

UNIVERSITÀ DEGLI STUDI DI SALERNO

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DOTTORATO DI RICERCA IN MATEMATICA

CICLO X

TESI DI DOTTORATO

**ON THE FORMULATION OF EINSTEIN
GENERAL RELATIVITY
IN A PHYSICAL REFERENCE SYSTEM**

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A.A. 2010 - 2011

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Chapter 1

The program of Einstein's theory of gravitation

1.1 Objections to the principles of Newtonian Me- chanics and special Relativity

A) Special Relativity provides a space-time vision of the Universe in evolution, but in the formulation of laws that allow to describe the physical phenomena uses, as in Newtonian mechanics, a class of privileged physical references, so-called “Galilean or inertial systems”, with respect to which the physical laws are expressed in the simplest form. One of them can be chosen with the origin in a “fixed” star and the three-axis, mutually orthogonal, oriented toward three other “fixed” stars of the same nebula. Such a choice does not have absolute validity because the nebula in which the physical reference is chosen, (of course

our “milky way”) as any other nebula, is not really motionless; but we can believe that its motion, taken as a whole, is essentially translational, and that choice is acceptable to our considerations. It is specified then that the class of inertial physical reference is constituted by all those references that are in translational uniform rectilinear motion relative to each other and to that initially chosen and it states the “Principle of Special Relativity”:

“All physical laws have the same form in every inertial physical reference”.

The our choice of a class of privileged physical references, however, is justified only if we attribute to space-time that physical property, considering the “absolute space-time”, independent in its physical properties, having a physical effect (the choice of a class of physical references) but not in turn influenced by the physical phenomena that take place in it.

Einstein argues that to conceive the space-time with a physical effect, without being able to act on it is not scientifically acceptable.

E. Mach already, in his critique of the Principles of Newtonian mechanics [?], felt the need to remove the space as an active cause in the motion of bodies, saying that any material particle in motion, such as near the Earth, is accelerated on a suitably defined average of all the matter that makes up the Universe; so between the forces acting on the particle must be included those performed by the stars and galaxies, that have an accelerated motion with respect to Earth. However, to develop this idea of Mach, which is already perceived and supported by Bishop Berkeley in his dispute with Newton on the replacement of absolute space with the material universe[?], and make the Mechanics self-consistent, it considered the properties of space-time that determine the inertia as field prop-

erty of space-time similar to what you do in the electromagnetic case, because really matter consists of electrically charged particles and must be considered itself as a part, indeed the principal part, of the electromagnetic field. The Principles of Newtonian Mechanics and Special Relativity does not allow this wider vision.

However E. Mach put out a substantial difference between the physical inertial references (according to the fixed stars) and physical non-inertial references, such as those that are in rotation with respect to the inertial references.

As is known, the fundamental law of motion, in its form valid in Newtonian Mechanics and Special Relativity in the physical inertial references,

$$\frac{d}{dt}(m\mathbf{v}) = \underline{F} \tag{1.1.1}$$

is no longer valid in the physical references with accelerated motion with respect to those inertial, unless so-called “apparent force fields” (driving and Coriolis’s field)[?] are added to the field of forces \underline{F} directly applied to the material point in motion. It is so obvious a substantial difference between physical inertial (or Galilean) references, indistinguishable from each other, and physical non-inertial references in which, in general, there are two “apparent force fields”.

What physical circumstances allow such difference? The only great diversity, already perceived by Berkeley, taken by E. Mach and further developed, is therefore given by the presence and behavior of the vast quantity of matter constituting the “fixed stars”, significantly at rest or in translational rectilinear uniform motion in Galilean physical references, where to the law (??) should

therefore be added the “apparent forces of the relative motion”.

Einstein ultimately accepts this view and states the following

General Principle of Relativity:

“All physical laws can be stated in a formally identical way in every physical reference, provided in such laws the distribution and motion of all bodies making up the universe in the same physical reference are taken into account.”

B) The Law of Universal Gravitation, perceived by Newton, is the following

$$\underline{F} = f \frac{Mm}{r^2} \quad (1.1.2)$$

where M and m show the masses of any two bodies, significantly point (material points) in the Universe, r is their distance, f is the gravitational constant¹, \underline{F} indicates the vector force with which the two bodies attract each other. As you can see, in this law is implicit the instantaneous propagation of the gravitational field, since there isn't dependence on time variable, while the Special Relativity rules out that the speed of propagation of any physical action can exceed the speed of light in vacuum.

The law (??) is susceptible to the following generalizations:

Given N material points P_i of respective masses M_i ($i = 1, 2, \dots, N$), however

¹With direct experience of laboratory in 1797 Cavendish led to the constant f value, expressed in units of the c.g.s. system of measures

$$f = 6,7 \cdot 10^{-8} gr^{-1}cm^3sec^{-2}$$

value subsequently obtained with other determinations carried out in an increasingly accurate.

located in the Universe, P is a any other material point of mass m , and $r_i = |PP_i|$ are the distances from their material points P_i of P ; the (attraction) gravitational action explicated by the N material points P_i on the material point P is defined as

$$\underline{F} = fm \sum_{i=1}^N \frac{M_i}{r_i^2} \tag{1.1.3}$$

whence it follows that the intensity of the gravitational field (ie the gravitational force acting on unit mass of P), generated by the N material points P_i in P has the form

$$\underline{G} = gradU, \tag{1.1.4}$$

where

$$U = f \sum_{i=1}^N \frac{M_i}{r} \tag{1.1.5}$$

is the corresponding gravitational potential.

If the gravitational field \underline{G} is generated by a quantity of matters dealt with continuously in a domain \mathcal{C} of the Universe, with density $\mu(Q)$, $\forall Q \in \mathcal{C}$, the corresponding gravitational potential has the form

$$U(P) = f \int_{\mathcal{C}} \frac{\mu(Q)}{r} d\mathcal{C}. \tag{1.1.6}$$

As is well known[?], the potential $U(P)$ satisfies the classical differential equation of Laplace-Poisson,

$$\Delta_2 U(P) = -4\pi f\mu(P) \begin{cases} = 0 \text{ in every empty portion of } \mathcal{C} \text{ (Laplace's equation)} \\ \neq 0 \text{ in every portion of } \mathcal{C} \text{ with matter (Poisson's equation)} \end{cases} \quad (1.1.7)$$

With appropriate regularity conditions on the boundary of \mathcal{C} the equation (??) uniquely identifies the function $U(P)$.

It is also in the differential formulation (??) absent the time variable, which, at the most, acts as a parameter.

Therefore, the Law of Universal Gravitation requires a radical change in order to take into account the finite speed of any physical action.

- C) As we have said before, the enormous amount of matter consisting of the countless galaxies that are evolving away from each other, and distant from (our) solar system, generates on each of the bodies of the solar system itself, and in a not-inertial physical reference, two force fields called “apparent”, in addition to the gravitational field generated by the sun and by the bodies of the solar system different from that is considered. Are they two different types of influence? It's desirable a conceptual unification of the two types of force fields, the ones generated, in a non-inertial physical reference, by the masses of the solar system, and those generated by all the galaxies (and other celestial bodies) greatly distant from the solar system.

Try to examine their influences.

Bearing in mind the mathematical expression of the two fields of apparent forces, we note that the motion of a free material point, with the same initial conditions, is independent of the value of its mass; it is precisely what happens also for the gravitational field generated by the sun and other solar system bodies on the same free material point.

The fact that the acceleration to a material point immersed in a gravitational force field is independent of the value of its mass is a consequence of the equality of inertial mass and gravitational mass of a particle, as Newton's law clearly shows [cfr.(??), (??)]

$$(\text{inertial mass}) \cdot \text{acceleration} = (\text{gravitational mass}) \cdot \text{gravitational field strength} \tag{1.1.8}$$

or equivalently

$${}_{(m)}\underline{a} = f \frac{M}{r^2} {}_{(m)}. \tag{1.1.9}$$

Only the numerical equality between inertial mass [in the first member of (??)] and gravitational mass [in the second member of (??) from (??)] ensures that the acceleration of a free material point immersed in a gravitational force field is independent of the value of its mass. This identity has also been accurately demonstrated in laboratory [?] neglects only quantities of order of 10^{-10} .

So we found a property common to the fields of "apparent" forces and the gravitational field generated by the sun and solar system bodies. But we can also

detect the following difference in behavior: instead of considering a non-inertial physical reference in which physical forces are “apparent”, we can think to make a change from the previous physical reference to another inertial physical reference (in which, as already mentioned, the “apparent” forces are zero). So we eliminated all the “apparent” forces. We can not do the same for the solar gravitational field (generated by the solar system), but you can do so only in small regions of space. For example, in a small cabin freely gravitating near the Earth, so only subject to a translational rectilinear uniform motion, with acceleration generated by the solar gravitational field; the resulting force field is zero: in fact if we consider a material point of mass m , free within the cabin, therefore subject to the intensity of the solar field as the cabin, which acceleration we denote by \underline{g} , the equation of motion of the particle (free within the cabin) is

$$m \underline{g} - m \underline{a}_\tau = m \underline{g} - m \underline{a}_g = 0 \quad (1.1.10)$$

being \underline{a}_τ the dragging acceleration to which is subject the material point (acceleration, by definition, is that of the point thought, moment by moment, in tune with the cabin, then \underline{g}).

Finally, even if restricted to small regions of space, we can consider identified the gravitational fields generated by “nearby” sources and those generated from extremely distant sources, whose total mass, that is not completely known, is vastly larger than the mass of all bodies in the solar system.

Their diversity seems solely due to the distribution of masses that we can de-

tect, as observers located within the solar system, and not to a different nature between them.

Should also keep in mind the following analogy suggested by electromagnetism. Let us consider a set of electric charges in translational uniform motion in a Galilean physical reference \mathcal{R} ; as is well known that set generates a magnetic field in the reference \mathcal{R} , but if we examine the behavior of the same charges in a Galilean physical reference \mathcal{R}' in which they are at rest, we find that the magnetic field is zero in the reference \mathcal{R}' .

However, if the electric charges are disorderly moving in a Galilean physical reference \mathcal{R} , where they generate a magnetic field, is no longer possible to identify a Galilean physical reference $\mathcal{R}' \neq \mathcal{R}$ in which it is entirely feasible the elimination of this magnetic field.

Therefore, the identification now acquired, intimately connected with the identification of inertial mass with the gravitational, is what Einstein has called the “Principle of equivalence”. The regions where we consider that the identification is valid are called by Einstein “Galilean regions”. In them a free mass can be considered subject to the action of the only inertia in a inertial physical reference, and simultaneously subject to inertia and gravitation in a non-inertial physical reference. Therefore, in a “Galilean region” we can take a inertial physical reference or a non-inertial physical reference depending on the advantage required by the particular problem of motion.

1.2 Evolution of the Minkowski space-time

As we have seen, in Special Relativity the Universe is represented by a space-time geometrically designed as a 4-dimensional Riemannian manifold, M_4 , with pseudo-Euclidean metric expressed by a indefinite quadratic form of type

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2. \quad (1.2.1)$$

In that space-time we could introduce a Galilean physical reference, \mathcal{R} , in which the introduced coordinates $\{x^h\}$ represent the generic event of M_4 , and the scalar ds^2 , invariant under Lorentz's transformations, expresses the so-called "space-time gap" between two infinitely neighbouring events,

$$E \equiv (x^1, x^2, x^3, x^4 = ct), \quad E' \equiv (x^1 + dx^1, x^2 + dx^2, x^3 + dx^3, x^4 + dx^4).$$

The General Relativity also takes a 4-dimensional Riemannian manifold, V_4 , to represent the totality of the events that make up the Universe in its evolution; but this variety is reduced to a Minkowski space-time M_4 only if it is totally lacking in the matter (and energy) and therefore lacks any gravitational field. Bearing in mind that in a freely gravitating cabin the gravitational field is zero [cfr. (??)], it makes sense to admit that the manifold V_4 has locally the structure of a Minkowski space-time, that is in every point of the manifold V_4 the tangent space has Minkowskian structure. It follows that the manifold V_4 is provided in each point of a metric that in general coordinates $\{x^h\}$ has the form

$$ds^2 = g_{hk}(x)dx^h dx^k \tag{1.2.2}$$

but that in any point of V_4 takes the algebraic structure of the Minkowski's metric,

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2. \tag{1.2.3}$$

where the ω^h are appropriate linear differential forms

$$\omega^h = A_k^h dx^k \tag{1.2.4}$$

integrable only in the regions of V_4 that are limited to Minkowskian regions ².

Therefore we exclude that in V_4 there is a coordinate system that allows to give to the metric (??) the pseudo-Pythagorean expression (??) everywhere. This is equivalent to saying that matter and energy constituents of the Universe in evolution make the space-time a Riemannian manifold with curvature³.

Let us now observe that by virtue of property attributed to the manifold V_4 of having in each point the corresponding tangent space with Minkowskian structure we can classify the vectors v^h and the directions u^h outgoing from the generic point of V_4 in three different ways, as in Special Relativity:

²Because of this characteristic of the metric (??) they say that it is normal hyperbolic according to Hadamard, with signature + + + -.

³The gravitational field generated by matter and energy that constitute it translates into a geometrical property of space-time. This fact, which is essentially a constraint imposed on the manifold V_4 , expresses the deletion in the Mechanics of the concept of force and its replacement with the concept of constraint. It should be noted that the idea of replacing forces with fit constraints had already been introduced by Hertz in the study of motion induced by constraints without friction, time or no-time dependent. It is also in this case that we consider motions independent from the value of the mass of the constrained body

Timelike vectors and directions, with negative norm,

$$g_{hk}v^h v^k < 0, \quad g_{hk}u^h u^k < 0; \quad (1.2.5)$$

Spacelike vectors and directions, with positive norm,

$$g_{hk}v^h v^k > 0, \quad g_{hk}u^h u^k > 0; \quad (1.2.6)$$

Lightlike vectors and directions, with null norm,

$$g_{hk}v^h v^k = 0, \quad g_{hk}u^h u^k = 0. \quad (1.2.7)$$

As in Special Relativity, the cone of the directions of lightlike vectors u^h , with vertex in the generic point $E \in V_4$ separates the events that are passed for E from those present to future, and the history of each material particle can be represented by a time-line with $ds^2 < 0$, while the element of proper time $d\tau$ of the particle is defined by the relation

$$d\tau^2 = -\frac{1}{c^2}ds^2, \quad [ds^2 = -(cd\tau)^2]. \quad (1.2.8)$$

Ultimately, always under the locally Minkowskian structure of the manifold V_4 , all the quantities represented by scalars, vectors, tensors that can be defined in Special Relativity can be introduced, in a legitimate way, in the manifold V_4 , as well as the physical laws of algebraic or differential nature expressed tensorially.

This place, let us try to deepen what we have seen until now on the translation of

the gravitational field, generated by matter and energy constituents the space-time, in geometric properties of the space-time itself.

Let us ask in what ways physical phenomena that occur in V_4 and the same V_4 influence each other, ie, in equivalent terms, in what way the curvature of the manifold V_4 influences the evolution of physical phenomena, and vice versa in what way matter and energy, which evolve constituting the manifold V_4 , act on the curvature of the space-time itself?

1.3 The curvature of the space-time V_4 and the evolution of the physical phenomena

As we have said before, the physical laws that are valid in each point of Minkowski space-time, are also confirmed in General Relativity. However, to take into account that the physical phenomena take place in a space-time with curvature it is postulated the following “Rule of transcription”:

- I. Every physical law, valid in each point of Minkowski space-time, and with algebraic structure, is unchanged in general Relativity;
- II. every physical law which has differential character, formulated in special Relativity, in Galilean coordinates, is transformed into a law valid in general Relativity by replacing the partial derivative ∂_h with the covariant derivative ∇_h , the ordinary differentiation with the absolute differentiation,

$$\partial_h \rightarrow \nabla_h, \quad d \rightarrow D. \quad (1.3.1)$$

This assumption is acceptable because the manifold V_4 is locally Minkowskian and we can then use local Cartesian coordinates in the neighborhood of each point of V_4 to write the differential laws valid in special Relativity; is therefore permitted the transcription of these laws in general coordinates as indicated by (??).

Having said this, we examine the law of motion of a free material point in special Relativity and in Galilean coordinates; it has the form

$$dU^h = 0 \tag{1.3.2}$$

where U^h is the 4-velocity of the material point, defined through the relation

$$U^h = \frac{dx^h}{d\tau} \tag{1.3.3}$$

and $d\tau$ is the elementary proper time interval of the material point (or material particle) invariant from the definition.

The equation (??) expresses the law of inertia in M_4 and suggests that the time-line of the material particle in M_4 is a straight.

Under the rule of transcription we deduce from (??) the equation

$$DU^h = 0 \tag{1.3.4}$$

that expresses the law of motion of a free material point in the gravitational field present in the manifold V_4 .

The equation (??) suggests that the time-line of a free material point in V_4 is a geodesic. As the manifold V_4 has a curvature, the geodesic isn't a straight line. We

see thus that the effect of the gravitational field in this manifold V_4 , and acting on the free material point in it, has been replaced by the curvature of the space-time in which the free material point evolves.

Let us note that even when the manifold V_4 has a small curvature, which is reflected in a small curvature of the geodesics, in the (3-dimensional) physical space these geodesics are projected in space trajectories that can have even large curvatures. Let us consider by analogy, for example, a finite right circular cylinder in ordinary Euclidean space, and on the cylinder a helix with a pitch much greater than the radius of the cylinder, and then curvature very small compared with the curvature of the cylinder; projecting this helix into the plane that belongs to the base of the cylinder we get a circumference, which owns a curvature, the same of the cylinder, which is much larger than the helix ⁴.

Therefore, in general Relativity the gravitational field generated by the vast amount of matter and energy that constitute the space-time V_4 is used to give a Riemannian structure to the space-time that is translated into (local) metric, and then in the functions g_{hk} which are therefore also called the gravitational potentials [cfr. (??), (??), (??)].

Let us now consider a body \mathcal{S} consisting of disintegrated matter that evolves in the space-time V_4 . Even in special Relativity, in the space-time M_4 we attributed to \mathcal{S} the material energy tensor

⁴Let us remember that if we indicate with θ the angle under which the helix intersects the generatrices of the cylinder, with p the pitch, with c the curvature of the helix, with r the radius of the cylinder, these relations exist

$$p = 2\pi r \cot \theta, \quad c = \frac{\sin^2 \theta}{r},$$

and $\frac{1}{r}$ gives the curvature of the cylinder.

$$T^{hk} = \mu_0 U^h U^k \quad (1.3.5)$$

where μ_0 is the proper density of proper mass of the body \mathcal{S} , and U^h the 4-velocity of the generic particle of \mathcal{S} , and we wrote the laws of evolution of \mathcal{S} , in Galilean coordinates, in the form

$$\partial_k T^{hk} = 0. \quad (1.3.6)$$

This equation, under the law of transcription, is transformed into the equation

$$\nabla_k T^{hk} = 0. \quad (1.3.7)$$

valid in the space-time V_4 and in general coordinates.

Obviously the equation (??) and its transformed (??) also express into their respective manifolds M_4, V_4 the law of evolution of a generic closed material system whose energy tensor T^{hk} has a structure less simple than the tensor written in (??).

The condition imposed by equation (??) to the material energy tensor is called “conservation condition”.

1.4 The evolution of matter and energy and their influence on the structure of the space-time V_4

As we stated before, the evolution of matter and energy that make up the space-time causes a curvature in it. We seek to deepen our understanding of this phenomena.

We have already seen that the evolution of every material freely gravitating particle describes a geodesic of the space-time V_4 , and is then characterized by the structure of V_4 and, taking into account the explicit form of the equation (??), by the gravitational potentials g_{hk} .

Besides remember that in Newtonian Mechanics the motion of a body immersed in a gravitational field is identified by the gravitational potential of the field through the Poisson equation (??).

However, Einstein was led to accept, by analogy with the equation (??), that in general relativity the evolution of matter and energy must act on the gravitational potentials too, and therefore on the structure of V_4 , and he formulated the following criteria:

1. Matter and energy that generate evolving the space-time act on the gravitational potentials g_{hk} by ordering that they should satisfy the second order differential equations, linear in second derivatives; these equations, under the Principle of general Relativity, must be invariant with respect to a generic coordinate transformation.
2. In the regions of spacetime where there are no matter and energy the above-mentioned equations reduce V_4 to portions of Minkowski space-time.
3. These equations admit as logical consequences the equations (??) that have been written only by applying the rule of transcription to the equations (??) valid in Minkowski space-time.

