Abstract

In this work we are interested in strong solutions of a Dirichlet problem for an elliptic linear operator. At this aim, let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. Given any $p \in]1, +\infty[$, a linear uniformly elliptic boundary value problem in non divergence form consists of

$$\begin{cases} Lu := -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} u + au = f \qquad \text{in} \;\; \Omega, \\ u = 0 \quad \text{on} \;\; \partial\Omega, \qquad f \in L^p(\Omega), \end{cases} \tag{1}$$

for the unknown function u defined on Ω.

We refer to the problem (1) as the homogeneus Dirichlet problem for the linear operator L and we are interested in *strong* solutions for it.

Namely, a strong solution of (1) is a twice weakly differentiable function, $u \in W^{2,p}(\Omega)$, $p \in]1, +\infty[$, that satisfies the equation $Lu = f$ *almost everywhere* (a.e.) in Ω and assumes the boundary values in the sense of $\overset{\circ}{W}{}^{1,p}(\Omega)$. This concept makes sense for $f \in L^p(\Omega)$ and when the

coefficients a_{ij} are measurable functions such that

$$a_{ij} = a_{ji} \in L^{\infty}(\Omega). \tag{2}$$

A reasonable strong solvability theory of (1) cannot be built up without suitable additional hypotheses on leading coefficients.

Indeed, if a_{ij} are continous functions in $\bar{\Omega}$

$$a_{ij} \in C^{0}(\bar{\Omega}) \tag{3}$$

a satisfactory theory (known "L^p-theory") exists. It provides solvability and regularity for (1) in Sobolev spaces $W^{2,p}(\Omega)$ for $p > 1$.

The next step of the theory deals with weakening the continuity assumption (3), that it generates boundary value problems for elliptic equations whose ellipticity is "*disturbed*" in the sense that some *degeneration* or *singularity* appears.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces.

We note that the role of a weight function consists in fixing the behaviour at infinity of the functions belonging to the weighted Sobolev space and of their derivatives and near the not regular part of boundary of the domain.

In this framework, we can insert our work. In chapter 1, we deal

with introducing the weight functions and their corresponding weighted Sobolev spaces to investigate, first of all, why to choose a weighted Sobolev space instead of classical Sobolev spaces and, after, how to select a certain type of weight functions than the other ones. This choice mainly depends by the necessity to obtain a new Sobolev space also Banach space. In this point of view, on a subset Ω di $\mathbb{R}^n$, $n \geq 2$, not necessary bounded, two new classes of weight functions are introduced and their properties are examined:

1. $\mathcal{G}(\Omega)$: this class, introduced yet by M. Troisi , is defined as the union of sets $\mathcal{G}_d(\Omega)$ for any $d \in \mathbb{R}_+$:

$$\mathcal{G}(\Omega) = \bigcup_{d \in \mathbb{R}_+} \mathcal{G}_d(\Omega),$$

   where $\mathcal{G}_d(\Omega)$ is the class of measurable functions $m : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y \in \Omega \\ |x-y|<d}} \frac{m(x)}{m(y)} < +\infty \,, \tag{4}$$

2. $\mathcal{C}^k(\overline{\Omega})$: this class is defined as the set of the functions $\rho : \overline{\Omega} \to \mathbb{R}_+$ such that $\rho \in C^k(\overline{\Omega})$, $k \in \mathbb{N}_0$, and

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall\, |\alpha| \leq k. \tag{5}$$

We stress the point that $\mathcal{C}^k(\overline{\Omega})$ weight functions are more regular than $\mathcal{G}(\Omega)$ - functions. Althought, $\mathcal{G}(\Omega)$ weights have the favourable

property to admit among its members a ***regularization function***, that is a function of the same weight type but also belonging to $C^\infty(\Omega)$, so a more regular function than a $\mathcal{C}^k(\overline{\Omega})$ weight.

Chapters 2 and 3 are devoted to the study of the solvability of the Dirichlet problem:

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega) \\ Lu = f, \;\; f \in L^p_s(\Omega), \end{cases} \tag{6}$$

where Ω is an unbounded and sufficiently regular open subset of $\mathbb{R}^n$ ($n \geq 2$), $p \in]1, +\infty[$, L is the uniform elliptic second order linear differential operator defined by

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \,, \tag{7}$$

with coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega)$, $i, j = 1, \ldots, n$, $s \in \mathbb{R}$, $p \in]1, +\infty[$, $W^{2,p}_s(\Omega)$, $\overset{\circ}{W}{}^{1,p}_s(\Omega)$ and $L^p_s(\Omega)$ suitable weighted Sobolev spaces on Ω.

