbb{R}^d)$: Space of functions f in $L^p_{loc}(\mathbb{R}^d)$ such that $\partial^\alpha f \in L^p_{loc}(\mathbb{R}^d)$, for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| \leq k$.
- $C_b(\mathbb{R}^d)$: Space of continuous bounded functions of $\mathbb{R}^d$.
- $C^\alpha_b(\mathbb{R}^d)$: Space of bounded α-Hölderian functions in $\mathbb{R}^d$.

- $C_b^{1+\alpha}(\mathbb{R}^d)$: Space of bounded functions u with bounded first-order derivatives such that u and $\partial_i u$ are α-Hölderian functions in $\mathbb{R}^d$, for all $i \in \{1, \ldots, m\}$.
- $C_c^\infty(\mathbb{R}^d)$: Space of infinitely many time derivable functions with compact support in $\mathbb{R}^d$.
- S_θ: Sector of angle $\theta \in (0, \pi)$.
- $\mathbb{C}^+$: Set of complex numbers with positive real part.
- $\mathbb{C}_\omega^+$: Set of complex numbers with real part greater than ω.
- $\mathcal{O}(S_\theta)$: Space of holomorphic function over S_θ.
- $\mathcal{M}(S_\theta)$: Space of meromorphic function over S_θ.
- $\mathcal{H}^\infty(S_\theta)$: Space of bounded holomorphic function over S_θ.
- $\mathcal{H}_0^\infty(S_\theta)$: Dunford–Riesz class over S_θ.
- $\rho(A)$: Resolvent set of A.
- $\sigma(A)$: Spectrum of the operator A.
- $R(\lambda, A)$: Resolvent operator of A at point λ.

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