Bearing in mind the conditions imposed by the criteria written now, the partial differential equations that can satisfy them, need to involve material energy tensor

T^{hk} that depends on the structure of V_4 , and the Laplace-Poisson equation (??) must be deduced, even approximately, by any of equations required. Well you can prove [?] that the above criteria can uniquely determine the following gravitational equations, written by Einstein

$$G_{hk} \equiv R_{hk} - \frac{1}{2} R g_{hk} - \lambda g_{hk} = -\chi T_{hk} \quad (1.4.1)$$

in which R_{hk} is the contract curvature tensor and $R \equiv g^{hk} R_{hk}$ is the scalar curvature of the space-time V_4 , λ and χ are two universal constants; the second related to the Newton's gravitational constant, T_{hk} is the material energy tensor, whose structure takes into account the density of energy of the electromagnetic field, the ponderable matter and the gravitational field: we must keep in mind that the gravitational field generated by matter and energy communicates momentum and energy to the same matter as exertes the forces on it and gives energy to it [?].

The equations (??) are ten as the functions g_{hk} , are linear in the second derivatives of g_{hk} , and the tensor G_{hk} has structure to satisfy the identity or "conservation condition"

$$\nabla_k G_h^k \equiv 0. \quad (1.4.2)$$

It follows from this identity the need to impose the "conservation condition" to the material energy tensor

$$\nabla_k T^{hk} = 0. \quad (1.4.3)$$

We then verify if the equations now written are in accordance with experience, and if in first approximation lead to the Newtonian theory.

Remark 1.4.1 *In this chapter we wrote the law of motion of a free material point and the law of evolution of a generic material system closed in the space-time V_4 by the Rule of transcription, as well as the equations of the gravitational field that Einstein, after a sharp criticism mentioned in the preceding pages was induced to formulate. The above-mentioned equations are written in absolute form, ie independently of any reference system, in general coordinates.*

In this regard we also recall the equations of the electromagnetic field set out by Maxwell , which through the Rule of transcription assume the form, which is also absolute:

a) *non-homogeneous equations*

$$\nabla_k F^{hk} = s^h ; \quad (1.4.4)$$

b) *homogeneous equations*

$$\nabla_h F_{kl} + \nabla_k F_{lh} + \nabla_l F_{hk} = 0. \quad (1.4.5)$$

In them F_{hk} is the antisymmetric electromagnetic field tensor, which merges into a single absolute entity the electric field \underline{E} and the magnetic field \underline{H} , s^h is the 4-current density vector, that is subject to the condition

$$s^h s_h = -\rho_0^2 \quad (1.4.6)$$

and expresses by the formula

$$s^h \equiv \left(\frac{\rho \bar{v}}{c}, \rho \right) = \rho_0 \frac{U^h}{c} \quad (1.4.7)$$

where $U^h = \frac{dx^h}{d\tau}$, ρ_0 is the proper charge density, which is an invariant, $d\tau$ is the elementary invariant proper time, in fact by definition:

$$ds^2 = -c^2 d\tau^2 = g_{hk} dx^h dx^k \quad (1.4.8)$$

is the fundamental invariant in V_4 , g_{hk} is the metric tensor in V_4 [cfr. (??)].

The absolute formulation is conceptually rigorous and rewarding; but when an observer examines any natural phenomenon, he works in a well-defined physical reference. For example by examining in a laboratory an electromagnetic phenomenon, there he is able to deduce from the tensor field F^{hk} the electric field \underline{E} and magnetic field \underline{H} ; and from the vector s^h he is able to deduce the current density \underline{j} and the charge density ρ : all variables essentially depending on the physical reference in which the observer works, ie all quantities associated to a physical frame in which the observer works.

Therefore arises, for the observer, the problem can deduce by absolute physical laws, expressed in terms of absolute physical quantities, the corresponding relative physical laws expressed on the basis of corresponding physical quantities relating to his particular reference. Indeed, as the observer can operate in physical references of various kinds, it is appropriate to pose the problem in condition of the highest generality: "To deduce from the absolute physical laws the corresponding physical laws related to more general physical reference".

To resolve this problem, after giving an accurate definition of physical reference in

general Relativity, first we should develop a projection Technique in a 4-dimensional Riemannian manifold V_4 . We are doing that in the next chapter.

Chapter 2

Projection technique in a 4-dimensional Riemannian manifold V_4

2.1 Definition of physical frame of reference. Natural decomposition of a vector

2.1.1 Definition of physical frame of reference in a normal hyperbolic Riemannian manifold with signature $+++ -$

Bearing in mind that, locally, the Riemannian manifold, spacetime of general Relativity, is mixed with an element of the tangent hyperplane, an element of Minkowski space time, let us define in general the space time environment of every physical phe-

nomenon as a 4-dimensional Riemannian manifold V_4 , with normal hyperbolic metric, with signature $+ + + -^1$, in which a 3-dimensional infinity of “ideal particles” trace (space-time) curves with regularity, constituting an “ideal fluid of reference”, a time-like congruence Γ^2 . This congruence will employ, from now in the future, as the physical frame of reference, and let $\underline{\gamma}$ be the field of unit vectors tangent to the stream lines of the ideal particles, that is to the space-time lines of the fluid of reference, set towards the future. Let us choose vector $\underline{\gamma}$ with the norm:

$$\|\underline{\gamma}\| = -1 . \quad (2.1.1)$$

Let us consider in a 4-dimensional normal hyperbolic Riemannian manifold V_4 , a coordinate system $\{x^h\}$ where x^1, x^2, x^3 are three-parameter label space-like curves, constituting the ideal fluid of reference, and x^4 is a time-like parameter that describes a co-ordinated time associated to a watch tied to every particles of the fluid of reference. These watches have never stopped and two infinitely near events, not belonged to the same time-line, have infinitely near time-lines and instants regulated by their respective watches.

This coordinate system haven't a strictly physical meaning, although it's suitable for coordinate every events in the physical world.

When we choose a congruence Γ in V_4 , we say that we introduce a “physical frame of reference” \mathcal{S} in V_4 . The coordinate system $\{x^1, x^2, x^3, x^4\}$, that can be general, is “adapted to the frame \mathcal{S} ” if these following conditions are true:

¹The chosen signature is such that where ds^2 assumes pseudo-Euclidean form, there it is reduced to the quadratic form (??) [cfr. chapter I, note 2].

²It should not hinder in any way the conduct of physical phenomena.

a) the coordinate lines equations

$$\begin{cases} x^\alpha = \text{const} & \alpha = 1, 2, 3 \\ x^4 = \text{var} \end{cases}$$

coincide with the space-time lines of the particles of the fluid of reference. That is these lines are temporal ones ($ds^2 < 0$);

b) the lines equations

$$\begin{cases} x^\alpha = \text{var} & \alpha = 1, 2, 3 \\ x^4 = \text{const} \end{cases}$$

are spatial lines ($ds^2 > 0$).

This coordinate system so introduced is said system of “physically admissible coordinates”.

Let us consider x^4 the temporal coordinate, except the velocity of light in vacuum c :

$$x^4 = ct$$

and the three-dimensional reference frame (x^1, x^2, x^3) identifies every particles of the reference fluid and is constant respect them during their evolution (their space-time lines).

By the admissible condition, we can deduce the following conditions:

$$ds^2 = g_{44} (dx^4)^2 < 0 \Rightarrow g_{44} < 0 \tag{2.1.2}$$

$$ds^2 = g_{\alpha\rho} dx^\alpha dx^\rho > 0 \quad \alpha, \rho = 1, 2, 3 \quad (2.1.3)$$

If we think to make a general transformation of coordinates in V_4

$$x^{h'} = x^{h'}(x^1, x^2, x^3, x^4) \quad (2.1.4)$$

the time lines $\{x^{4'} = var, x^{\alpha'} = const\}$ are generally different from the original ones in the new coordinates, that means a change of physical reference generally takes place.

If we want that the transformation of coordinates (??) lets unchanged the time lines $\{x^4 = var\}$, that is an internal transformations at the physical reference S , we have to choose the transformation of coordinates in the following way:

$$\begin{cases} x^{\rho'} = x^\rho(x^1, x^2, x^3) \text{ the name of the particles of the fluid is changed} \\ x^{4'} = x^4(x^1, x^2, x^3, x^4) \end{cases} \quad (2.1.5)$$

This transformation is the product of the only spatial coordinates of a transformation

$$x^{\rho'} = x^{\rho'}(x^1, x^2, x^3) \quad x^{4'} = x^4 \quad (2.1.6)$$

for a transformation of this form:

$$x^{\rho'} = x^\rho \quad x^{4'} = x^4(x^1, x^2, x^3, x^4) \quad (2.1.7)$$

that modifies the coordinated clock of every particle of the fluid of reference.

2.1.2 Natural decomposition of a vector

Let us consider a physical frame of reference \mathcal{S} through a field of unitary vectors $\underline{\gamma}(x)$, with the norm $\|\underline{\gamma}\| = -1$, tangent to the stream lines of a time-like congruence Γ , set towards the future. So, in every point–event $P \in V_4$, we introduce a natural basis $\underline{e}_i = \partial_i P$ in adapted coordinates with

$$g_{hk} = \underline{e}_h \cdot \underline{e}_k = \partial_h P \cdot \partial_k P \quad (2.1.8)$$

So, in our system of adapted coordinates, we have:

$$\underline{\gamma} = \gamma^4 \underline{e}_4, \quad \|\underline{\gamma}\| = g_{hk} \gamma^h \gamma^k = g_{44} (\gamma^4)^2 = -1 \quad (2.1.9)$$

thus

$$\gamma^\rho = 0 \quad (\rho = 1, 2, 3), \quad \gamma^4 = \frac{1}{\sqrt{-g_{44}}}. \quad (2.1.10)$$

We deduce from these the following formulas:

$$\gamma_h = g_{h4} \gamma^4 = \frac{g_{h4}}{\sqrt{-g_{44}}} \quad h = 1, 2, 3, 4. \quad (2.1.11)$$

In every general point–event $P \equiv \{x^h\} \in V_4$, let us consider the hyperplane T_x tangent to V_4 as sum of two subspaces, supplementary of each other, Σ_x and Θ_x with Θ_x 1–dimensional like–time space, tangent to the stream line $x^4 = var.$, and Σ_x 3–dimensional space orthogonal to Θ_x . We will say that Θ_x and Σ_x give respectively the temporal direction and the spatial platform locally associated to the physical frame

of reference \mathcal{S} . So, we can write:

$$T_x = \Theta_x + \Sigma_x \quad (2.1.12)$$

The vectors belonging to Θ_x are *purely temporal* vectors, while the vectors belonging to Σ_x are *purely spatial* ones. In adapted coordinates, every purely temporal vector \underline{V} , as $\underline{\gamma}(x)$, has the first three contravariant components V^ρ equal to zero, that is

$$\underline{V} \in \Theta_x, \quad V^\rho = 0 \quad \rho = 1, 2, 3 \quad (2.1.13)$$

while every purely spatial vector, being orthogonal to $\underline{\gamma}(x)$, has the fourth covariant component equal to zero, that is

$$\underline{V} \cdot \underline{\gamma} = \gamma^r V_r = \gamma^4 V_4 = 0 \Rightarrow V_4 = 0, \quad \underline{V} \in \Sigma_x. \quad (2.1.14)$$

Let us consider a general vector $\underline{V} \in T_x$; it can be uniquely decomposed into two vectors for the (??)

$$\underline{V} = \underline{V}_\Theta + \underline{V}_\Sigma \quad (2.1.15)$$

with

$$\underline{V}_\Theta = -(\underline{V} \cdot \underline{\gamma})\underline{\gamma}, \quad \underline{V}_\Sigma = \underline{V} - \underline{V}_\Theta = \underline{V} + (\underline{V} \cdot \underline{\gamma})\underline{\gamma} \quad (2.1.16)$$

By

$$\underline{V}_\Theta \cdot \underline{\gamma} = -(\underline{V} \cdot \underline{\gamma})\underline{\gamma} \cdot \underline{\gamma} = \underline{V} \cdot \underline{\gamma} \quad (2.1.17)$$

these remarkable formulas follow:

$$(\underline{V}_\Theta)_h = -(V^r \gamma_r) \gamma_h = -\gamma_h \gamma_r V^r \quad (2.1.18)$$

$$(\underline{V}_\Sigma)_h = V_h + V_r \gamma^r \gamma_h = g_{hr} V^r + V^r \gamma_r \gamma_h = (g_{hr} + \gamma_r \gamma_h) V^r = \gamma_{hr} V^r \quad (2.1.19)$$

where

$$\gamma_{hr} = g_{hr} + \gamma_r \gamma_h \quad (2.1.20)$$

The eq.(??) is called natural decomposition of a general vector $\underline{V} \in T_x$ and the two component vectors \underline{V}_Θ , \underline{V}_Σ are respectively the temporal projection and the spatial projection of the vector \underline{V} . They are label as follows:

$$\underline{V}_\Theta = \mathcal{P}_\Theta(\underline{V}) \quad , \quad \underline{V}_\Sigma = \mathcal{P}_\Sigma(\underline{V}) \quad (2.1.21)$$

From the eqs. (??) and (??), we called the two tensors

$$-\gamma_h \gamma_r \quad , \quad \gamma_{hr} \equiv g_{hr} + \gamma_r \gamma_h \quad (2.1.22)$$

time projector and space projector respectively. We can see that the following condition is true for the tensor γ_{hr}

$$\gamma_{4u} = 0, \quad u = 1, 2, 3, 4 \quad (2.1.23)$$

because

$$g_{4u} + \gamma_4 \gamma_u = g_{4u} + \frac{g_{4u} g_{44}}{-g_{44}} = 0.$$

The eqs. (??) and (??) remark the purely temporal character of the tensor $-\gamma_h \gamma_r$ and the purely spatial character of the tensor γ_{hr} .

Let us now consider the norm of the vectors \underline{V}_Θ and \underline{V}_Σ , we obtain:

$$\|\underline{V}_\Theta\| = g_{hr}(\underline{V}_\Theta)^h(\underline{V}_\Theta)^r = g_{hr}(V^r \gamma_k) \gamma^h (V^s \gamma_s) \gamma^r = V^k \gamma_k V^s \gamma_s g_{hr} \gamma^h \gamma^r = -(V^k \gamma_k)^2$$

so

$$\|\underline{V}_\Theta\| = -(V^k \gamma_k)^2 < 0 \quad (2.1.24)$$

$$\begin{aligned} \|\underline{V}_\Sigma\| &= g_{hr}(\underline{V}_\Sigma)^h(\underline{V}_\Sigma)^r = g_{hr} \gamma_u^h V^u \gamma_z^r V^z = g_{hr} (g_u^h + \gamma^h \gamma_u) (g_z^r + \gamma^r \gamma_z) V^u V^z = \\ &= (g_{ru} + \gamma_r \gamma_u) V^u (g_z^r + \gamma^r \gamma_z) V^z = \gamma_{ru} V^u (\delta_z^r + \gamma^r \gamma_z) V^z = \\ &= \gamma_{ru} V^u V^r + (\gamma^{ru} \gamma^r) \gamma_z V^u V^z = \gamma_{ru} V^r V^u \end{aligned}$$

so

$$\|\underline{V}_\Sigma\| = \gamma_{ru} V^r V^u. \quad (2.1.25)$$

These two last formulas remark the purely temporal character of the tensor $-\gamma_h \gamma_r$ and the purely spatial character of the tensor γ_{hr} again. In eqs.(??) (??) they act as time-projector and space-projector of the vector V respectively, while in the eqs. (??) (??) they perform a metric function. For this reason, they are called temporal metric tensor and spatial metric tensor respectively. Moreover these two last equations suggest that the temporal norm of a vector is negative, while the spa-

tial one is positive, in fact by eq.(??)

$$\gamma_{ru}V^rV^u = \gamma_{\alpha\rho}V^\alpha V^\rho = (g_{\alpha\rho} + \gamma_\alpha\gamma_\rho)V^\alpha V^\rho = g_{\alpha\rho}V^\alpha V^\rho + (\gamma_\alpha V^\alpha)^2 > 0.$$

2.2 Natural decomposition of a general tensor

2.2.1 Time and space projections of a tensor

It's known from the tensorial algebra that, if we consider the tensorial product of T_x for itself (or for its dual)[?] and, so, every double tensor in this tangent product space, every general double tensor can be uniquely decomposed into the sum of four tensors by the decomposition (??):

$$T_x \otimes T_x = (\Theta_x + \Sigma_x) \otimes (\Theta_x + \Sigma_x) = \Theta_x \otimes \Theta_x + \Theta_x \otimes \Sigma_x + \Sigma_x \otimes \Theta_x + \Sigma_x \otimes \Sigma_x \quad (2.2.1)$$

By this decomposition formula, it follows that every general double tensor $t_{hk} \in T_x \otimes T_x$ can be uniquely decomposed into the sum of four tensors belonging to the four subspaces of the (??), that are orthogonal to each other:

$$t_{hk} = \mathcal{P}_{\Theta\Theta}(t_{hk}) + \mathcal{P}_{\Theta\Sigma}(t_{hk}) + \mathcal{P}_{\Sigma\Theta}(t_{hk}) + \mathcal{P}_{\Sigma\Sigma}(t_{hk}) \quad (2.2.2)$$

This is called *natural decomposition* of the tensor t_{hk} .

We must consider the eqs. (??) (??) to project formally: we must use the time-projector $-\gamma_h\gamma_r$ to obtain the time projection, involving the only index r of the vector \underline{V} , written in contravariant position, to obtain the covariant component of index h ;

we must use the space-projector $-\gamma_{hr}$ to obtain the space projection, involving the only index r of the vector \underline{V} , written in contravariant position, to obtain the covariant component of index h .

We act on the indices of a double tensor in the same way: we will use the time-projector $-\gamma_i\gamma_j$ on both indexes of the tensor t_{hk} , written in contravariant position, to obtain the purely time projection:

$$\mathcal{P}_{\Theta\Theta}(t_{hk}) = (-\gamma_i\gamma_h)(-\gamma_j\gamma_k)t^{hk} = \gamma_i\gamma_j(\gamma_h\gamma_k t^{hk}) \equiv \tau_{ij} \quad (2.2.3)$$

We will use the time-projector $-\gamma_i\gamma_j$ on the index h and the space-projector $-\gamma_{ij}$ on the index k to obtain

$$\mathcal{P}_{\Theta\Sigma}(t_{hk}) = -\gamma_i\gamma_h t^{hk} \gamma_{jk} = -\gamma_i\gamma_h \gamma_{jk} t^{hk} \equiv \theta_{ij} \quad (2.2.4)$$

By this time, it's obvious that:

$$\mathcal{P}_{\Sigma\Theta}(t_{hk}) = -\gamma_{ih} \gamma_j \gamma_k t^{hk} \equiv \sigma_{ij} \quad (2.2.5)$$

$$\mathcal{P}_{\Sigma\Sigma}(t_{hk}) = \gamma_{ih} \gamma_j \gamma_k t^{hk} \equiv s_{ij}. \quad (2.2.6)$$

It's clear that:

$$t_{ij} = \tau_{ij} + \theta_{ij} + \sigma_{ij} + s_{ij} \quad (2.2.7)$$

in consequence of:

$$\begin{aligned}
& \tau_{ij} + \theta_{ij} + \sigma_{ij} + s_{ij} = \\
& = \gamma_i \gamma_j \gamma_h \gamma_k t^{hk} - \gamma_i \gamma_h \gamma_j \gamma_k t^{hk} - \gamma_j \gamma_k \gamma_i \gamma_h t^{hk} + \gamma_{ih} \gamma_j \gamma_k t^{hk} = \\
& = \gamma_i \gamma_j \gamma_h \gamma_k t^{hk} - \gamma_i \gamma_h (g_{jk} + \gamma_j \gamma_k) t^{hk} - \gamma_j \gamma_k (g_{ih} + \gamma_i \gamma_h) t^{hk} + (g_{ih} + \gamma_i \gamma_h) (g_{jk} + \gamma_j \gamma_k) t^{hk} = t_{ij}.
\end{aligned}$$

It can sometimes be useful to decompose $T_x \otimes T_x$ in the following way:

$$T_x \otimes T_x = (\Theta_x + \Sigma_x) \otimes T_x = \Theta_x \otimes T_x + \Sigma \otimes T_x \quad (2.2.8)$$

or

$$T_x \otimes T_x = T_x \otimes (\Theta_x + \Sigma_x) = T_x \otimes \Theta_x + T_x \otimes \Sigma_x. \quad (2.2.9)$$

In these cases the general tensor t_{ij} is uniquely decomposed into the sum of two tensors orthogonal to each other: we act on the first index of t_{ij} with the time-projector and then we add a tensor derived acting on the first index with the space-projector; we obtain:

$$t_{ij} = \mathcal{P}_\Theta(t_{\dot{i}j}) + \mathcal{P}_\Sigma(t_{\dot{i}j}) = -\gamma_i \gamma_h t_j^h + \gamma_i^h t_{hj} \equiv -\gamma_i \gamma^h t_{hj} + \gamma_{ih} t_j^h. \quad (2.2.10)$$

We use a point to indicate the index that is interested by the projector.