In particular, we confine the problem to $\mathcal{G}(\Omega)$ - weighted Sobolev space. In detail we assume that:

- in chapter 2, Ω is an unbounded domain of $\mathbb{R}^n$, for any $n \geq 3$;
- in chapter 3, Ω is an unbounded domain of the plane ($n = 2$).

Instead, in chapter 4, we deal with the solvability in $\mathcal{C}^k(\overline{\Omega})$ - weighted

Sobolev spaces

$$\begin{cases} u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega) \\ Lu = f\,, \quad f \in L^2_s(\Omega)\,, \end{cases} \tag{8}$$

where Ω is an unbounded domain of $\mathbb{R}^n$, for any $n \geq 2$.

In chapter 2, we start with certain a priori estimates for the operator L, obtained by means of the following properties, just introduced in chapter 1:

(I) ***topological isomorphism***:

$$u \longrightarrow \sigma^s u$$

(from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ or from $\overset{\circ}{W}{}^{1,p}_s(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$). It leads to go from weighted spaces to no-weighted spaces and to get their properties.

(II) ***compactness and boundedness***: of multiplying operator

$$u \longrightarrow \beta u \tag{9}$$

defined in a weighted Sobolev space and which takes values in a weighted Lebesgue space.

Actually, we do that just assuming the following hypotheses, listed below, on the coefficients and on the weight functions:

- a_{ij} (in addition to simmetry and boundedness) locally $VMO(\Omega)$

and at infinity close to certain e_{ij}, belonging to a suitable subset of $VMO(\Omega)$,

- a_i and a having sommability conditions of local character,
- weight function, s-th power of a function $m \in \mathcal{G}(\Omega)$, not bounded at infinity and with derivates of its regularization function having suitable infinity conditions,we get the following a priori bound:

$$||u||_{W^{2,p}_s(\Omega)} \leq c\Big(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p(\Omega_1)} \Big) \quad \forall\, u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega), \tag{10}$$

where $s \in \mathbb{R}$, Ω is sufficiently regular and Ω_1 is a bounded open subset of Ω. This a priori bound allows to deduce that L is a semi-Fredholm operator, that is it has close range and finite - dimensional kernel, which is an essential property to state the solvability of the problem (6).

After this, by a method of continuity along a parameter, using a priori estimate (10) and the topological isomorphism, it is possible taking an advantage of an existence and uniqueness result for the following no-weighted problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f\,, \quad f \in L^p(\Omega)\,, \end{cases} \tag{11}$$

in order to establish a uniqueness and existence theorem for $\mathcal{G}(\Omega)$ - problem (6) for any $n \geq 3$.

In chapter 3, the solvability of the $\mathcal{G}(\Omega)$ - problem (6) for unbounded domains of the plane is presented. Indeed, we show that a priori estimate (10) for the solutions of (6), when Ω is an unbounded $C^{1,1}$ domains of the plane for the solutions, leads to an existence and uniqueness theorem.

In chapter 4, we deal with $\mathcal{C}^k(\overline{\Omega})$ - weighted Sobolev spaces on unbounded domains of $\mathbb{R}^n$, $n \geq 2$. As a main result we describe a weighted and a not-weighted a priori $W^{2,2}$-bound. These are obtained under hypotheses of Miranda's type on the leading coefficients and supposing that their derivatives $(a_{ij})_{x_k}$ belong to a suitable Morrey type space, which is a generalization to unbounded domains of the classical Morrey space. Notice that the existence of the derivatives is of crucial relevance in our analysis, since it allows us to rewrite the operator L in divergence form and to use some known results concerning variational operators. A straightforward consequence of our argument is the following $W^{2,2}$-bound, having the only term $\|Lu\|_{L^2(\Omega)}$ in the right hand side,

$$\|u\|_{W^{2,2}(\Omega)} \leq c\|Lu\|_{L^2(\Omega)} \quad \forall\, u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega), \tag{12}$$

where the dependence of the constant c is explicitly described. This kind of estimate often cannot be obtained when dealing with unbounded domains and clearly immediately takes to the uniqueness of the solution of problem (11) for $p = 2$.

In the framework of unbounded domains, we show that the $W^{2,2}$-bound obtained in (12) allows us to extend our result passing to the $\mathcal{C}^2(\overline{\Omega})$

weighted case. Infact, using (12) we get the following $\mathcal{C}^2(\overline{\Omega})$ weighted $W^{2,2}_s$-bound:

$$\|u\|_{W^{2,2}_s(\Omega)} \leq c\|Lu\|_{L^2_s(\Omega)} \quad \forall\, u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega).$$

From this a priori estimate, assuming that the weight function satisfies also conditions at infinity

$$\lim_{|x|\to+\infty} \left(\rho(x) + \frac{1}{\rho(x)}\right) = +\infty \quad \text{and} \lim_{|x|\to+\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0,$$

we deduce the solvability of problem (8).