In the same way, from (??), we have:

$$t_{ij} = \mathcal{P}_\Theta(t_{i\dot{j}}) + \mathcal{P}_\Sigma(t_{i\dot{j}}) = -\gamma_j \gamma_h t_i^h + \gamma_j^h t_{ih} \equiv -\gamma_j \gamma^h t_{ih} + \gamma_{jh} t_i^h. \quad (2.2.11)$$

It's now easy to extend the projection operations on the tensors of order > 2 .

Besides, it's advisable to remark the following assertions:

- a) An index of t_{ij} involved in a space projection verifies the condition that the obtained tensor components which have that index in covariant position equal to four are all null.
- b) An index of t_{ij} involved in a time projection verifies the condition that the obtained tensor components which have that index in contravariant position not equal to four are all null.
- c) A tensor with all spatial indices (which means that the components having the indices in covariant position equal to four are all null) is called totally spatial tensor; as an example the tensor γ_{hk} (??) is totally spatial tensor.
- d) A tensor with all temporal indices (which means that the components having the indices in contravariant position not equal to four are all null) is called totally temporal tensor; as an example the tensor $\gamma^h\gamma^k$ (??) is totally temporal tensor.

2.2.2 Remarkable algebraic properties of the projection operation

It's advisable to remark some properties of the projection operation that sometimes allow to make calculations more quickly.

- 1) If the projection Θ is applied on a spatial index, null components are obtained.
- 2) The projection Σ on a temporal index of a tensor gives null components.

- 3) If a projection is applied on the indices of the product of two tensors (without contractions), it's possible to put the projection on the separate factors on condition that we maintain order initially indicated; for example:

$$\mathcal{P}_{\Sigma\Theta}(u_{\mathfrak{h}k}v_{\mathfrak{l}r}) = \mathcal{P}_{\Sigma}(u_{\mathfrak{h}k})\mathcal{P}_{\Theta}(v_{\mathfrak{l}r}), \quad (2.2.12)$$

$$\mathcal{P}_{\Sigma\Sigma}(u_{\mathfrak{h}k}v_{\mathfrak{l}r}) = \mathcal{P}_{\Sigma}(u_{\mathfrak{h}k})\mathcal{P}_{\Sigma}(v_{\mathfrak{l}r}). \quad (2.2.13)$$

- 4) Projection and sum are permutable to each other if the sum not involves the indices interested in the projection operation:

$$\mathcal{P}_{\Sigma}(u_{\mathfrak{h}k}v^{kl}) = \mathcal{P}_{\Sigma}(u_{\mathfrak{h}k})v^{kl}. \quad (2.2.14)$$

- 5) Put:

$$\left\{ \begin{array}{l} t = \gamma_h\gamma_k t^{hk} \\ \tilde{t}'_j = -\gamma_{jk}\gamma_h t^{hk} \quad \text{purely spatial vector} \\ \tilde{t}_i = -\gamma_{ih}\gamma_k t^{hk} \\ \tilde{t}_{ij} = \gamma_{ih}\gamma_{jk} t^{hk} \equiv s_{ij} \quad \text{purely spatial tensor} \end{array} \right. \quad (2.2.15)$$

it can be useful in the calculations to rewrite the formula (??) in the following way:

$$t_{ij} = \gamma_i\gamma_j t + \gamma_i\tilde{t}'_j + \gamma_j\tilde{t}_i + \tilde{t}_{ij}. \quad (2.2.16)$$

This last formula shows that all double tensor t_{hk} is totally characterized by a scalar t , two purely spatial vectors \tilde{t}'_j and \tilde{t}_i and a totally spatial double tensor \tilde{t}_{ij} .

From (??) it's drawn that if the tensor t_{ij} is symmetric, the tensor \tilde{t}_{ij} also results symmetric while the two vectors \tilde{t}'_i and \tilde{t}_i are the same; in that case the Eq. (??) becomes:

$$t_{ij} = t_{ji} = \gamma_i \gamma_j t + \gamma_i \tilde{t}'_j + \gamma_j \tilde{t}'_i + \tilde{t}_{ij}. \quad (2.2.17)$$

If on the contrary the tensor t_{ij} is antisymmetric ($t_{ij} = -t_{ji}$), also its totally spatial projection \tilde{t}_{ij} is antisymmetric ($\tilde{t}_{ij} = -\tilde{t}_{ji}$) while the two spatial vectors \tilde{t}'_i and \tilde{t}_i are opposite

$$\tilde{t}'_j = \gamma_{jk} \gamma_h t^{hk} = -\tilde{t}_j$$

and the scalar t is equal to zero. Therefore the natural decomposition of a double antisymmetric tensor is:

$$t_{ij} = \gamma_j \tilde{t}'_i - \gamma_i \tilde{t}'_j + \tilde{t}_{ij} \quad (t_{ij} = -t_{ji}, \quad \tilde{t}_{ij} = -\tilde{t}_{ji}). \quad (2.2.18)$$

- 6) When a purely spatial index becomes saturated with a purely temporal one and they are in opposite position of variance, the result is zero and the other indices that don't involve in the saturation are not considered.

- 7) It's immediate to verify that for the metric tensor of V_4 , $g_{hk} = \gamma_{hk} - \gamma_h \gamma_k$, the

following conditions are true:

$$\begin{cases} \mathcal{P}_{\Theta\Theta}(g_{ij}) = -\gamma_i\gamma_j \\ \mathcal{P}_{\Theta\Sigma}(g_{ij}) = 0, \quad \mathcal{P}_{\Sigma\Theta}(g_{ij}) = 0 \\ \mathcal{P}_{\Sigma\Sigma}(g_{ij}) = \gamma_{ij} \end{cases} \quad (2.2.19)$$

2.3 Transverse partial derivative. Longitudinal derivative. Lie derivative

2.3.1 Transverse partial derivative

Let us define a scalar field $\varphi(x)$ in a domain \mathcal{C} of the manifold V_4 and let us consider in every general point–event $P \equiv \{x^h\} \in \mathcal{C}$ the set of vectors dP tangent to the spatial platform Σ in P ; this condition translates into the following mathematical relation:

$$\underline{\gamma}(x) \cdot dP = \gamma_i dx^i = 0 \quad (2.3.1)$$

\Rightarrow

$$dx^4 = -\gamma_\rho dx^\rho \frac{1}{\gamma_4} = \gamma^4 \gamma_\rho dx^\rho \quad \rho = 1, 2, 3 (\gamma_4 \gamma^4 = -1). \quad (2.3.2)$$

Now we can calculate the total differential of the scalar field $\varphi(x)$ performed according to the vector dP ; we obtain:

$$\begin{aligned} d\varphi &= \partial_i \varphi dx^i = \partial_\rho \varphi dx^\rho + \partial_4 \varphi dx^4 = \partial_\rho \varphi dx^\rho + \gamma^4 \partial_4 \varphi \gamma_\rho dx^\rho = \\ &= (\partial_\rho \varphi + \gamma_\rho \gamma^4 \partial_4 \varphi) dx^\rho. \end{aligned} \quad (2.3.3)$$

So it's introduced the operator:

$$\tilde{\partial}_\rho \equiv \partial_\rho + \gamma_\rho \gamma^4 \partial_4 \quad \left[\tilde{\partial}_4 \equiv \partial_4 + \gamma_4 \gamma^4 \partial_4 \equiv 0 \right] \quad (2.3.4)$$

that will be called transverse partial derivative, that allows to make the total differential of a scalar field $\varphi(x)$ according to a direction orthogonal to Σ

$$d\varphi = \tilde{\partial}_\rho \varphi dx^\rho. \quad (2.3.5)$$

The quantities $\tilde{\partial}_\rho \varphi$ are the covariant components of a vector orthogonal to $\underline{\gamma}$ belonging to the spatial platform in P , Σ_P , that we will call the transverse gradient of the scalar field $\varphi(x)$:

$$g\tilde{r}ad\varphi \equiv \tilde{\partial}_\rho \varphi. \quad (2.3.6)$$

It's obvious that the operator $\tilde{\partial}_\rho$ is invariant under coordinate changes in the physical frame of reference \mathcal{S} , as the (??) shows.

It's easy to verify that the transverse partial differentiation has the formal properties of the ordinary partial differentiation, as the derivative of a sum, of a product, of a quotient; it can be also applied subsequently but the order of the following partial differentiations is not permutable as it is not generally changed with the operator ∂_4 .

It's immediate to point out that the projection of the vector $grad\varphi \equiv \partial_h \varphi$ on the spatial platform gives the spatial vector $g\tilde{r}ad\varphi$:

$$\mathcal{P}_\Sigma(\partial_h \varphi) = \gamma_h^k \partial_k \varphi = (\delta_h^k + \gamma_h \gamma^k) \partial_k \varphi = \partial_h \varphi + \gamma_h \gamma^4 \partial_4 \varphi \equiv \tilde{\partial}_h \varphi. \quad (2.3.7)$$

2.3.2 Longitudinal derivative or Local temporal derivative

If we consider an orthogonal projection of the field $grad\varphi$ on Σ_x , we obtain the field of purely temporal vectors [(??), (??)]

$$-\gamma_h\gamma^k\partial_k\varphi = -\gamma_h\gamma^4\partial_4\varphi. \quad (2.3.8)$$

We will call longitudinal derivative or local temporal derivative in the general point-event $P \in \mathcal{C}$ of the scalar field $\varphi(x)$ the quantity

$$\gamma^4\partial_4\varphi. \quad (2.3.9)$$

This operator isn't commutable, in general, with the operator $\tilde{\partial}_h$ ($h = 1, 2, 3, 4$), as it can verify directly.

Bearing in mind the invariance of the scalar

$$\underline{\gamma} \cdot grad\varphi = \gamma^4\partial_4\varphi$$

we can affirm that the operator $\gamma^4\partial_4$ is invariant with respect to every change of coordinates in the physical frame of reference \mathcal{S} .

In the following section we will show that the longitudinal derivative is a particular case of an operation more general, the Lie derivative, when the physical system of reference \mathcal{S} is interpreted as the set of the trajectories of a group of one-parameter transformations, having the field of vectors $\underline{\gamma}(x)$ with the norm -1 as generator field.

2.3.3 Lie derivative of a tensor field

Let us assign an open domain \mathcal{C} of a differential manifold V_4 with a system of local coordinates x^k , when we define a field of timelike unit vectors $\underline{u}(x)$ and a system of ordinary differential equations

$$\frac{d}{dt}x_t^h = u^h(x_t) \quad (2.3.10)$$

where x_t^h are functions of one parameter t [$x^h = x_t^h$ for $t = 0$].

We suppose that the differential system (??) has a regular solution in \mathcal{C} corresponding with the initial data x^h ,

$$x_t^h = \varphi^h(t|x^1, x^2, x^3, x^4) \quad h = 1, 2, 3, 4 \quad (2.3.11)$$

which can be written in the condensed form

$$x_t = T_t x . \quad (2.3.12)$$

We can observe that the Eqs. (??), which are invertible, define a transformation of V_4 into itself in the domain \mathcal{C} , for every t that is $|t| < \eta$ with $\eta > 0$, for every variation of the x^h in \mathcal{C} .

On the contrary if we choose an initial point $P \equiv [x^h, t = 0]$ in \mathcal{C} and change with continuity the parameter t , we obtain a continuum set of points P_t constituting a straight line that can be interpreted as trajectory of a particle, initially in P , with instantaneous velocity $\underline{u}(x_t)$ [$dx_t^h = u^h(x_t)dt$]. Concerning this, it's advisable to remind that the theory of the systems of ordinary differential equations assure the following results:

- For every point $Q \in \mathcal{C}$ it can to determinate a neighborhood $I_Q \in \mathcal{C}$ and a positive scalar η_Q for them, for every $P \in I_Q$ and $|t| < \eta_Q$, there is a transformation $T_t x$ that has the following properties:[?]

$$\text{a) } T_{t'} T_{t''} x = T_{t'+t''} x \quad |t'|, |t''| < \eta_Q, |t'| + |t''| < \eta_Q$$

$$\text{b) } T_t^{-1} x = T_{-t} x$$

$$\text{c) } T_0 x = x.$$

This set of transformations is a group of local transformations of V_4 in \mathcal{C} , with t -parameter, produced from the field of unit vectors $\underline{u}(t, x)$, $\{u^h(x_t) = u^h(t|x)\}$. Let us indicate this group with $G_1(\underline{u})$ and call canonical parameter t , generator field $\underline{u}(t|x)$.

If from every point $Q \in \mathcal{C}$ we change with continuity the parameter t , it is generated a line, the trajectory of $G_1(\underline{u})$ for Q in \mathcal{C} . The set of the trajectories of $G_1(\underline{u})$ is a congruence Γ of V_4 in \mathcal{C} . Every transformation T_t of $G_1(\underline{u})$ introduces an isomorphism between the vectorial spaces tangent to V_4 respectively in x and in x_t , and consequently between the corresponding dual spaces. It follows that there are these relations between two isomorphic vector fields $\underline{u}(x)$ and $\underline{w}(x_t)$

$$\begin{cases} w^h(x_t) \equiv T_t v^h(x) = \frac{\partial}{\partial x^k} x_t^h v^k(x) & [v^{h'} = \theta_{h'}^{h'} v^h, \theta_{h'}^{h'} \equiv \frac{\partial}{\partial x^h} x_t^h] \\ w_h(x_t) \equiv T_t v_h(x) = \frac{\partial}{\partial x_t^h} x^k v_k(x) & [v_{h'} = \theta_{h'}^h v_h, \theta_{h'}^h \equiv \frac{\partial}{\partial x_t^h} x^h] \end{cases} \quad (2.3.13)$$

that can be synthesized using the only vector relation

$$\underline{w}(x_t) \equiv T_t \underline{v}(x). \quad (2.3.14)$$

Now we can define Lie derivative of an assigned field of vectors $\underline{v}(x)$, defined into the domain \mathcal{C} , with respect to the group $G_1(\underline{u})$.

For this reason we estimate the incremental quotient

$$\frac{T_t^{-1}\underline{v}(x_t) - \underline{v}(x)}{t} \quad (2.3.15)$$

in the point x , because of $T_t^{-1}\underline{v}(x_t)$ is the image of the vector $\underline{v}(x_t)$ in x [that is the transformed vector in x of the vector $\underline{v}(x_t)$ through the transformation T_t^{-1}].

We admit the existence of the incremental quotient for $t \rightarrow 0$ and obtain a vector, defined in x , that is called Lie derivative with respect to the group $G_1(\underline{u})$ of $\underline{v}(x)$:

$$\mathcal{L}_{\underline{u}}v = \lim_{t \rightarrow 0} \frac{T_t^{-1}\underline{v}(x_t) - \underline{v}(x)}{t}. \quad (2.3.16)$$

In scalar form, working on the contravariant components v^k ,

$$\begin{aligned}
 \mathcal{L}_{\underline{u}}v^k &= \lim_{t \rightarrow 0} \frac{T_t^{-1}v^k(x_t) - v^k(x)}{t} = \\
 &= \lim_{t \rightarrow 0} \frac{\frac{\partial x^k}{\partial x^h} v^h(x_t) - v^k(x)}{t} = \\
 &= \lim_{t \rightarrow 0} \frac{1}{t} \left\{ \left[\delta_h^k - \frac{\partial u^k}{\partial x^h}(x_t), t \right] v^h(x_t) - v^k(x) \right\} = \\
 &= \lim_{t \rightarrow 0} \frac{1}{t} \left\{ v^k(x_t) - \frac{\partial u^k}{\partial x^h}(x_t) \cdot t \cdot v^h(x_t) - v^k(x) \right\} = \\
 &= \lim_{t \rightarrow 0} \frac{1}{t} \left\{ v^k(x_t) - v^k(x) \right\} - \lim_{t \rightarrow 0} \frac{\partial u^k}{\partial x^h}(x_t) v^h(x_t) = \\
 &= \lim_{t \rightarrow 0} \frac{1}{t} \left\{ v^k[x^r + u^r(x_t)t + \dots] - v^k(x^r) \right\} - \frac{\partial u^k}{\partial x^h}(x) v^h(x) = \\
 &= \lim_{t \rightarrow 0} \frac{1}{t} \left\{ v^k(x^r) + \frac{\partial v^k}{\partial x^r} u^r(x_t)t - v^k(x^r) \right\} - v^h \partial_h u^k = \\
 &= \lim_{t \rightarrow 0} \frac{\partial v^k}{\partial x^r} u^r(x_t) - v^h \partial_h u^k = \partial_r v^k \cdot u^r - v^h \partial_h u^k
 \end{aligned}$$

so in conclusion:

$$\mathcal{L}_{\underline{u}}v^k = u^r \partial_r v^k - v^h \partial_h u^k \quad (r, h, k = 1, 2, 3, 4). \quad (2.3.17)$$

In a similar way, but working on the covariant components v_k , we obtain the formula:

$$\mathcal{L}_{\underline{u}}v_k = u^r \partial_r v_k + v_r \partial_k u^r \quad (r, h, k = 1, 2, 3, 4). \quad (2.3.18)$$

In particular for a scalar function $f(x)$:

$$\mathcal{L}_{\underline{u}}f(x) = u^r \partial_r f(x) \quad (2.3.19)$$

Working on every tensor field of order ≥ 2 in a similar way, for example on the field of order 3 A_{hk}^r , we can establish the following formula:

$$\mathcal{L}_{\underline{u}}A_{hk}^r = u^s \partial_s A_{hk}^r + A_{sk}^r \partial_h u^s + A_{hs}^r \partial_k u^s - A_{hk}^s \partial_s u^r. \quad (2.3.20)$$

If the differential manifold V_4 is Riemannian too, and we suppose that it has a hyperbolic metric that is, according to Hadamard,

$$ds^2 = g_{hk} dx^h dx^k$$

with signature $\{+ + + -\}$, the Eqs. (??), (??), (??), that have a tensorial character for their definitions, can be completed by the introduction of the covariant derivative.

It has to add and subtract in (??) and (??) respectively

$$u^r \Gamma_{rs}^k v^s, \quad u^r \Gamma_{rk}^s v_s$$

to obtain

$$\begin{aligned}
 \mathcal{L}_{\underline{u}}v^k &= u^r [\partial_r v^k + \Gamma_{rs}^k v^s] - v^r \partial_r u^k - u^r \Gamma_{rs}^k v^s = \\
 &= u^r \nabla_r v^k - v^r \partial_r u^k - u^p \Gamma_{pr}^k v^r = u^r \nabla_r v^k - v^r \nabla_r u^k \\
 \mathcal{L}_{\underline{u}}v^k &= u^r \nabla_r v^k - v^r \nabla_r u^k
 \end{aligned} \tag{2.3.21}$$

$$\begin{aligned}
 \mathcal{L}_{\underline{u}}v_k &= u^r [\partial_r v_k - \Gamma_{rk}^s v_s] + v_r \partial_k u^r + u^r \Gamma_{rk}^s v_s = \\
 &= u^r \nabla_r v_k + v_r [\partial_k u^r + u^p \Gamma_{pk}^r] = u^r \nabla_r v_k + v_r \nabla_k u^r \\
 \mathcal{L}_{\underline{u}}v_k &= u^r \nabla_r v_k + v_r \nabla_k u^r.
 \end{aligned} \tag{2.3.22}$$

In a similar way, we have from (??)

$$\mathcal{L}_{\underline{u}}A_{hk}^r = u^s \nabla_s A_{hk}^r + A_{sk}^r \nabla_h u^s + A_{hs}^r \nabla_k u^s - A_{hk}^s \nabla_s u^r. \tag{2.3.23}$$

Calculating, for example, the Lie derivative of the metric tensor g_{hk} of V_4 , we obtain

$$\begin{aligned}
 \mathcal{L}_{\underline{u}}g_{hk} &= u^s \nabla_s g_{hk} + g_{sk} \nabla_h u^s + g_{hs} \nabla_k u^s = 0 + \nabla_h (g_{sk} u^s) + \nabla_k (g_{hs} u^s) \\
 \mathcal{L}_{\underline{u}}g_{hk} &= \nabla_h u_k + \nabla_k u_h \equiv K_{hk}.
 \end{aligned} \tag{2.3.24}$$

We consider now the following properties of the Lie derivative:

- a) The Lie derivative and the contraction are permutable operations;
- b) The Lie derivative of the product of two tensors performs as covariant derivative;
- c) The Lie derivative and a transformation of the group $G_1(\underline{u})$ are permutable;
- d) The Lie derivative performed on an antisymmetric covariant tensor field is permutable with the operation of external differentiation;

e) The Lie derivative and the projection operation are permutable.

We give now the definition of invariance of a vector field $\underline{v}(x)$ with respect to a group $G_1(\underline{u})$.

If for every point P in a neighborhood $I_Q \in \mathcal{C}$ and $|t| < \eta_Q$ it results

$$T_t v^h(x) = v^h(x_t) \quad (2.3.25)$$

we call the vector field $\underline{v}^h(x)$ invariant with respect to the group $G_1(\underline{u})$ or more brevity \underline{u} -invariant in \mathcal{C} .

In that case it follows from (??)

$$\mathcal{L}_{\underline{u}} v^h = 0. \quad (2.3.26)$$

If, vice versa, the vector field $\underline{v}^h(x)$ verifies the condition (??) and we calculate the total derivative of the vector field $T_t^{-1} v^h(x_t)$, we get directly from the definition

$$\begin{aligned} \frac{d}{dt} [T_t^{-1} v^h(x_t)] &= \lim_{s \rightarrow 0} \frac{T_{t+s}^{-1} v^h(x_{t+s}) - T_t^{-1} v^h(x_t)}{s} = \\ &= \lim_{s \rightarrow 0} \frac{T_s^{-1} T_t^{-1} v^h(x_{t+s}) - T_t^{-1} v^h(x_t)}{s} = \mathcal{L}_{\underline{u}} [T_t^{-1} v^h(x_t)] = \\ &= T_t^{-1} (\mathcal{L}_{\underline{u}} v^h)_{x=x_t} = 0 \end{aligned}$$

that is

$$T_t^{-1} v^h(x_t) = \text{constant varying } t \text{ and so equal to the} \\ \text{some expression calculated for } t = 0:$$

$$T_t^{-1}v^h(x_t) = v^h(x).$$

Therefore we have obtained the following result:

Proposition 2.3.1 *A vector field $v^h(x)$ is invariant with respect to a group $G_1(\underline{u})$ in \mathcal{C} if and only if*

$$\mathcal{L}_{\underline{u}}v^h = 0 \text{ in } \mathcal{C}.$$

It's immediate to verify from (??) the following identity:

$$\mathcal{L}_{\underline{u}}u^k = u^r \partial_r u^k(x) - u^r \partial_r u^k(x) \equiv 0. \quad (2.3.27)$$

The definition of invariance with respect to a group $G_1(\underline{u})$ can be generalized to a tensor field of every order.

It's also common to define the Γ -invariance for a tensor field in the following way:

Definition 2.3.2 *A vector field is Γ -invariant if it is invariant with respect to every vector field $\underline{v}(x) = a\underline{u}(x)$ tangent to the congruence Γ , where $a(x)$ is a general scalar function.*

Likewise the precedent case (the invariance with respect to a group), it can be demonstrated that the necessary and sufficient condition for the Γ -invariance of a vector field $\underline{v}(x)$ is

$$\mathcal{L}_{\underline{v}}v^h = 0 \quad \forall a(x). \quad (2.3.28)$$

2.3.4 Systems of coordinates adapted to a congruence

As we have already seen, if we assign a group of transformations $G_1(\underline{u})$ in V_4 then it is individuated by a well established congruence Γ . Introducing a system of local coordinates, the field of unit vectors $\underline{u}(x)$ that generates the group has components like that its norm is unitary. But we can choose coordinate systems that admit the trajectories of Γ as coordinate-lines, for example as lines of equations

$$\begin{cases} x^\rho = \text{const.} & \rho = 1, 2, 3 \\ x^4 = \text{var.} \end{cases} \quad (2.3.29)$$

In that case the vector field $\underline{u}(x)$ has components

$$u^\rho(x) \equiv 0, \quad u^4(x) \neq 0 \quad (2.3.30)$$

and the coordinate system is called Γ -adapted.

More in particular if the coordinate system $\{x^h\}$ is like that the contravariant components of the vector field $\underline{u}(x)$ verify the conditions

$$\begin{cases} x^\rho = 0 & \rho = 1, 2, 3 \\ x^4 = 1 \end{cases} \quad (2.3.31)$$

then this coordinate system is called \underline{u} -adapted.

Using Γ -adapted coordinates, the Eqs. (??), (??), (??) become respectively

$$\mathcal{L}_{\underline{u}}v^k = u^4 \nabla_4 v^k - v^r \nabla_r u^k \quad (2.3.32)$$

$$\mathcal{L}_{\underline{u}}v_k = u^4\nabla_4v_k + v_r\nabla_ku^r \quad (2.3.33)$$

$$\mathcal{L}_{\underline{u}}A_{hk}^r = u^4\nabla_4A_{hk}^r + A_{sk}^r\nabla_hu^s + A_{hs}^r\nabla_ku^s - A_{hk}^s\nabla_su^r \quad (2.3.34)$$

with

$$\nabla_ru^k = \partial_ru^k + \left\{r^k_s\right\}u^s$$

and from there

$$\nabla_ru^\rho = \left\{r^\rho_4\right\}u^4, \quad \nabla_ru^4 = \partial_ru^4 + \left\{r^4_4\right\}u^4. \quad (2.3.35)$$

It can also be proved that there is an infinite set of systems of Γ -adapted coordinates; in fact, being

$$u^{\rho'} = u^r\partial_rx^{\rho'} = u^4\partial_4x^{\rho'} = 0$$

it follows that $x^{\rho'}$ have to verify the conditions

$$\partial_4x^{\rho'} = 0 \quad (2.3.36)$$

hence the $x^{\rho'}$ have to be like that

$$x^{\rho'} = x^{\rho'}(x^1, x^2, x^3) \quad (2.3.37)$$

while for $u^{4'}$

$$u^{4'} = u^r\partial_rx^{4'} = u^4\partial_4x^{4'}$$

hence $x^{4'}$ has to be like that

$$x^{4'} = x^{4'}(x^1, x^2, x^3, x^4). \quad (2.3.38)$$

In conclusion the transformations of Γ -adapted coordinates are like that

$$\begin{cases} x^{\rho'} = x^{\rho'}(x^1, x^2, x^3) & \rho' = 1, 2, 3 \\ x^{4'} = x^{4'}(x^1, x^2, x^3, x^4) \end{cases} \quad (2.3.39)$$

with $x^{\rho'}$ and $x^{4'}$ arbitrary regular functions. In particular if we want move from a system of Γ -adapted coordinates to a system of \underline{u} -adapted coordinates, we have to consider the condition (??) and the following condition [(??)]

$$u^{4'} = u^r \partial_r x^{4'} = u^4 \partial_4 x^{4'}. \quad (2.3.40)$$

So we have to choose the $x^{r'}$ like that

$$\begin{cases} x^{\rho'} = x^{\rho'}(x^1, x^2, x^3) & \rho' = 1, 2, 3 \\ x^{4'} = \int [u^4(x)]^{-1} dx^4 + \varphi(x^1, x^2, x^3) \end{cases} \quad (2.3.41)$$

where the $x^{\rho'}(x^1, x^2, x^3)$ and $\varphi(x^1, x^2, x^3)$ are arbitrary regular functions.

If then we want all the transformations that permit to change from \underline{u} -adapted coordinates to \underline{u} -adapted coordinates, we have to consider the conditions

$$\begin{cases} \partial_4 x^{\rho'} = 0 \\ u^{4'} = u^r \partial_r x^{4'} = u^4 \partial_4 x^{4'} = 1, \quad \partial_4 x^{4'} = 1. \end{cases} \quad (2.3.42)$$

It follows that the transformations which take \underline{u} -adapted coordinates are like that

$$\begin{cases} x^{\rho'} = x^{\rho'}(x^1, x^2, x^3) & \rho' = 1, 2, 3 \\ x^{4'} = x^4 + \varphi(x^1, x^2, x^3). \end{cases} \quad (2.3.43)$$

2.4 The Christoffel symbols and their formal projections

2.4.1 The Christoffel symbols

Let us consider a riemannian manifold V_4 , the Christoffel symbols of first and second kind, known as coefficients of the riemannian connection of V_4 too, are defined through the metric tensor g_{hk} from the following formulas:

$$(hk, i) \equiv \frac{1}{2} (\partial_h g_{ki} + \partial_k g_{ih} - \partial_i g_{hk}) \quad \text{Christoffel symbols of first kind} \quad (2.4.1)$$

$$\left\{ \begin{matrix} l \\ hk \end{matrix} \right\} = g^{li} (hk, i) \quad \text{Christoffel symbols of second kind.} \quad (2.4.2)$$

It can be proved that they are in biunivocal correspondence, in fact multiplying both sides of (??) by g_{ls} we obtain the Christoffel symbols of first kind:

$$g_{ls} \left\{ \begin{matrix} l \\ hk \end{matrix} \right\} = g_{ls} g^{li} (hk, i) = \delta_s^i (hk, i) = (hk, s)$$

and vice versa.

They act like tensors only in the class of the linear transformations; but they don't transform with tensorial law in a general coordinate change.

In other word

$$g_{ki} = \theta_k^{k'} \theta_i^{i'} g_{k'i'} \quad (2.4.3)$$

\Rightarrow

$$\partial_h g_{ki} = \left[\theta_{kh}^{k'} \theta_i^{i'} + \theta_{ih}^{i'} \theta_k^{k'} \right] g_{k'i'} + \theta_k^{k'} \theta_i^{i'} \theta_h^{h'} \partial_{h'} g_{k'i'} \quad \text{where } (\theta_h^{h'} \equiv \partial_h x^{h'}) \quad (2.4.4)$$

Now If we consider a circular permutation of the indices h, k, i in (??), we obtain

$$\begin{aligned} \partial_k g_{ih} &= [\theta_{ki}^{i'} \theta_h^{h'} + \theta_{hk}^{h'} \theta_i^{i'}] g_{i'h'} + \theta_k^{k'} \theta_i^{i'} \theta_h^{h'} \partial_{k'} g_{i'h'} \\ \partial_i g_{hk} &= [\theta_{ih}^{h'} \theta_k^{k'} + \theta_{ki}^{k'} \theta_i^{i'}] g_{h'k'} + \theta_k^{k'} \theta_i^{i'} \theta_h^{h'} \partial_{i'} g_{h'k'} \end{aligned}$$

hence it follows

$$(hk, i) = \theta_h^{h'} \theta_k^{k'} \theta_i^{i'} (h'k', i') + \theta_{hk}^{k'} \theta_i^{i'} g_{k'i'}. \quad (2.4.5)$$

From this last formula, we can see that only when the quantities $\theta_h^{h'}$ are constant, that is when the coordinate transformation is linear, the Christoffel symbols act like tensors.

We can also deduce the transformation law of the Christoffel symbols of second kind in the following way

$$\begin{aligned} \left\{ \begin{matrix} l \\ hk \end{matrix} \right\} &= g^{li} [\theta_h^{h'} \theta_k^{k'} \theta_i^{i'} (h'k', i') + \theta_{hk}^{k'} \theta_i^{i'} g_{k'i'}] = \\ &= \theta_h^{h'} \theta_k^{k'} \theta_i^{i'} \left\{ \begin{matrix} l' \\ h'k' \end{matrix} \right\} + \theta_{hk}^{k'} \theta_i^{i'} g_{k'i'}. \end{aligned} \quad (2.4.6)$$

The following symmetry properties are true for both symbols that we have introduced:

$$(hk, i) = (kh, i) \quad \left\{ \begin{matrix} l \\ hk \end{matrix} \right\} = \left\{ \begin{matrix} l \\ kh \end{matrix} \right\}. \quad (2.4.7)$$

According to these properties the number of the different symbols both of first kind and of second kind in V_4 are

$$4 \cdot C_{4,2} = \frac{4^2(4+1)}{2} = 40$$

as much as the different quantities $\partial_h g_{kl}$ are. It follows that these derivatives can be expressed by the Christoffel symbols; in fact

$$\partial_h g_{ki} = (hk, i) + (ih, k) \quad (2.4.8)$$

et cetera.

2.4.2 The projections of the Christoffel symbols

The projections on the tensor fields can be also make on geometric objects that are different from tensors, as, for example, on the riemannian connection of the manifold V_4 , which is expressed with the Christoffel symbols in local coordinates. It's interesting that the projections of the riemannian connection are traslated in algebraic relations between geometric objects that characterize the geometric structure of the physical frame of reference \mathcal{S} introduced in V_4 with the congruence Γ .

For more detail, we can consider to calculate easily

$$g_{hk} = \gamma_{hk} + \nu_{hk} \quad \nu_{hk} = -\gamma_h \gamma_k \quad (2.4.9)$$

so the Christoffel symbols of first kind can be written in the form

$$(hk, i) = (hk, i)_\gamma + (hk, i)_\nu \quad (2.4.10)$$

where

$$\begin{cases} (hk, i)_\gamma = \frac{1}{2} (\partial_h \gamma_{ki} + \partial_k \gamma_{ih} - \partial_i \gamma_{hk}) \\ (hk, i)_\nu = \frac{1}{2} (\partial_h \nu_{ki} + \partial_k \nu_{ih} - \partial_i \nu_{hk}). \end{cases} \quad (2.4.11)$$

It's advisable to make the $(??)_1$ a total spatial equation. It's possible with the use of the transverse partial derivatives instead of the ordinary partial derivatives. From $(??)$

$$\partial_\rho = \tilde{\partial}_\rho - \gamma_\rho \gamma^4 \partial_4 \quad (\tilde{\partial}_4 \equiv 0)$$

it follows from $(??)_1$

$$\begin{aligned} (hk, i)_\gamma &= \frac{1}{2} \left[(\tilde{\partial}_h - \gamma_h \gamma^4 \partial_4) \gamma_{ki} + (\tilde{\partial}_k - \gamma_k \gamma^4 \partial_4) \gamma_{ih} - (\tilde{\partial}_i - \gamma_i \gamma^4 \partial_4) \gamma_{hk} \right] = \\ &= \frac{1}{2} \left[\tilde{\partial}_h \gamma_{ki} + \tilde{\partial}_k \gamma_{ih} - \tilde{\partial}_i \gamma_{hk} \right] - \frac{1}{2} \left[\gamma_h \tilde{K}_{ki} + \gamma_k \tilde{K}_{ih} - \gamma_i \tilde{K}_{hk} \right] = \\ &= (h\tilde{k}, i)^* - \frac{1}{2} \left[\gamma_h \tilde{K}_{ki} + \gamma_k \tilde{K}_{ih} - \gamma_i \tilde{K}_{hk} \right] \end{aligned} \quad (2.4.12)$$

where

$$\gamma^4 \partial_4 \gamma_{hk} \equiv \tilde{K}_{hk}, \quad (h\tilde{k}, i)^* = \frac{1}{2} \left[\tilde{\partial}_h \gamma_{ki} + \tilde{\partial}_k \gamma_{ih} - \tilde{\partial}_i \gamma_{hk} \right]. \quad (2.4.13)$$

We can observe that the symbol $(h\tilde{k}, i)^*$ has all like spatial indices and the symmetric tensor \tilde{K}_{hk} too.

We can now interest in the relation $(??)_2$; we have:

$$\left\{ \begin{array}{l} \partial_h \nu_{ki} = -\partial_h (\gamma_k \gamma_i) = -[\partial_h \gamma_k \cdot \gamma_i + \gamma_k \cdot \partial_h \gamma_i] \\ \partial_k \nu_{ih} = -\partial_k (\gamma_i \gamma_h) = -[\partial_k \gamma_i \cdot \gamma_h + \gamma_i \cdot \partial_k \gamma_h] \\ \partial_i \nu_{hk} = -\partial_i (\gamma_h \gamma_k) = -[\partial_i \gamma_h \cdot \gamma_k + \gamma_h \cdot \partial_i \gamma_k] \end{array} \right.$$

so it follows

$$\begin{aligned} (hk, i)_\nu &= \frac{1}{2} [-\gamma_k (\partial_h \gamma_i - \partial_i \gamma_h) - \gamma_i (\partial_h \gamma_k + \partial_k \gamma_h) - \gamma_h (\partial_k \gamma_i - \partial_i \gamma_k)] = \\ &= -\frac{1}{2} [\gamma_h \Omega_{ki} + \gamma_k \Omega_{hi} + \gamma_i Q_{hk}] \end{aligned} \quad (2.4.14)$$

where

$$\Omega_{ki} \equiv \partial_k \gamma_i - \partial_i \gamma_k, \quad Q_{hk} \equiv \partial_h \gamma_k + \partial_k \gamma_h. \quad (2.4.15)$$

Therefore the Eq. $(??)$ can be written in the form

$$(hk, i) = (h\tilde{k}, i)^* - \frac{1}{2} \left[\gamma_h (\tilde{K}_{ki} + \Omega_{ki}) + \gamma_k (\tilde{K}_{hi} + \Omega_{hi}) + \gamma_i (Q_{hk} - \tilde{K}_{hk}) \right] \quad (2.4.16)$$

From this equation we can deduce that we have to do the projections of the quantities $\partial_h \gamma_k$ (with $h, k = 1, 2, 3, 4$) to complete the projections of the Christoffel symbols of the first kind, according to the positions $(??)$. With this intention we observe that we can obtain the following two relations projecting the index of partial differentiation,

in adapted coordinates:

$$\mathcal{P}_\Sigma(\partial_{\mathbf{h}}\gamma_k) = \gamma_h^r \partial_r \gamma_k = (\delta_h^r + \gamma_h \gamma^r) \partial_r \gamma_k = \partial_h \gamma_k + \gamma_h \gamma^4 \partial_4 \gamma_k \quad (2.4.17)$$

$$\mathcal{P}_\Theta(\partial_{\mathbf{h}}\gamma_k) = -\gamma_h \gamma^r \partial_r \gamma_k = -\gamma_h \gamma^4 \partial_4 \gamma_k \quad \gamma^4 \gamma_4 = -1 \quad (2.4.18)$$

then

$$\mathcal{P}_\Sigma(\partial_{\mathbf{h}}\gamma_k) = \partial_h \gamma_k - \mathcal{P}_\Theta(\partial_{\mathbf{h}}\gamma_k) \quad (2.4.19)$$

The relations (??), (??) aren't purposeful for the future developments of the theory. It's more convenient to examine the formulas that obtain projecting the index of the vector field γ_k . The spatial projection takes this remarkable form

$$\begin{aligned} \mathcal{P}_\Sigma(\partial_h \gamma_k) &= \gamma_k^r \partial_h \gamma_r = \delta_k^r \partial_h \gamma_r + \gamma_k \gamma^r \partial_h \gamma_r = \\ &= \partial_h \gamma_k + \gamma_k (\gamma^4 \partial_h \gamma_4) \end{aligned} \quad (2.4.20)$$

or, considering the following condition

$$\gamma_k \gamma^k = \gamma_4 \gamma^4 = -1,$$

it follows

$$\partial_h (\gamma_4 \gamma^4) = \partial_h \gamma_4 \cdot \gamma^4 + \gamma_4 \cdot \partial_h \gamma^4 = 0 \quad (2.4.21)$$

and so we obtain

$$\mathcal{P}_\Sigma(\partial_h \gamma_k) = \partial_h \gamma_k - \gamma_k \gamma_4 \partial_h \gamma^4 = \partial_h \gamma_k + \gamma_k \gamma_4 \partial_h \left(\frac{1}{\gamma_4} \right) = \gamma_4 \partial_h \left(\frac{\gamma_k}{\gamma_4} \right)$$

\Rightarrow

$$\mathcal{P}_\Sigma(\partial_h \gamma_k) = \gamma_4 \partial_h \left(\frac{\gamma_k}{\gamma_4} \right). \quad (2.4.22)$$

From (??), we obtain

$$\mathcal{P}_\Theta(\partial_h \gamma_k) = -\gamma_k \gamma^r \partial_h \gamma_r = -\gamma_k \gamma^4 \partial_h \gamma_4 = \gamma_k \gamma_4 \partial_h \gamma^4. \quad (2.4.23)$$

In conclusion it's true the relation

$$\partial_h \gamma_k = \gamma_4 \partial_h \left(\frac{\gamma_k}{\gamma_4} \right) + \gamma_k \gamma_4 \partial_h \gamma^4. \quad (2.4.24)$$

It's advisable to introduce the transverse partial derivatives in this formula too, so we can write from (??)

$$\begin{aligned} \partial_h \gamma_k &= \gamma_4 \left[\tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) - \gamma_h \gamma^4 \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) \right] + \gamma_k \gamma_4 \left[\tilde{\partial}_h (\gamma^4) - \gamma_h \gamma^4 \partial_4 (\gamma^4) \right] = \\ &= \gamma_4 \tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) + \gamma_h \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) + \gamma_k \gamma_4 \tilde{\partial}_h \gamma^4 + \gamma_h \gamma_k \partial_4 \gamma^4, \end{aligned} \quad (2.4.25)$$

and applying a permutation of h with k

$$\partial_k \gamma_h = \gamma_4 \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_4} \right) + \gamma_k \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) + \gamma_h \gamma_4 \tilde{\partial}_k \gamma^4 + \gamma_h \gamma_k \partial_4 \gamma^4. \quad (2.4.26)$$

Through these two last formulas we can write the projections of the tensor Ω_{hk}

and of the quantities Q_{hk} expressed in (?). We obtain

$$\begin{aligned}
\Omega_{hk} &= \partial_h \gamma_k - \partial_k \gamma_h = \gamma_4 \tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) + \gamma_h \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) + \gamma_k \gamma_4 \tilde{\partial}_h \gamma^4 + \gamma_h \gamma_k \partial_4 \gamma^4 + \\
&\quad - \gamma_4 \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_4} \right) - \gamma_k \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) - \gamma_h \gamma_4 \tilde{\partial}_k \gamma^4 - \gamma_h \gamma_k \partial_4 \gamma^4 = \\
&= \gamma_4 \left[\tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) - \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_4} \right) \right] + \gamma_4 \left[\gamma_k \tilde{\partial}_h \gamma^4 - \gamma_h \tilde{\partial}_k \gamma^4 \right] + \gamma_h \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) - \gamma_k \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) = \\
&= \tilde{\Omega}_{hk} + \gamma_k \gamma_4 \tilde{\partial}_h \left(-\frac{1}{\gamma_4} \right) - \gamma_k \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) + \gamma_h \gamma_4 \tilde{\partial}_k \left(\frac{1}{\gamma_4} \right) + \gamma_h \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) = \\
&= \tilde{\Omega}_{hk} - \gamma_k \left[\gamma_4 \tilde{\partial}_h \left(\frac{1}{\gamma_4} \right) + \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) \right] + \gamma_h \left[\gamma_4 \tilde{\partial}_k \left(\frac{1}{\gamma_4} \right) + \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) \right]
\end{aligned}$$

so

$$\Omega_{hk} = \tilde{\Omega}_{hk} + \gamma_k \tilde{\Omega}_h - \gamma_h \tilde{\Omega}_k \quad (2.4.27)$$

where

$$\begin{cases} \tilde{\Omega}_{hk} = \gamma_4 \left[\tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) - \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_4} \right) \right] \\ \tilde{\Omega}_h = -\gamma_4 \tilde{\partial}_h \left(\frac{1}{\gamma_4} \right) - \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) \end{cases} \quad (2.4.28)$$

and

$$\begin{aligned}
Q_{hk} &\equiv \partial_h \gamma_k + \partial_k \gamma_h = \\
&= \gamma_4 \left[\tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) + \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_4} \right) \right] + 2\gamma_h \gamma_k \partial_4 \gamma^4 + \gamma_h \left[\tilde{\partial}_k (\gamma^4) \cdot \gamma_4 + \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) \right] + \\
&\quad + \gamma_k \left[\gamma_4 \tilde{\partial}_h (\gamma^4) + \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) \right]
\end{aligned}$$

so

$$Q_{hk} = \tilde{Q}_{hk} + \gamma_h \tilde{Q}_k + \gamma_k \tilde{Q}_h + 2\gamma_h \gamma_k \partial_4 \gamma^4 \quad (2.4.29)$$

where

$$\begin{cases} \tilde{Q}_{hk} \equiv \gamma_4 \left[\tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) + \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_4} \right) \right] \\ \tilde{Q}_k \equiv -\gamma_4 \tilde{\partial}_k \left(\frac{1}{\gamma_4} \right) + \partial_4 \left(\frac{\gamma_k}{\gamma_4} \right) \\ \tilde{Q}_h \equiv -\gamma_4 \tilde{\partial}_h \left(\frac{1}{\gamma_4} \right) + \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) \end{cases} \quad (2.4.30)$$

We can easily verify that the vector $\tilde{\Omega}_h$ is the curvature vector of the lines of the congruence Γ (of equation $x^4 = var.$); for this purpose we rewrite $\tilde{\Omega}_h$:

$$\tilde{\Omega}_h = \gamma_4 \tilde{\partial}_h \gamma^4 + \partial_4 (\gamma_h \gamma^4) = \gamma_4 [\partial_h \gamma^4 + \gamma_h \gamma^4 \partial_4 \gamma^4] + \partial_4 (\gamma_h \gamma^4) \quad (2.4.31)$$

Since

$$\gamma^4 \gamma_4 = -1, \quad \partial_4 (\gamma^4 \gamma_4) = \gamma^4 \partial_4 \gamma_4 + \gamma_4 \partial_4 \gamma^4 = 0 \quad (2.4.32)$$

It follows from (??)

$$\begin{aligned} \tilde{\Omega}_h &= -\gamma^4 \partial_h \gamma_4 - \gamma_h \partial_4 \gamma^4 + \gamma^4 \partial_4 \gamma_h + \gamma_h \partial_4 \gamma^4 = \gamma^4 (\partial_4 \gamma_h - \partial_h \gamma_4) = \\ &= \gamma^r (\partial_r \gamma_h - \partial_h \gamma_r) = \gamma^r (\nabla_r \gamma_h - \nabla_h \gamma_r). \end{aligned} \quad (2.4.33)$$

Since it results also

$$\nabla_h (\gamma^r \gamma_r) = \gamma^r \nabla_h \gamma_r + \gamma_r \nabla_h \gamma^r = \gamma^r \nabla_h \gamma_r + \gamma^r \nabla_h \gamma_r = 2\gamma^r \nabla_h \gamma_r = 0$$

the Eq. (??) is reduced to the following equation:

$$\tilde{\Omega}_h = \gamma^r \nabla_r \gamma_h. \quad (2.4.34)$$

So the curvature vector C_h of the general trajectory of the congruence Γ is for definition, as it is known,

$$C_h = \frac{D\gamma_h}{ds} = \nabla_r \gamma_h \cdot \frac{dx^r}{ds} = \gamma^r \nabla_r \gamma_h \quad (2.4.35)$$

and it is equal to the vector $\tilde{\Omega}_h$ expressed in (??).

If now, through the (??), we make the Σ -projection and the Θ -projection of the first index of the tensor Ω_{hk} , we obtain:

$$\begin{cases} \mathcal{P}_\Sigma(\Omega_{\mathfrak{h}\mathfrak{k}}) = \tilde{\Omega}_{hk} + \gamma_k \tilde{\Omega}_h = \tilde{\Omega}_{hk} + C_h \gamma_k \\ \mathcal{P}_\Theta(\Omega_{\mathfrak{h}\mathfrak{k}}) = -\gamma_h \tilde{\Omega}_k = -\gamma_h C_k. \end{cases} \quad (2.4.36)$$

At this point we are able to write easily the projections of the two first indices of the Christoffel symbols of the first kind given in (??). Considering the Eq. (??), we have

$$\begin{aligned} \mathcal{P}_{\Sigma\Sigma}(\mathfrak{h}\mathfrak{k}, i) &= (h\tilde{k}, i)^* - \frac{1}{2} \mathcal{P}_{\Sigma\Sigma} \left[\gamma_i (Q_{\mathfrak{h}\mathfrak{k}} - \tilde{K}_{\mathfrak{h}\mathfrak{k}}) \right] = \\ &= (h\tilde{k}, i)^* - \frac{1}{2} \gamma_i (\tilde{Q}_{hk} - \tilde{K}_{hk}) \end{aligned} \quad (2.4.37)$$

$$\begin{aligned} \mathcal{P}_{\Sigma\Theta}(\mathfrak{h}\mathfrak{k}, i) &= -\frac{1}{2} \left\{ \mathcal{P}_{\Sigma\Theta} \left[\gamma_{\mathfrak{k}} \left(\tilde{K}_{\mathfrak{h}\mathfrak{i}} + \Omega_{\mathfrak{h}\mathfrak{i}} \right) \right] + \gamma_i \mathcal{P}_{\Sigma\Theta}(Q_{\mathfrak{h}\mathfrak{k}}) \right\} = \\ &= -\frac{1}{2} \left\{ \gamma_k \tilde{K}_{hi} + \gamma_k \mathcal{P}_\Sigma \left(\tilde{\Omega}_{\mathfrak{h}\mathfrak{i}} + \gamma_i \tilde{\Omega}_{\mathfrak{h}} - \gamma_{\mathfrak{h}} \tilde{\Omega}_i \right) \right\} + \\ &= -\frac{1}{2} \left\{ \gamma_i \mathcal{P}_{\Sigma\Theta} \left(\tilde{Q}_{\mathfrak{h}\mathfrak{k}} + \gamma_{\mathfrak{h}} \tilde{Q}_{\mathfrak{k}} + \gamma_{\mathfrak{k}} \tilde{Q}_{\mathfrak{h}} + 2\gamma_{\mathfrak{h}} \gamma_{\mathfrak{k}} \partial_4 \gamma^4 \right) \right\} = \\ &= -\frac{1}{2} \left\{ \gamma_k \tilde{K}_{hi} + \gamma_k \tilde{\Omega}_{hi} + \gamma_k \gamma_i \tilde{\Omega}_h + \gamma_k \gamma_i \tilde{Q}_h \right\} = \\ &= -\frac{1}{2} \gamma_k \left\{ \tilde{K}_{hi} + \tilde{\Omega}_{hi} + \gamma_i \tilde{\Omega}_h + \gamma_i \tilde{Q}_h \right\}. \end{aligned}$$

Since it results

$$\gamma_i \left(\tilde{\Omega}_h + \tilde{Q}_h \right) = -2\gamma_i \gamma_4 \tilde{\partial}_h \left(\frac{1}{\gamma_4} \right) = 2\gamma_i \gamma_4 \tilde{\partial}_h \gamma^4$$

it follows

$$\mathcal{P}_{\Sigma\Theta}(\mathfrak{h}\mathfrak{k}, i) = -\frac{1}{2} \gamma_k \left[\tilde{K}_{hi} + \tilde{\Omega}_{hi} + 2\gamma_i \gamma_4 \tilde{\partial}_h \gamma^4 \right]. \quad (2.4.38)$$

Inverting the two projections Σ and Θ , it results

$$\mathcal{P}_{\Theta\Sigma}(\mathfrak{h}\mathfrak{k}, i) = -\frac{1}{2} \gamma_h \left[\tilde{K}_{ki} + \tilde{\Omega}_{ki} + 2\gamma_i \gamma_4 \tilde{\partial}_k \gamma^4 \right]. \quad (2.4.39)$$

In conclusion the following relation is valid [cfr. (??), (??), (??)]

$$\begin{aligned}
 \mathcal{P}_{\Theta\Theta}(\mathfrak{h}\mathfrak{k}, i) &= -\frac{1}{2} \left[\gamma_h \mathcal{P}_{\Theta}(\Omega_{\mathfrak{k}i}) + \gamma_k \mathcal{P}_{\Theta}(\Omega_{\mathfrak{h}i}) + \gamma_i \mathcal{P}_{\Theta\Theta}(Q_{hk}) \right] = \\
 &= -\frac{1}{2} \left[\gamma_h (-\gamma_k C_i) + \gamma_k (-\gamma_h C_i) + \gamma_i \cdot 2\gamma_h \gamma_k \partial_4 \gamma^4 \right] = \\
 &= \gamma_h \gamma_k (C_i - \gamma_i \partial_4 \gamma^4).
 \end{aligned} \tag{2.4.40}$$

It can be useful to have the totally spatial projection of the two Christoffel symbols, so we make this calculus; from (??) it follows:

$$\mathcal{P}_{\Sigma\Sigma\Sigma}(\mathfrak{h}\mathfrak{k}, i) = (h\tilde{k}, i)^*; \tag{2.4.41}$$

$$\begin{aligned}
 \mathcal{P}_{\Sigma\Sigma\Sigma} \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} &= \left\{ \begin{matrix} \mathfrak{i} \\ \mathfrak{h}\mathfrak{k} \end{matrix} \right\}^* = \mathcal{P}_{\Sigma}(g^{\mathfrak{i}r}) \mathcal{P}_{\Sigma\Sigma}(\mathfrak{h}\mathfrak{k}, r) = \\
 &= \gamma^{ir} \mathcal{P}_{\Sigma\Sigma}(\mathfrak{h}\mathfrak{k}, r) = \\
 &= \gamma^{ir} (h\tilde{k}, r)^* - \frac{1}{2} \gamma_i \mathcal{P}_{\Sigma\Sigma}(Q_{\mathfrak{h}\mathfrak{k}} - \tilde{K}_{\mathfrak{h}\mathfrak{k}}) \gamma^{ir} = \\
 &= \gamma^{ir} (h\tilde{k}, r)^* - \frac{1}{2} \gamma_i \gamma^{ir} (\tilde{Q}_{hk} - \tilde{K}_{hk}) = \\
 &= \gamma^{ir} (h\tilde{k}, r)^*
 \end{aligned}$$

that is

$$\mathcal{P}_{\Sigma\Sigma\Sigma} \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} = \gamma^{ir} (h\tilde{k}, r)^* \equiv \left\{ \begin{matrix} \mathfrak{i} \\ \mathfrak{h}\mathfrak{k} \end{matrix} \right\}^*. \tag{2.4.42}$$

The symbol \sim means that it makes the transverse derivatives and the symbol $*$ is to remember the metric tensor is the spatial one γ_{hk} .

2.5 The covariant transverse derivative of a field of totally spatial tensors and its formal properties

2.5.1 The covariant transverse derivative of a field of totally spatial tensors

Let us consider a general field of spatial vectors $\underline{v}(x)$ in V_4 , it's valid the condition

$$v_4 = 0 \quad (2.5.1)$$

for them. Let us make them the covariant derivative $\nabla_h v_k$ and then consider the totally spatial projection of this derivative, that we say covariant transverse derivative of the field of spatial vectors $\underline{v}(x)$, and we can write:

$$\mathcal{P}_{\Sigma\Sigma}(\nabla_h v_k) \equiv \tilde{\nabla}_h^* v_k. \quad (2.5.2)$$

We estimate the explicit expression of this derivative; it results:

$$\begin{aligned} \mathcal{P}_{\Sigma\Sigma}(\nabla_h v_k) &= \mathcal{P}_{\Sigma\Sigma} \left[\partial_h v_k - \left\{ \begin{matrix} r \\ hk \end{matrix} \right\} v_r \right] = \gamma_r^h \partial_r v_k - \mathcal{P}_{\Sigma\Sigma} \left[\left\{ \begin{matrix} r \\ \underline{hk} \end{matrix} \right\} v_r \right] = \\ &= (\delta_h^r + \gamma_h \gamma^r) \partial_r v_k - \mathcal{P}_{\Sigma\Sigma} [g^{ri}(hk, i)v_r] = \\ &= \partial_h v_k + \gamma_h \gamma^4 \partial_4 v_k - \mathcal{P}_{\Sigma\Sigma} [(\underline{hk}, i)v^i] = \\ &= \tilde{\partial}_h v_k - v^i \left\{ (h\tilde{k}, i)^* - \frac{1}{2} \gamma_i (\tilde{Q}_{hk} - \tilde{K}_{hk}) \right\} = \\ &= \tilde{\partial}_h v_k - v^i (h\tilde{k}, i)^*. \end{aligned}$$

So we deduce the formula [cfr. (??)]

$$\tilde{\nabla}_h^* v_k = \tilde{\partial}_h v_k - v^i (h\tilde{k}, i)^*. \quad (2.5.3)$$

Put

$$\widetilde{\left\{ \begin{matrix} u \\ hk \end{matrix} \right\}}^* = \gamma^{ui} (h\tilde{k}, i)^* \quad (2.5.4)$$

the (??) takes the form

$$\tilde{\nabla}_h^* v_k = \tilde{\partial}_h v_k - \widetilde{\left\{ \begin{matrix} u \\ hk \end{matrix} \right\}}^* v_u. \quad (2.5.5)$$

It's easy to verify that both the transverse Christoffel symbols $(h\tilde{k}, i)$, $\widetilde{\left\{ \begin{matrix} u \\ hk \end{matrix} \right\}}^*$ and the covariant transverse derivative (??) are totally spatial objects; it only has to assign the value 4 to an index in covariant position to discover that the corresponding term is equal to zero.

For example it results

$$(4\tilde{k}, i)^* = \frac{1}{2} \left(\tilde{\partial}_4 \gamma_{ki} + \tilde{\partial}_k \gamma_{i4} - \tilde{\partial}_i \gamma_{4k} \right) = 0 \quad (2.5.6)$$

Let us now consider a field of double totally spatial tensors v_{hk} ($v_{4k} = v_{h4}$) and extend the covariant transverse derivative to this field. For this purpose it only has

to make the totally spatial projection of the field of triple tensors $\nabla_i v_{hk}$; we obtain

$$\begin{aligned}
\tilde{\nabla}_i^* v_{hk} &\equiv \mathcal{P}_{\Sigma\Sigma\Sigma}(\nabla_i v_{hk}) = \mathcal{P}_{\Sigma\Sigma\Sigma} \left[\partial_i v_{hk} - \left\{ \begin{matrix} r \\ ih \end{matrix} \right\} v_{rk} - \left\{ \begin{matrix} r \\ ik \end{matrix} \right\} v_{hr} \right] = \\
&= \mathcal{P}_{\Sigma} \left(\partial_i v_{hk} \right) - \mathcal{P}_{\Sigma\Sigma\Sigma} [(ih, u)g^{ru}v_{rk}] - \mathcal{P}_{\Sigma\Sigma\Sigma} [(ik, u)g^{ru}v_{hr}] = \\
&= \tilde{\partial}_i v_{hk} - \mathcal{P}_{\Sigma\Sigma} [(i\tilde{h}, u)v_k^u] - \mathcal{P}_{\Sigma\Sigma} [(i\tilde{k}, u)v_h^u] = \\
&= \tilde{\partial}_i v_{hk} - \left[(i\tilde{h}, u) - \frac{1}{2}\gamma_u(\tilde{Q}_{ih} - \tilde{K}_{ih}) \right] v_k^u + \\
&\quad - \left[(i\tilde{k}, u) - \frac{1}{2}\gamma_u(\tilde{Q}_{ik} - \tilde{K}_{ik}) \right] v_h^u = \\
&= \tilde{\partial}_i v_{hk} - (i\tilde{h}, u)v_k^u - (i\tilde{k}, u)v_h^u = \\
&= \tilde{\partial}_i v_{hk} - (i\tilde{h}, u)\gamma^{ur}v_{rk} - (i\tilde{k}, u)\gamma^{ur}v_{hr}
\end{aligned}$$

in conclusion

$$\tilde{\nabla}_i^* v_{hk} = \tilde{\partial}_i v_{hk} - \left\{ \begin{matrix} \tilde{r} \\ ih \end{matrix} \right\}^* v_{rk} - \left\{ \begin{matrix} \tilde{r} \\ ik \end{matrix} \right\}^* v_{hr}. \quad (2.5.7)$$

In this formula it's evident that the covariant transverse derivative performs themselves like the ordinary covariant derivative, with the only replacement of the transverse partial derivative instead of the ordinary partial derivative and of the transverse Christoffel symbols made with the metric spatial tensor γ_{hk} and the transverse partial derivative instead of the space-time metric tensor g_{hk} and the ordinary partial derivative.

It's obvious that we can do similar operations if we replace a field of tensors > 2 with a field of double tensors.

2.5.2 The more easy formal properties of the covariant transverse derivative

It's easy to verify that the covariant transverse differentiation has the following formal properties. Let us enunciate some of these:

- a) The covariant transverse derivative of the sum of more totally spatial tensors is equal to the sum of the covariant transverse derivatives of the single tensors;
- b) The covariant transverse derivative of the product of two or more totally spatial tensors is like the ordinary covariant derivative;
- c) The covariant transverse differentiation commutes with the saturation operation as long as it involves two purely spatial indices;
- d) It is valid the Ricci Theorem for the covariant transverse derivative:
the spatial metric tensor γ_{hk} has covariant transverse derivative equal to zero,

$$\tilde{\nabla}_h^* \gamma_{ki} = 0. \tag{2.5.8}$$

It's so noticed the fundamental character of the spatial metric tensor γ_{hk} as the space-time metric tensor g_{hk} .

- e) In general two successive covariant transverse differentiations cannot invert each other as two successive covariant differentiations.

2.6 Some differential properties of the like-time congruences of a Riemannian manifold V_4

2.6.1 Differential properties of the first order of a physical frame of reference in V_4

In Einstein's general Relativity and in a 4-dimensional normal hyperbolic Riemannian manifold V_4 , the like-time congruences that represent physical frame of reference are very important. For this reason now we will look over which informations we can draw from the application of the theory of the projections.

In a manifold V_4 let us consider a like-time congruence Γ that locates a physical frame of reference \mathcal{S} in V_4 , and suppose to choose a system of local coordinates $\{x^h\}$ adapted to the congruence Γ (cfr. sect. 1). Since this congruence is univocally defined by the field of unit vectors $\underline{\gamma}(x)$ tangent to its space-time trajectories, we can analyze the covariant derivative of the field of the vectors $\underline{\gamma}(x)$, $\nabla_h \gamma_k$ with the projection technique, and more in detail the field of double symmetric tensors

$$K_{hk} \equiv \nabla_h \gamma_k + \nabla_k \gamma_h \quad \text{Killing tensor} \quad (2.6.1)$$

and the field of double antisymmetric tensors

$$\Omega_{hk} \equiv \nabla_h \gamma_k - \nabla_k \gamma_h = \partial_h \gamma_k - \partial_k \gamma_h. \quad (2.6.2)$$

Let us decompose $\nabla_h \gamma_k$ bearing in mind the (??), (??); we obtain the relation

$$\begin{aligned} \nabla_h \gamma_k &= \partial_h \gamma_k - (hk, r) \gamma^r = \partial_h \gamma_k - \gamma^r (h\tilde{k}, r) + \\ &+ \frac{1}{2} \gamma^r \left[\gamma_h (\tilde{K}_{kr} + \Omega_{kr}) + \gamma_k (\tilde{K}_{hr} + \Omega_{hr}) + \gamma_r (Q_{hk} - \tilde{K}_{hk}) \right] = \\ &= \partial_h \gamma_k - \frac{1}{2} \gamma_r \gamma^r \tilde{K}_{hk} + \frac{1}{2} [\gamma_h \gamma^r \Omega_{kr} + \gamma_k \gamma^r \Omega_{hr} - Q_{hk}] = \\ &= \frac{1}{2} \tilde{K}_{hk} + \frac{1}{2} \gamma_h \Omega_{kr} \gamma^r + \frac{1}{2} \gamma_k \Omega_{hr} \gamma^r + (\partial_h \gamma_k - \frac{1}{2} Q_{hk}) = \\ &= \frac{1}{2} (\tilde{K}_{hk} + \Omega_{hk}) + \frac{1}{2} [\gamma_h \Omega_{kr} \gamma^r + \gamma_k \Omega_{hr} \gamma^r] \quad (Q_{hk} = \partial_h \gamma_k - \partial_k \gamma_h) \end{aligned}$$

from (??) and (??) it follows

$$\begin{aligned} \nabla_h \gamma_k &= \frac{1}{2} \left[\tilde{K}_{hk} + \tilde{\Omega}_{hk} + \gamma_k C_h - \gamma_h C_k \right] + \\ &+ \frac{1}{2} \left[\gamma_h (\tilde{\Omega}_{kr} + \gamma_r C_k - \gamma_k C_r) \gamma^r + \gamma_k (\tilde{\Omega}_{hr} + \gamma_r C_h - \gamma_h C_r) \gamma^r \right] \end{aligned}$$

\Rightarrow

$$\nabla_h \gamma_k = \frac{1}{2} \left[\tilde{K}_{hk} + \tilde{\Omega}_{hk} \right] - \gamma_h C_k. \quad (2.6.3)$$

We can observe that this last formula shows clearly the spatial character of the index k of the tensor $\nabla_h \gamma_k$. Besides we can draw the decomposition formula for the Killing tensor

$$K_{hk} = \tilde{K}_{hk} - \gamma_h C_k - \gamma_k C_h. \quad (2.6.4)$$

It's so evident that the tensor $\tilde{K}_{hk} = \gamma^4 \partial_4 \gamma_{hk}$ [cfr. (??)] is the totally spatial projection of the Killing tensor.

Now we go back to the tensor Ω_{hk} and its formula of decomposition (??) that we rewrite in consideration of (??),

$$\Omega_{hk} = \tilde{\Omega}_{hk} + \gamma_k C_h - \gamma_h C_k \quad (2.6.5)$$

and are more precise about the totally spatial tensor [cfr. (??)]

$$\tilde{\Omega}_{hk} = \gamma_4 \left[\tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_4} \right) - \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_4} \right) \right] \quad (2.6.6)$$

is called spatial vortex tensor or transverse rotor of the congruence Γ .

The reason of this denomination is in the interpretation of the physical frame of reference \mathcal{S} associated to the congruence Γ as generated by an ideal fluid in motion (the trajectories of Γ as current lines of the fluid), next to the antisymmetric spatial tensor $\tilde{\Omega}_{hk}$ in the subspace Σ_x we can locally consider the added vector

$$\omega^\alpha \equiv \frac{1}{2} \eta^{\alpha\rho\sigma} \tilde{\Omega}_{\rho\sigma} \quad (\alpha, \rho, \sigma = 1, 2, 3 \quad \eta^{\alpha\rho\sigma} \text{ antisymmetrical Ricci tensor in } \Sigma_x) \quad (2.6.7)$$

that, multiplying for $\frac{1}{2}c$ (for dimensional motives) gives (like in the classical fluid mechanics) the local angular velocity of the ideal fluid that generates the reference \mathcal{S} .

The space-time antisymmetric tensor Ω_{hk} is called, in consequence, space-time vortex tensor of \mathcal{S} .

If in all reference \mathcal{S} it results

$$\tilde{\Omega}_{hk} = 0 \quad (\text{and then } \omega^\alpha = 0) \quad (2.6.8)$$

the motion of the fluid of reference is called irrotational.

If in addition to the condition (??) we add the other condition

$$C_h = 0 \quad \forall P \in S \quad (2.6.9)$$

the motion of the fluid of reference is called irrotational and geodetic.

In this case the space-time vortex tensor Ω_{hk} is identically null.

2.6.2 Some differential properties about the geometric structure of a physical frame of reference in V_4

Let us now deepen the geometric meaning of the tensors $\tilde{\Omega}_{hk}, \tilde{K}_{hk}$. We start considering the condition (??) which characterizes an irrotational physical frame. Choosing local coordinates $\{x^h\}$ adapted to the congruence Γ , associated to the physical frame of reference \mathcal{S} , in general the components $g_{4\rho} = g_{\rho 4}$ ($\rho = 1, 2, 3$) of the space-time metric tensor of V_4 aren't all null. But if it happens, the lines of the congruence, of equation $x^4 = var.$ $x^\rho = const.$ could verify the following condition: being dP an elementary vector tangent to a line of Γ (or space-time trajectory of the associated physical frame of reference \mathcal{S}); it has contravariant components $\{0, 0, 0, dx^4\}$. Another vector δP , tangent to the hypersurface of equation $x^4 = const.$ (both with origin in the same point-event of \mathcal{S}), that has for this reason contravariant components $\{\delta x^1, \delta x^2, \delta x^3, 0\}$, multiplyed inner by dP gives

$$dP \cdot \delta P = g_{hk} dx^h \delta x^k = g_{4\rho} dx^4 \delta x^\rho = 0, \quad (g_{4\rho} = 0, \rho = 1, 2, 3). \quad (2.6.10)$$

If this happens in every point of Γ the lines of Γ are orthogonal to the hypersurfaces of equations $x^4 = const.$

Let us suppose that in the local coordinates $\{x^k\}$ adapted to the congruence Γ the components $g_{4\rho}(x)$ aren't all equal to zero and ask ourselves if we can individuate another system of local coordinates $\{x^{k'}\}$ adapted to the congruence Γ , that allows

to satisfy the conditions

$$g_{4'\rho'} = 0, \quad \forall P \in \Gamma. \quad (2.6.11)$$

For this reason we remember that an internal transformation in the physical frame of reference \mathcal{S} is in the form (??) and it is the product of a transformation like (??)[that leaves unchanged the hypersurfaces of equation $x^4 = const.$] for one of the form (??), that introduces an only arbitrary function; this arbitrariness can't, in general, allows to satisfy the three conditions (??).

Really, for a transformation of the form (??), that we rewrite for convenience

$$x^{\rho'} = x^{\rho'}(x^1, x^2, x^3) \quad x^{4'} = x^4 \quad (2.6.12)$$

it results

$$g_{4\rho} = \theta_4^{h'} \theta_\rho^{k'} g_{h'k'} = \theta_4^{4'} \theta_\rho^{\rho'} g_{4'\rho'} = \theta_\rho^{\rho'} g_{4'\rho'} \quad \left(\theta_4^{4'} = \frac{\partial x^{4'}}{\partial x^4} \right) \quad (2.6.13)$$

and it's visible that the components $g_{4\rho}$ of the space-time metric tensor of V_4 transform themselves like the components of a vector; this is equivalent to maintain that if they aren't all equal to zero in a coordinate system then they cannot become all null by a transformation like (??).

For a transformation of the form (??) that we rewrite for convenience

$$x^{\rho'} = x^\rho \quad x^{4'} = x^{4'}(x^1, x^2, x^3, x^4) \quad (2.6.14)$$

we obtain

$$\begin{aligned} g_{4\rho} &= \theta_4^{h'} \theta_\rho^{k'} g_{h'k'} = \theta_4^{h'} \theta_\rho^{\rho'} g_{h'\rho'} + \theta_4^{h'} \theta_\rho^{4'} g_{h'4'} = \\ &= \theta_4^{4'} \theta_\rho^{\rho'} g_{4'\rho'} + \theta_4^{4'} \theta_\rho^{4'} g_{4'4'} = \theta_4^{4'} g_{4'\rho'} + \theta_4^{4'} \theta_\rho^{4'} g_{4'4'} \end{aligned} \quad (2.6.15)$$

$$g_{44} = \theta_4^{h'} \theta_4^{k'} g_{h'k'} = \left(\theta_4^{4'}\right)^2 g_{4'4'}, \quad (2.6.16)$$

that is

$$\theta_4^{4'} \equiv \frac{\partial x^{4'}}{\partial x^4} = \sqrt{\frac{+g_{44}}{g_{4'4'}}}. \quad (2.6.17)$$

As we can put

$$g_{4'4'} = -1, \quad (2.6.18)$$

from (??) it follows that we can choose the function $x^{4'}(x^1, x^2, x^3, x^4)$ so that it results

$$x^{4'} = \int \sqrt{-g_{44}} dx^4 + F(x^1, x^2, x^3) \quad (2.6.19)$$

with F an arbitrary regular function of the spatial coordinates x^1, x^2, x^3 .

Besides the Eq. (??) suggests that if we want all the components $g_{4'4'}$ equal to zero, considering the position (??), it must result

$$g_{4\rho} = -\theta_4^{4'} \theta_\rho^{4'}; \quad (2.6.20)$$

this last equation, considering the Eq. (??), assumes the form

$$g_{4\rho} = -\theta_\rho^{4'} \frac{(\theta_4^{4'})^2}{\theta_4^{4'}} = \theta_\rho^{4'} \frac{g_{44}}{\theta_4^{4'}}$$

that can be translated in the other

$$\partial_\rho x^{4'} - \frac{g_{4\rho}}{g_{44}} \partial_4 x^{4'} = 0. \quad (2.6.21)$$

If now we remember the formulas (??); that we rewrite,

$$\gamma_h = \frac{g_{4h}}{\sqrt{-g_{44}}} \quad \left[\gamma_\rho = \frac{g_{4\rho}}{\sqrt{-g_{44}}}, \quad \gamma_4 = -\sqrt{-g_{44}} \right] \quad (2.6.22)$$

the Eq. (??) becomes

$$\partial_\rho x^{4'} - \frac{\gamma_\rho}{\gamma_4} \partial_4 x^{4'} \equiv \partial_\rho x^{4'} + \gamma_\rho \gamma^4 \partial_4 x^{4'} = 0 \quad (2.6.23)$$

that is [cfr. (??)]

$$\tilde{\partial}_\rho x^{4'} = 0. \quad (2.6.24)$$

Therefore if we want that the conditions (??) are identically satisfied the coordinate transformation in the physical frame of reference \mathcal{S} (??)-(??) must be such that the function $x^{4'}(x^1, x^2, x^3, x^4)$ satisfies the conditions (??), what is in general not to come.

On the contrary if it happens, that is every physical frame of reference \mathcal{S} admits a system of local coordinates $\{x^h\}$ for which the components $g_{4\rho}$ of the space-time metric tensor of V_4 are identically equal to zero, \mathcal{S} is called frame of reference “orthogonal time” since the trajectories of the physical frame of reference \mathcal{S} result orthogonal to the hypersurfaces of equation $x^4 = const.$, as we have already seen.

At this point we have the following problem:

What conditions must be satisfied so that the physical frame of reference \mathcal{S} is orthogonal-time, and then the function $x^{4'}$ satisfies the conditions (??)? From the analytical point of view what are the compatibility conditions of the system of partial differential equations (??)?

For this purpose we can observe that if the Eqs.(??) are satisfied then the other conditions must be satisfied too

$$\tilde{\partial}_\tau \tilde{\partial}_\rho x^{4'} = 0 \quad (\rho, \tau = 1, 2, 3) \quad (2.6.25)$$

And obviously the conditions that follow from (??) changing ρ to τ must be true too, so these following equations must be valid

$$\tilde{\partial}_\tau \tilde{\partial}_\rho x^{4'} - \tilde{\partial}_\rho \tilde{\partial}_\tau x^{4'} = 0, \quad (\rho, \tau = 1, 2, 3). \quad (2.6.26)$$

Since in general two different transverse differentiations aren't permutable with each other, the (??) gives an effective condition which the function $x^{4'}$ must submit to; it introduces the operator $\tilde{\partial}_\tau \tilde{\partial}_\rho - \tilde{\partial}_\rho \tilde{\partial}_\tau$ which can be expressed by the tensor $\tilde{\Omega}_{\rho\tau}$. We'll see this.

Let us consider a general regular function $f(x^1, x^2, x^3, x^4)$ in V_4 and calculate the second partial transverse derivative $\tilde{\partial}_\rho \tilde{\partial}_\tau f$; we obtain:

$$\begin{aligned}
\tilde{\partial}_\rho \tilde{\partial}_\tau f &= \partial_\rho (\partial_\tau f + \gamma_\tau \gamma^4 \partial_4 f) + \gamma_\rho \gamma^4 \partial_4 (\partial_\tau f + \gamma_\tau \gamma^4 \partial_4 f) = \\
&= \partial_\rho \partial_\tau f + \partial_\rho \gamma_\tau \gamma^4 \partial_4 f + \gamma_\tau \partial_\rho \gamma^4 \partial_4 f + \gamma_\tau \gamma^4 \partial_\rho \partial_4 f + \\
&+ \gamma_\rho \gamma^4 [\partial_4 \partial_\tau f + \partial_4 \gamma_\tau \gamma^4 \partial_4 f + \gamma_\tau \partial_4 \gamma^4 \partial_4 f + \gamma_\tau \gamma^4 \partial_4 \partial_4 f] = \\
&= \partial_\rho \partial_\tau f + \gamma_\tau \gamma^4 \partial_\rho \partial_4 f + \gamma_\rho \gamma^4 \partial_4 \partial_\tau f + \gamma_\rho \gamma_\tau (\gamma^4)^2 \partial_4 \partial_4 f + \\
&+ \partial_4 f \left[\partial_\rho \gamma_\tau \gamma^4 + \gamma_\tau \partial_\rho \gamma^4 + \gamma_\rho (\gamma^4)^2 \partial_4 \gamma_\tau + \gamma_\tau \gamma_\rho \gamma^4 \partial_4 \gamma^4 \right] = \\
&= \partial_\rho \partial_\tau f + \gamma_\tau \gamma^4 \partial_\rho \partial_4 f + \gamma_\rho \gamma^4 \partial_4 \partial_\tau f + \gamma_\rho \gamma_\tau (\gamma^4)^2 \partial_4 \partial_4 f + \partial_4 f \tilde{\partial}_\rho (\gamma_\tau \gamma^4);
\end{aligned} \tag{2.6.27}$$

inverting ρ and τ , it follows from (??)

$$\tilde{\partial}_\tau \tilde{\partial}_\rho f = \partial_\tau \partial_\rho f + \gamma_\rho \gamma^4 \partial_\tau \partial_4 f + \gamma_\tau \gamma^4 \partial_4 \partial_\rho f + \gamma_\tau \gamma_\rho (\gamma^4)^2 \partial_4 \partial_4 f + \partial_4 f \tilde{\partial}_\tau (\gamma_\rho \gamma^4). \tag{2.6.28}$$

Estimating the difference $\tilde{\partial}_\rho \tilde{\partial}_\tau f - \tilde{\partial}_\tau \tilde{\partial}_\rho f$ it follows

$$\begin{aligned}
\tilde{\partial}_\rho \tilde{\partial}_\tau f - \tilde{\partial}_\tau \tilde{\partial}_\rho f &= \partial_4 f \left[\tilde{\partial}_\rho (\gamma_\tau \gamma^4) - \tilde{\partial}_\tau (\gamma_\rho \gamma^4) \right] = \\
&= \partial_4 f \left[\tilde{\partial}_\rho \left(\frac{\gamma_\tau}{-\gamma^4} \right) - \tilde{\partial}_\tau \left(\frac{\gamma_\rho}{-\gamma^4} \right) \right] = \\
&= \partial_4 f \left[\tilde{\partial}_\tau \left(\frac{\gamma_\rho}{\gamma^4} \right) - \tilde{\partial}_\rho \left(\frac{\gamma_\tau}{\gamma^4} \right) \right];
\end{aligned} \tag{2.6.29}$$

considering the definition (??)₁ di $\tilde{\Omega}_{hk}$ the (??) can become

$$\tilde{\partial}_\rho \tilde{\partial}_\tau f - \tilde{\partial}_\tau \tilde{\partial}_\rho f = -\tilde{\Omega}_{\tau\rho} \gamma^4 \partial_4 f. \tag{2.6.30}$$

So the condition (??), considering the (??) becomes

$$\tilde{\Omega}_{\tau\rho}\gamma^4 \cdot \partial_4 x^{4'} = 0. \tag{2.6.31}$$

To obtain this last identity, we can consider directly the function $x^{4'}(x^1, x^2, x^3, x^4)$, instead of a general function $f(x^1, x^2, x^3, x^4)$.

So a physical frame of reference \mathcal{S} is orthogonal-time if the double totally spatial tensor $\tilde{\Omega}_{\tau\rho}$ is identically equal to zero

$$\tilde{\Omega}_{\tau\rho} = 0 \quad \forall P \in \Gamma. \tag{2.6.32}$$

It's easy to verify that this condition is also sufficient, so we can state that

Theorem 2.6.1 *A physical frame of reference \mathcal{S} is orthogonal-time (and so the associated congruence Γ results normal) if and only if the correspondent spatial vortex tensor is identically equal to zero.*

Let us now consider the symmetric totally spatial tensor $\tilde{K}_{hk} \equiv \gamma^4 \partial_4 \gamma_{hk}$ [cfr. (??)] and look for what happens to the physical frame of reference if it becomes equal to zero.

For this purpose let us consider the vector $dP \equiv \{dx^h\}$ that joins two infinitely neighbouring events and consider the spatial norm

$$d\sigma^2 \equiv \gamma_{\rho\tau} dx^\rho dx^\tau; \tag{2.6.33}$$

that can be interpreted as the square of the distance between two infinitely neighbouring particles of the (ideal) fluid that generates the reference \mathcal{S} (both taken

on the same hypersurface of eq. $x^4 = \text{const.}$). If we want that this distance is invariable along the space-time trajectories of the two particles we must impose the following condition to the spatial norm $d\sigma^2$

$$\gamma^4 \partial_4 (d\sigma^2) = 0 \quad (2.6.34)$$

that becomes in explicit form

$$\gamma^4 \partial_4 (\gamma_{\rho\tau} dx^\rho dx^\tau) = 0 \quad (2.6.35)$$

Since the only coordinate x^4 varies along the space-time trajectories of the two particles, the spatial components dx^ρ of the vector dP result constant, it follows that the condition (??) becomes

$$\gamma^4 \partial_4 \gamma_{\rho\tau} \cdot dx^\rho dx^\tau \equiv \tilde{K}_{\rho\tau} dx^\rho dx^\tau = 0. \quad (2.6.36)$$

This condition is valid for all pair of particles of the fluid of reference if it follows from (??)

$$\tilde{K}_{\rho\tau} = 0 \quad \forall P \in S. \quad (2.6.37)$$

This last condition characterizes the invariability of the spatial distance between two general trajectories of the physical frame of reference \mathcal{S} and says that the fluid that generates the physical frame of reference \mathcal{S} is in “Born rigid motion”. For this reason the tensor $\tilde{K}_{\rho\tau}$ is called Born spatial tensor or deformation tensor of the physical frame of reference \mathcal{S} , since the motion of the fluid isn’t rigid but it is subjected to a deformation if it is not equal to zero. In conclusion if the tensor $\nabla_h \gamma_k$ is identically equal

to zero in the physical frame \mathcal{S} , from the natural decomposition formula (??) must be identically equal to zero the tensors $\tilde{K}_{hk}, \tilde{\Omega}_{hk}, C_k$, then the motion of the fluid of reference is rigid, irrotational and geodetic, so it is called uniform, traslatory, rigid motion.

2.6.3 Geometric characteristics of the class of frames of reference associated to Levi-Civita's curvature coordinates and gaussian polar coordinates

Levi-Civita's curvature coordinates

Let us choose a metric of the following form:

$$ds^2 = e^{2\lambda(r,t)} dr^2 + Y^2(r,t) d\Omega^2 - e^{2v(r,t)} dt^2 \quad (2.6.38)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$.

Using Levi-Civita's curvature coordinates the metric (??), as it is well known, can be given the form³:

$$ds^2 = e^{2\lambda(r,t)} dr_c^2 + r_c^2 d\Omega^2 - e^{2v(r,t)} dt_c^2 \quad (2.6.39)$$

by using the following transformation of coordinates:

$$r_c = Y(r,t), \quad \theta' = \theta, \quad \varphi' = \varphi, \quad t_c = \xi[Y(r,t), t] \quad (2.6.40)$$

where ξ is a solution of $\frac{\partial \xi}{\partial Y} = 0$, see [?].

This change is not always internal to the original physical frame of reference, since

³We prefer the exponential form instead of Levi-Civita's one

the function $Y(r, t)$ is, in general, also time dependent. Therefore, not all reference frames admit curvature coordinates. Being furthermore possible to see, [?], that there are no transformations that leave unchanged the form of the metric and are external to R_c , we conclude: there is one and only one frame of reference associated to the chosen curvature coordinates.

With regard to the geometrical properties of R_c , we observe that a system of curvature coordinates can be constructed, in a frame of reference R , if and only if the transformation $r_c = Y(r, t)$ is time-independent: that is to say if and only if the distance between the neighbouring points with coordinates (r, θ, φ, t) and $(r, \theta + d\theta, \varphi + d\varphi, t)$ is time-independent.

Gaussian polar coordinates

Let $(\rho, \theta, \varphi, t)$ be a system of gaussian polar coordinates. The form of the correspondent metric is:

$$ds^2 = d\rho^2 + Y^2 d\Omega^2 - e^{2v} dt^2 \quad (2.6.41)$$

It is not difficult to see that not all systems of reference admits these coordinates. The metric (??), in fact, can be given the form (??), through an internal change of coordinates, only if the first invariant parameter ⁴ does not depend on time.

⁴We recall that the orthogonal form of the metric on a hypersurface $\theta = \text{const}$ $\varphi = \text{const}$:

$$g_{11} dr^2 - g_{44} dt^2$$

This circumstance cannot be always true if the function $\lambda(r, t)$ depends both on r and t .

It can also be proved that there are infinite frames of reference, different from each other, associated to gaussian polar coordinates. In fact, let $(\rho, \theta, \varphi, t)$ be a system of gaussian polar coordinates and \bar{R}_p an associated frame of reference.

Another frame of reference, both different from \bar{R}_p and associated to gaussian polar coordinates, can be obtained by considering a transformation of the type:

$$\rho' = f_1(\rho, t) \quad t' = f_2(\rho, t) \tag{2.6.42}$$

with $\Delta_1 f_1 = F(\rho) \quad \Delta_1(f_1 f_2) = 0$.

These transformations are possible and can be constructed in infinite ways, by choosing as coordinates lines on the hypersurfaces $\theta = \text{const} \quad \varphi = \text{const}$, an orthogonal net whose lines $t' = \text{var}$ are geodesic. Therefore, there are infinite different systems of reference \bar{R}_p associated to gaussian polar coordinates. A frame of reference belongs to \bar{R}_p if and only if $\dot{\lambda} = 0$: i.e. if and only if the distance between the neighbouring points with coordinates $(\rho, \theta, \varphi, t)$ and $(\rho + d\rho, \theta, \varphi, t)$ is time independent.

is transformed by an orthogonal transformation of the type:

$$r' = f_1(r, t); \quad t' = f_2(r, t); \quad \Delta_1(f_1 f_2) = 0$$

into the:

$$\frac{1}{\Delta_1 f_1} dr^2 - \frac{1}{\Delta_1 f_2} dt^2$$

2.6.4 Geometric characteristics of the class of frames of reference associated to isotropic coordinates and harmonic coordinates

Isotropic coordinates

Let $(r_i, \theta, \varphi, t)$ be a system of isotropic coordinates. In the case of spherical symmetry, the form of the correspondent metric is ⁵

$$ds^2 = Y^2(r, t) d\Omega^2 + e^{2\lambda(r, t)} dr^2 - e^{2v(r, t)} dt^2 \quad (2.6.43)$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$$

Then we get: $Y = re^\lambda$; where the r -coordinate is defined up to a transformation $r_i = 1/\bar{r}_i$. Accordingly, e^λ transforms in the following way: $e^\lambda = e^\lambda/r^2$. It is not difficult to see that not all systems of reference admit these coordinates. In fact, the line-element (??) can be given the form $Y = re^\lambda$, through an internal change of coordinates, if and only if the ratio e^λ/Y is not dependent of the radius.

It can also be proved that there are infinite systems of reference, different from each other, associated to isotropic coordinates. Let $(r_i, \theta, \varphi, t)$ be, in fact, a system of isotropic coordinates and \bar{R}_i an associated system of reference. Another system of reference, both different from \bar{R}_i and associated to isotropic coordinates, can be obtained by considering a transformation of the type:

$$r'_i = f_1(r_i, t), \quad t' = f_2(r_i, t) \quad (2.6.44)$$

⁵ In order to make the reading easier, we will follow, as much as possible, the notations of reference [?].

with $e^\lambda/Y = F(r_i)$, $\Delta_1(f_1 f_2) = 0$.

These transformations are possible and can be constructed in infinite ways, by choosing as coordinates lines on the hypersurfaces $\theta = const.$ $\varphi = const.$, an orthogonal net whose lines $t' = var$ are geodesics.

Let R_i be the class we have thus established. In the following we shall discuss fluid distributions comoving with R_i , in other words fluid distribution comoving with isotropic coordinates. Two immediate consequences of this assumption, respectively of kinematical and dynamical nature, are the following:

In the first place the isotropic coordinates are comoving, as it is well know, if and only if the shear vanished: i.e. if and only if $\dot{\lambda} = \frac{\dot{Y}}{Y}$.

In the second place, the shear free condition and the conservation equation written in the equivalent form [?]

$$\dot{\lambda} = \frac{e^v}{Y'} \partial_r \left(\frac{\dot{Y}}{e^v} \right) \tag{2.6.45}$$

give:

$$\frac{\dot{Y}}{e^v} = Y e^{f(t)} \tag{2.6.46}$$

where $f(t)$ is an integration function. This relation is the well know velocity-distance relation (Hubble's law) valid in newtonian as well as in relativistic cosmology. On the contrary, one can see, in view of Eq. (??), that the shear vanishes where the Hubble's law is valid (provided $Y' \neq 0$). Since a solution with $Y = const.$ is irregular at the origin, we can conclude:

Proposition 2.6.2 *The shear free motions of a spherically symmetric distribution in general relativity, being regular at the origin, are the ones and only the ones for*

which the Hubble's law holds.

Harmonic coordinates

Let x^h be a system of harmonic coordinates and g_{ij} the coefficient of the corresponding line-element; the condition of harmonicity can be expressed either by the use of the following equations:

$$\nabla x^h = 0 \quad (2.6.47)$$

where ∇ is the invariant d'Alembertian; or by the use of the following equations:

$$\Gamma^h = 0 \quad (2.6.48)$$

where $\Gamma^h = g^{lm}\Gamma_{lm}^h$, Γ_{lm}^h being the Christoffel symbols.

Denoted with \mathcal{R} the class of the systems of reference associated to harmonic coordinates, we are going to prove that this class is formed neither by only one system of reference nor by the totality of possible physical systems.

In order to prove the first part of this assertion, let us start from an harmonic system of reference \mathcal{R}' . It is sufficient to show the existence of coordinates transformations, external to \mathcal{R}' , that enable one to pass from a system of harmonic coordinates to another of the same type. On this purpose, let us recall the transformation equations of the quantities Γ^i in any coordinates transformation $x^i \rightarrow x'^i$:

$$\Gamma'^l = \frac{\partial x'^l}{\partial x^s} \Gamma^s - g'^{sm} \frac{\partial^2 x'^l}{\partial x^s \partial x^m} \quad (2.6.49)$$

Writing these equations for two systems of harmonic coordinates (i.e. $\Gamma^s = \Gamma'^l = 0$) we obtain:

$$\frac{\partial^2 x'^l}{\partial x'^s \partial x'^m} = 0.$$

Consequently, it is sufficient to take a linear transformation, involving spatial and temporal coordinates, in order to obtain a transformation that is external to \mathcal{R}' and preserves the harmonic character of \mathcal{R}' .

In order to prove the second part of the above assertion, let us consider a reference frame $\bar{\mathcal{R}}$ which is not harmonic. We are going to show that it is not always possible to construct, inside $\bar{\mathcal{R}}$, a system of admissible harmonic coordinates. On this purpose, let us observe that an internal transformation of coordinates

$$x^\alpha = x^\alpha(x'^\alpha) \quad x^4 = x^4(x'^4, x'^\alpha)$$

inside $\bar{\mathcal{R}}$, from coordinates x^l to harmonic coordinates x'^l , must satisfy the following equations (see (??) for $\Gamma'^l = 0$):

$$\frac{\partial x'^l}{\partial x'^s} \Gamma^s - g^{sm} \frac{\partial^2 x'^l}{\partial x'^s \partial x'^m} = 0. \tag{2.6.50}$$

The coefficients of these equations generally depend also on time coordinate. Hence, Eqs. (??) admit no always solutions of the type

$$x^\alpha = x^\alpha(x'^\alpha) \quad x^4 = x^4(x'^4, x'^\alpha).$$

Hence a frame of reference admits harmonic coordinates (in other words, a frame of reference is harmonic) if and only if Eqs (??) hold for unknowns $x^\alpha = x^\alpha(x'^\alpha)$, $x^4 = x^4(x'^4, x'^\alpha)$.

Chapter 3

Particle dynamics in a general physical frame of reference

3.1 Relative standard quantities for a material particle

As it is well known, in relativistic Kinematics the absolute fundamental quantities of a material particle, without an internal structure, are the following:

$$\text{proper mass } m_0, \quad \text{constant}; \quad (3.1.1)$$

$$\text{proper time elementary interval } d\tau = \frac{1}{c}\sqrt{-ds^2}; \quad (3.1.2)$$

$$\text{4-velocity } U^h = \frac{dx^h}{d\tau} = c \frac{dx^h}{\sqrt{-ds^2}} \quad \text{tangent to the time-line of the particle;} \quad (3.1.3)$$

$$\text{4-acceleration } A^h = \frac{DU^h}{d\tau}; \quad (3.1.4)$$

$$\begin{aligned} \text{spatial norm of the vector } dx^h \\ d\sigma^2 = \gamma_{hk} dx^h dx^k = \gamma_{\alpha\rho} dx^\alpha dx^\rho \quad (\alpha, \rho = 1, 2, 3); \end{aligned} \quad (3.1.5)$$

$$\begin{aligned} \text{relative standard time interval between two infinitely neighbouring} \\ \text{events } \{x^h\}, \{x^h + dx^h\} \\ dT = -\frac{1}{c} \gamma_h dx^h; \end{aligned} \quad (3.1.6)$$

where $\underline{\gamma}$ is versor tangent to a line of the congruence Γ , or equivalently to a line of the physical frame of reference associated to Γ .

The scalar (??) isn't generally an exact differential and this prevents the interlock of the standard clocks in \mathcal{S} , unless the reference \mathcal{S} doesn't result irrotational and geodesic [$\Omega_{hk} = \partial_h \gamma_k - \partial_k \gamma_h = 0 \rightarrow \gamma_h = \partial_h f$, where f is a regular scalar function].

Moreover we can define, for the particle,

$$\begin{aligned} \text{relative standard velocity} \\ v^\rho = \frac{dx^\rho}{dT} = v^\rho(x^1, x^2, x^3, x^4) \end{aligned} \quad (3.1.7)$$

spatial vector, which (spatial) norm is

$$\|\underline{v}\| \equiv v^2 = \gamma_{\alpha\rho} v^\alpha v^\rho = \gamma_{\alpha\rho} \frac{dx^\alpha}{dT} \frac{dx^\rho}{dT} = \frac{d\sigma^2}{dT^2}; \quad (3.1.8)$$

relative standard momentum

$$p_h = \mathcal{P}_\Sigma(P_h) \equiv \gamma_{hk} P^k \equiv m_0 \gamma_{hk} \frac{dx^k}{d\tau} \quad (3.1.9)$$

where P_h is the 4-absolute momentum $m_0 U$ of the particle;

relative standard material energy

$$E = -c\gamma_h P^h = -m_0 c \gamma_h \frac{dx^h}{d\tau}. \quad (3.1.10)$$

Let us observe that the proper time elementary interval $d\tau$ and the relative standard time interval dT are defined throughout the world lines of the material particles; but, beside them, the coordinate time interval dt is also defined

$$dt = \frac{dx^4}{c}, \quad (3.1.11)$$

which doesn't have a real physical sense. Moreover let us observe that the quantities $d\sigma^2$ and dT are invariant with respect to every coordinate transformations into the chosen physical frame of reference \mathcal{S} , that is indispensable for a correct formulation

of the physical laws. This invariance allows to use the formulas ¹

in every system of adapted coordinates in the fixed physical frame of reference \mathcal{S}

$$ds^2 = d\sigma^2 - c^2 dT^2 = (-c^2 d\tau^2) \quad (3.1.12)$$

$$\frac{dT}{d\tau} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad (3.1.13)$$

$$\frac{dt}{dT} = \frac{1 + \gamma_\rho \frac{v^\rho}{c}}{\sqrt{-g_{44}}} \quad (3.1.14)$$

If we introduce the relative standard mass m of the particle by the formula

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad (3.1.15)$$

the relative standard material energy defined in (??), considering (??) and (??) as well as (??) and (??) becomes

¹

$$d\sigma^2 = c^2 dT^2 - c^2 d\tau^2; \quad v^2 = \frac{d\sigma^2}{dT^2} = c^2 - c^2 \left(\frac{d\tau}{dT}\right)^2 \Rightarrow \frac{v^2}{c^2} = 1 - \left(\frac{d\tau}{dT}\right)^2 \Rightarrow$$

$$\frac{dT}{d\tau} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}};$$

$$dT = -\frac{1}{c} \gamma_h dx^h = -\frac{1}{c} \gamma_4 dx^4 - \frac{1}{c} \gamma_\rho dx^\rho = -\gamma_4 dt - \frac{1}{c} \gamma_\rho \frac{dx^\rho}{dT} dT,$$

$$dT + \gamma_\rho \frac{v^\rho}{c} dT = -\gamma_4 dt \rightarrow dT \left(1 + \gamma_\rho \frac{v^\rho}{c}\right) = -\gamma_4 dt \rightarrow$$

$$\frac{dt}{dT} = \frac{1 + \gamma_\rho \frac{v^\rho}{c}}{-\gamma_4} = \frac{1 + \gamma_\rho \frac{v^\rho}{c}}{\sqrt{-g_{44}}} \text{ (??)}.$$

$$ds^2 = g_{hk} dx^h dx^k = (\gamma_{hk} - \gamma_h \gamma_k) dx^h dx^k = \gamma_{hk} dx^h dx^k - (\gamma_h dx^h)^2$$

$$E = -m_0 c \gamma_h \frac{dx^h}{d\tau} = -m_0 c \left(-c \frac{dT}{d\tau} \right) = m c^2 \quad (3.1.16)$$

and the relative standard momentum (??) becomes

$$p_h = m_0 \gamma_{hk} \frac{dx^k}{dT} \frac{dT}{d\tau} = m \gamma_{hk} v^k = m v_h. \quad (3.1.17)$$

The (??) can be replaced with the following expression

$$E = -c \gamma^h P_h = -c \gamma^4 P_4 = c \frac{P_4}{\gamma_4} \quad (3.1.18)$$

(in adapted coordinates $\gamma^\rho = 0$, $\gamma^4 \gamma_4 = -1$)

Being besides

$$\mathcal{P}_\Theta(P_h) = -\gamma_h \gamma_k P^k = -\gamma_h \gamma^4 P_4 = \gamma_h \frac{E}{c} \quad (3.1.19)$$

or also

$$\gamma^4 P_4 = -\frac{E}{c} \quad (3.1.20)$$

we can establish that the orthogonal projection onto $\underline{\gamma}$ of \underline{P} coincides, unless the sign, with the quantity

$$-\frac{E}{c}.$$

3.2 Relative law of motion for a material particle

As it is well known, the absolute law of motion for a freely gravitating material particle, with constant proper mass m_0 has the form

$$\frac{DP^h}{d\sigma} = 0. \quad (3.2.1)$$

Being

$$\begin{aligned} \|\underline{P}\| &= P_h P^h = m_0 U_h \cdot m_0 U^h = m_0^2 U_h U^h = m_0^2 g_{hk} \frac{dx^k}{d\tau} \cdot \frac{dx^h}{d\tau} = \\ &= -m_0^2 c^2. \end{aligned}$$

This equation is the generalization of the inertia's law of the special Relativity to the Riemannian manifold V_4 and expresses that the time-line of the particle is a geodetic of the manifold V_4 , space-time environment of every physical phenomena [cfr. n.1].

Being true the relation [cfr. (??), (??)]

$$P_h = \mathcal{P}_\Sigma(P_h) + \mathcal{P}_\Theta(P_h) = p_h + \frac{E}{c} \gamma_h \quad (3.2.2)$$

the equation (??), or the equivalent

$$\frac{DP_h}{dT} = 0 \quad (3.2.3)$$

becomes

$$\frac{DP_h}{dT} = \frac{D}{dT}p_h + \frac{E}{c} \frac{D}{dT}\gamma_h + \gamma_h \frac{1}{c} \frac{dE}{dt} = 0. \quad (3.2.4)$$

Projecting onto Σ this last equation we get

$$\mathcal{P}_\Sigma \left(\frac{DP_h}{dT} \right) = \mathcal{P}_\Sigma \left(\frac{D}{dT}p_h \right) + \frac{E}{c} \mathcal{P}_\Sigma \left(\frac{D}{dT}\gamma_h \right) = 0 \quad (3.2.5)$$

so it follows, being p_h a spatial vector,

$$\begin{aligned} \mathcal{P}_\Sigma \left(\frac{DP_h}{dT} \right) &= \mathcal{P}_\Sigma \left(\nabla_k p_{\dot{h}} \frac{dx^k}{dT} \right) + \frac{E}{c} \mathcal{P}_\Sigma \left(\nabla_k \gamma_{\dot{h}} \frac{dx^k}{dT} \right) = \\ &= \mathcal{P}_\Sigma \left[\left(\partial_k p_h - \left\{ \begin{matrix} r \\ k\dot{h} \end{matrix} \right\} p_r \right) \frac{dx^k}{dT} \right] + \frac{E}{c} \mathcal{P}_\Sigma \left[\left(\partial_k \gamma_h - \left\{ \begin{matrix} r \\ k\dot{h} \end{matrix} \right\} \gamma_r \right) \frac{dx^k}{dT} \right] = \\ &= \mathcal{P}_\Sigma \left[\frac{d}{dT}p_h - (k\dot{h}, s) g^{sr} p_r \frac{dx^k}{dT} \right] + \frac{E}{c} \mathcal{P}_\Sigma \left(\nabla_k \gamma_{\dot{h}} \frac{dx^k}{dT} \right) = \\ &= \frac{d}{dT}p_h - \mathcal{P}_\Sigma (k\dot{h}, s) p^s \frac{dx^k}{dT} + \frac{E}{c} \mathcal{P}_\Sigma \left(\nabla_k \gamma_{\dot{h}} \frac{dx^k}{dT} \right) = 0 \end{aligned} \quad (3.2.6)$$

If now we consider the relations (??), (??), (??), (??), these following equations

$$\mathcal{P}_\Sigma (k\dot{h}, s) p^s \frac{dx^k}{dT} = (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} - \frac{1}{2} \gamma_k \tilde{K}_{hs} p^s \frac{dx^k}{dT} - \frac{1}{2} \mathcal{P}_\Sigma \left(\gamma_k \Omega_{\dot{h}s} \right) p^s \frac{dx^k}{dT}$$

$$\mathcal{P}_\Sigma \left(\gamma_k \Omega_{\dot{h}s} \right) = \gamma_k \mathcal{P}_\Sigma \left(\Omega_{\dot{h}s} \right) = \gamma_k \left[\tilde{\Omega}_{hs} + \gamma_s \tilde{\Omega}_h \right]$$

$$\mathcal{P}_\Sigma \left(\nabla_k \gamma_{\dot{h}} \right) = \frac{1}{2} \left[\tilde{K}_{kh} + \tilde{\Omega}_{kh} \right] - \gamma_k C_h$$

are valid and hence the equation (??) can assume the form

$$\begin{aligned}
\mathcal{P}_\Sigma \left(\frac{DP_h}{dT} \right) &= \frac{d}{dT} p_h - \left[(k\tilde{h}, s)^* - \frac{1}{2} \gamma_k \tilde{K}_{hs} - \frac{1}{2} \gamma_k \left(\tilde{\Omega}_{hs} + \gamma_s \tilde{\Omega}_h \right) \right] p^s \frac{dx^k}{dT} + \\
&\quad + \frac{E}{c} \left[\frac{1}{2} \left(\tilde{K}_{kh} + \tilde{\Omega}_{kh} \right) - \gamma_k C_h \right] \frac{dx^k}{dT} = \\
&= \frac{d}{dT} p_h - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} + \frac{1}{2} \gamma_k \tilde{K}_{hs} p^s \frac{dx^k}{dT} + \frac{1}{2} \gamma_k \tilde{\Omega}_{hs} p^s \frac{dx^k}{dT} + \\
&\quad + \frac{E}{2c} \left(\tilde{K}_{kh} + \tilde{\Omega}_{kh} \right) \frac{dx^k}{dT} - \frac{E}{c} \gamma_k \frac{dx^k}{dT} C_h = 0
\end{aligned}$$

so it follows, considering (??) and (??), (??)

$$\begin{aligned}
\mathcal{P}_\Sigma \left(\frac{DP_h}{dT} \right) &= \frac{d}{dT} p_h - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} + \frac{1}{2} \gamma_k \tilde{K}_{hs} p^s \frac{dx^k}{dT} - \frac{1}{2} \tilde{\Omega}_{hs} p^s c + \\
&\quad + \frac{E}{2c} \left(\tilde{K}_{kh} + \tilde{\Omega}_{kh} \right) \frac{dx^k}{dT} + EC_h = \\
&= \frac{d}{dT} p_h - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} + \frac{1}{2} \tilde{K}_{hs} m v^s (-c) + \frac{1}{2} m c \tilde{K}_{hs} v^k - \frac{1}{2} \tilde{\Omega}_{hs} c m v^s + \\
&\quad + \frac{E}{2c} + \tilde{\Omega}_{kh} v^k + EC_h = \\
&= \frac{d}{dT} p_h - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} - \frac{1}{2} m c \tilde{\Omega}_{hs} v^s + \frac{1}{2} m c \tilde{\Omega}_{kh} v^k + EC_h = \\
&= \left[\frac{d}{dT} p_h - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} \right] - m c \tilde{\Omega}_{hs} v^s + m c^2 C_h = 0.
\end{aligned} \tag{3.2.7}$$

We have to point out that the expression

$$\frac{d}{dT}p_h - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} \quad (3.2.8)$$

can be rewritten in the form

$$\begin{aligned} \partial_r p_h \frac{dx^r}{dT} - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} &= \left(\tilde{\partial}_r - \gamma_r \gamma^4 \partial_4 \right) p_h \frac{dx^r}{dT} - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} = \\ &= \tilde{\partial}_r p_h \frac{dx^r}{dT} - \left(\gamma_r \frac{dx^r}{dT} \right) \gamma^4 \partial_4 p_h - (k\tilde{h}, s)^* p^s \frac{dx^k}{dT} = \\ &= \left[\tilde{\partial}_r p_h - (r\tilde{h}, s)^* p^s \right] \frac{dx^r}{dT} + c\gamma^4 \partial_4 p_h = \\ &= \left[\tilde{\partial}_r p_h - (r\tilde{h}, s)^* \gamma^{su} p_u \right] \frac{dx^r}{dT} + c\gamma^4 \partial_4 p_h = \\ &= \left[\tilde{\partial}_r p_h - \left\{ r\tilde{h} \right\}^* p_u \right] \frac{dx^r}{dT} + c\gamma^4 \partial_4 p_h = \tilde{\nabla}_r^* p_h \frac{dx^r}{dT} + c\gamma^4 \partial_4 p_h. \end{aligned} \quad (3.2.9)$$

Put

$$\begin{cases} \hat{D} p_h \equiv \tilde{\nabla}_r^* p_h \frac{dx^r}{dT} + c\gamma^4 \partial_4 p_h \\ G_h \equiv c\tilde{\Omega}_{hs} v^s - c^2 C_h \end{cases} \quad (3.2.10)$$

the equation (??) assumes the remarkable form

$$\frac{\hat{D}}{dT} p_h = m G_h. \quad (3.2.11)$$

Let us observe that the first member of this equation, which its explicit form is given by (??), expresses a real derivative since it follows the particle throughout its world line (its time–line); it is invariant respect to every transformation into the chosen physical frame of reference \mathcal{S} .

The second member represents the product of the relative standard mass [cfr.(??)] for the vector G_h which expresses the standard gravitational field into the physical frame of reference \mathcal{S} ; such force field, by virtue of (??), results the sum of two spatial vectors, invariant into the frame \mathcal{S} ,

$$-c^2 C_h \equiv G'_h, \quad c\tilde{\Omega}_{hs} v^s \equiv G''_h. \quad (3.2.12)$$

If the particle is at rest into the physical frame of reference \mathcal{S} , it results $v^s = 0$, and hence the standard gravitational field is given by the only vector G'_h , that is, unless the factor $-c^2$, equal to the curvature vector of the line of the reference described by the particle; for this reason it's fair to call G'_h the dragging gravitational field by analogy with the newtonian Mechanics. Therefore also in general Relativity the dragging gravitational field expresses, unless the sign, the absolute acceleration of the particle of the fluid of reference on which, at the general instant T , stays the material particle that is taken in consideration (the dragging acceleration).

So we can put also

$$G'_h = -A_h. \quad (3.2.13)$$

With regards to G''_h expressed in (??)₂, if we remember the added vector of $\tilde{\Omega}_{hs}$ in Σ_x [cfr. (??)]

$$\omega^\alpha = \frac{c}{4} \eta^{\alpha\beta\gamma} \tilde{\Omega}_{\beta\gamma} \quad \alpha, \beta, \gamma = 1, 2, 3 \quad (3.2.14)$$

and use locally euclidean coordinates in the spatial platform Σ_x , thus the anti-symmetrical Ricci tensor

$$\eta^{\alpha\beta\gamma} = \frac{1}{\sqrt{\gamma}} \epsilon^{\alpha\beta\gamma} \quad (3.2.15)$$

is equal to

$$\eta^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma} = \pm 1 \quad (3.2.16)$$

and we obtain

$$\left\{ \begin{array}{l} 2(\omega^2 v^3 - \omega^3 v^2) = 2(\underline{\omega} \times \underline{v})_1 = -c(\tilde{\Omega}_{12} v^2 + \tilde{\Omega}_{13} v^3) \\ 2(\omega^3 v^1 - \omega^1 v^3) = 2(\underline{\omega} \times \underline{v})_2 = -c(\tilde{\Omega}_{21} v^1 + \tilde{\Omega}_{23} v^3) \\ 2(\omega^1 v^2 - \omega^2 v^1) = 2(\underline{\omega} \times \underline{v})_3 = -c(\tilde{\Omega}_{31} v^1 + \tilde{\Omega}_{32} v^2) \end{array} \right. \quad (3.2.17)$$

In conclusion it results

$$G''_h = -2(\underline{\omega} \times \underline{v})_h \quad (3.2.18)$$

typical expression of the Coriolis gravitational field in newtonian Mechanics, so it's fair to extend the same denomination to the (??); and the equation (??) can assume the remarkable formula

$$\frac{\hat{D}}{dT}p_h = -m\underline{A} - 2m(\underline{\omega} \times \underline{v}). \quad (3.2.19)$$

It suggests that the relative law of motion for a freely gravitating material particle coincides formally with the law of newtonian motion of a material particle subjected just to an “apparent field”. Therefore the equation (??) expresses the equivalence principle between “apparent fields” and “real fields” introduced into general Relativity.

Also we can observe that the gravitational drag G'_h can assume the other following form [cfr. (??)₂, (??), (??)]

$$\begin{aligned} G'_h &= -c^2 C_h = -c^2 \gamma^r \nabla_r \gamma_h = -c^2 \left\{ \gamma_4 \tilde{\partial}_h \gamma^4 + \partial_4 (\gamma_h \gamma^4) \right\} = \\ &= -c^2 \left\{ \tilde{\partial}_h \log(-\gamma_4) - \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) \right\} = -c^2 \tilde{\partial}_h \log(-\gamma_4) + c^2 \partial_4 \left(\frac{\gamma_h}{\gamma_4} \right) \end{aligned} \quad (3.2.20)$$

where if we put

$$U = -c^2 \log \sqrt{-g_{44}}, \quad (3.2.21)$$

it's evident that G'_h is the sum of a term that depends on the scalar potential U (by transverse partial derivation) and a term that depends on the vectorial potential

$$c^2 \frac{\gamma_h}{\gamma_4}.$$

This situation reminds the electromagnetic case in which the 4–density of driving

force generated by a electromagnetic field results sum of two alike terms, as it is expressed by the space-time divergence of the energy tensor of the electromagnetic field S_h^k :

$$f_h = -\partial_k S_h^k \equiv -\partial_\rho S_h^\rho - \partial_4 S_h^4.$$

However let us observe that the two terms that break the field G'_h in the (??) are just invariant with respect to a transformation of purely spatial coordinates into the physical frame of reference \mathcal{S} , while G'_h and G''_h are invariant with respect to every transformation of coordinates into the physical frame of reference \mathcal{S} .

3.3 Energy law for a freely gravitating material particle

Let us now consider the time projection of the equation (??), that is the equation

$$\mathcal{P}_\Theta \left(\frac{DP_h}{dT} \right) = 0; \quad (3.3.1)$$

considering the decomposition (??) it assumes the form

$$\mathcal{P}_\Theta \left(\frac{DP_h}{dT} \right) + \frac{1}{c} \gamma_h \frac{dE}{dt} = 0, \quad (3.3.2)$$

since the vector $\frac{D\gamma_h}{dT}$ is spatial, being $\gamma^h D\gamma_h = 0$; hence from (??) it follows

$$\mathcal{P}_\Theta \left[\partial_r p_h - \left\{ \begin{matrix} u \\ rh \end{matrix} \right\} p_u \right] \frac{dx^r}{dT} + \frac{1}{c} \gamma_h \frac{dE}{dt} = \mathcal{P}_\Theta \left(\frac{d}{dT} p_h \right) - \mathcal{P}_\Theta(r\dot{h}, s) p^s \frac{dx^r}{dT} + \frac{1}{c} \gamma_h \frac{dE}{dt} = 0.$$

But the vector $\frac{d}{dT} p_h$ is purely spatial, so its time projection is equal to zero, and the equation before becomes

$$-\mathcal{P}_\Theta(r\dot{h}, s) p^s \frac{dx^r}{dT} + \frac{1}{c} \gamma_h \frac{dE}{dt} = 0. \quad (3.3.3)$$

In addition this equation can be transformed using the decomposition (??); considering (??), (??) we obtain

$$\begin{aligned} & \frac{1}{2} \left[\gamma_h \left(\tilde{K}_{rs} + \Omega_{rs} \right) + \gamma_r \mathcal{P}_\Theta(\Omega_{\dot{h}s}) + \gamma_s \mathcal{P}_\Theta(Q_{\dot{h}r}) \right] p^s \frac{dx^r}{dT} + \frac{1}{c} \gamma_h \frac{dE}{dt} = \\ & = \frac{1}{2} \left[\gamma_h \left(\tilde{K}_{rs} + \Omega_{rs} \right) + \gamma_r \mathcal{P}_\Theta(\Omega_{\dot{h}s}) \right] p^s \frac{dx^r}{dT} + \frac{1}{c} \gamma_h \frac{dE}{dt} = \\ & = \frac{1}{2} \left[\gamma_h \left(\tilde{K}_{rs} + \tilde{\Omega}_{rs} + \gamma_s C_r - \gamma_r C_s \right) + \gamma_r (-\gamma_h C_s) \right] p^s \frac{dx^r}{dT} + \frac{1}{c} \gamma_h \frac{dE}{dt} = \\ & = \frac{1}{2} \left[\gamma_h \left(\tilde{K}_{rs} + \tilde{\Omega}_{rs} \right) p^s \frac{dx^r}{dT} - 2\gamma_h C_s \gamma_r \frac{dx^r}{dT} p^s \right] + \frac{1}{c} \gamma_h \frac{dE}{dt} = \\ & = \frac{1}{2} \gamma_h \left(\tilde{K}_{rs} + \tilde{\Omega}_{rs} \right) p^s \frac{dx^r}{dT} - \gamma_h C_s p^s (-c) + \frac{1}{c} \gamma_h \frac{dE}{dt} = 0. \end{aligned}$$

In conclusion from this last equation, dividing by γ_h ,

$$\frac{dE}{dt} = -\frac{1}{2} \left(\tilde{K}_{rs} + \tilde{\Omega}_{rs} \right) c p^s \frac{dx^r}{dT} - c^2 C_s p^s$$

or

$$\frac{dE}{dt} = -\frac{1}{2} mc \left(\tilde{K}_{\rho\sigma} + \tilde{\Omega}_{\rho\sigma} \right) \frac{dx^\rho}{dT} \frac{dx^\sigma}{dT} - mc^2 C_\rho \frac{dx^\rho}{dT} \quad (3.3.4)$$

Since the spatial tensorial field $\tilde{\Omega}_{\rho\sigma}$ is antisymmetrical, it results

$$\tilde{\Omega}_{\rho\sigma} v^\rho v^\sigma = 0$$

and the (??), remembering the (??), becomes

$$\frac{dE}{dt} = mG_\rho v^\rho - \frac{1}{2} mc \tilde{K}_{\rho\sigma} v^\rho v^\sigma. \quad (3.3.5)$$

This equation also shows a strict analogy with the energy equation in the newtonian Mechanics: the first member expresses the derivative of the relative energy of material respect to the relative standard time; the second member is the power of the standard gravitational field G_ρ multiplied for the relative mass of the particle (like in the newtonian case) plus the term

$$-\frac{1}{2} mc \tilde{K}_{\rho\sigma} v^\rho v^\sigma$$

which expresses the product of the relative mass of the particle for the power of the tensor $\tilde{K}_{\rho\sigma} \equiv \gamma^4 \partial_4 \gamma_{\rho\sigma}$ that characterizes the deformation of the physical frame of reference \mathcal{S}

$$-\frac{1}{2}c\tilde{K}_{\rho\sigma}v^\rho v^\sigma.$$

This term gives the energy given in the unit of relative standard time to the evolution of the particle by the deformation of the physical frame of reference \mathcal{S} ; it is null if the physical frame of reference \mathcal{S} is rigid ($\tilde{K}_{\rho\sigma} \equiv 0$), as it happens in newtonian Mechanics where the physical reference is rigid. However if it considers the motion of a material particle subjected to a constrain depending on the time, in newtonian Mechanics there is also in the energy equation the work performed by the action of the constrain [?].

In particular let us suppose that the chosen physical frame of reference \mathcal{S} is stationary, that is it admitts a system of adapted coordinates $\{x^h\}$ for which the following conditions are valid

$$\frac{\partial}{\partial x^4}g_{hk} = 0 \tag{3.3.6}$$

and hence the others too

$$\frac{\partial}{\partial x^4}\gamma_{hk} = 0, \quad \frac{\partial}{\partial x^4}\gamma_h = 0, \quad \tilde{K}_{\rho\sigma} = 0. \tag{3.3.7}$$

Let us consider the equation (??) that can assume the form too

$$\mathcal{P}_\Theta \left(\frac{DP_h}{dT} \right) = -\gamma_h \gamma^r \frac{DP_r}{dT} = -\gamma_h \gamma^4 \frac{DP_4}{dT} = \frac{DP_4}{dT} \cdot \frac{\gamma_h}{\gamma_4} = 0 \tag{3.3.8}$$

or equivalently

$$\frac{DP_4}{dT} = 0 \quad (3.3.9)$$

since the ratios $\frac{\gamma_h}{\gamma_4}$ cannot be all equal to zero.

We can also substitute the following equation for (??)

$$DP_4 \equiv dP_4 - (4h, r) P^h dx^r = 0. \quad (3.3.10)$$

In consideration of the vector P^h is tangent to the time–line of the particle, it's valid the condition

$$P^h = a dx^h \quad (a \text{ is a scalar}) \quad (3.3.11)$$

the (??) can be written in the following way:

$$\begin{aligned} dP_4 - \frac{1}{2} (\partial_4 g_{hr} + \partial_h g_{4r} - \partial_r g_{4h}) P^h dx^r &= \\ &= dP_4 - \frac{1}{2} \partial_4 g_{hr} P^h dx^r - \frac{1}{2} \partial_h g_{4r} P^h dx^r + \frac{1}{2} \partial_r g_{4h} P^h dx^r = \\ &= dP_4 - \frac{1}{2} \partial_4 g_{hr} P^h dx^r - \frac{1}{2} a \partial_h g_{4r} dx^h dx^r + \frac{1}{2} a \partial_r g_{4h} dx^h dx^r = \\ &= dP_4 - \frac{1}{2} \partial_4 g_{hr} P^h dx^r = 0 \end{aligned}$$

that, considering (??), becomes

$$dP_4 = 0 \quad (3.3.12)$$

so it follows

$$P_4 = \text{const.} \quad (3.3.13)$$

along the time–line of the freely gravitating material particle.

If we now introduce the relative total energy of the particle with the function

$$H \equiv -cP_4 \equiv -E\gamma_4 \quad (3.3.14)$$

the (??) suggests that for the material particle there is the first integral of the relative total energy

$$H = \text{const.} \quad (3.3.15)$$

since the function H contains not only the relative material energy $E = mc^2$ but also the potential energy of the gravitational field: in fact, if the physical reference \mathcal{S} is stationary, what it has been assumed, the gravitational drag G'_h expressed by (??) now takes the form

$$G'_h = -c^2 \partial_h \log(-\gamma_4) \quad (3.3.16)$$

and therefore depends on the scalar potential (??), which for convenience we rewrite,

$$U = -c^2 \log(-\gamma_4) \quad (3.3.17)$$

whence it follows

$$-\gamma_4 = e^{-U/c^2}. \quad (3.3.18)$$

Therefore the first integral (??) becomes

$$H = mc^2 e^{-U/c^2} = \frac{m_0 c^2}{\sqrt{1 - (v/c)^2}} e^{-U/c^2} = \text{const.} \quad (3.3.19)$$

assuming the particle motion very slowly, so the restrictions are valid

$$\frac{v}{c} \ll 1, \quad \frac{U}{c^2} \ll 1, \quad (3.3.20)$$

and considering expansion series $\sqrt{1 - (v/c)^2}$, e^{-U/c^2} , we get

$$H \cong m_0 c^2 \left[1 + \frac{1}{2} \left(\frac{v}{c} \right)^2 \right] \left(1 - \frac{U}{c^2} \right) = \text{const.},$$

$$H \cong m_0 c^2 + \frac{1}{2} m_0 v^2 - m_0 U = \text{const.} \quad (3.3.21)$$

which does not differ formally from the Newtonian case: the sum of the rest energy $m_0 c^2$, of the kinetic energy $\frac{1}{2} m_0 v^2$ and of the potential energy $-m_0 U$ is constant.

3.4 The evolution of a free photon in a gravitational field

Let us consider the photon as a null proper mass particle, and therefore, with a 4-momentum P^h of norm equal to zero; put

$$P^h = a dx^h \quad (a \text{ is a scalar function, } P^h \text{ tangent to the time-line of the photon}) \quad (3.4.1)$$

is therefore assumed to

$$\|\underline{P}\| = a^2 g_{hk} dx^h dx^k = 0 \quad (3.4.2)$$

what is to admit that the time-line of the photon (we indicate it with l) is zero-length. If now we consider two infinitely neighbouring events on l , both the corresponding relative standard time interval dT and the corresponding space projection $d\sigma$ of the interval dl between the two events will be nonzero. Let us assume as characteristic property of the photon to have a (relative) wavelength, and with dn indicating the number of wave crests in the (elementary) range of length $d\sigma$ contained, we can put the relations

$$\lambda = \frac{d\sigma}{dn}, \text{ relative wavelength of the photon;} \quad (3.4.3)$$

$$\nu = \frac{dn}{dT}, \text{ frequency of the photon.} \quad (3.4.4)$$

It follows from them, because of l has zero length ($ds^2 = d\sigma^2 - c^2 dT^2 = 0$)

$$\lambda\nu = \frac{d\sigma}{dn} \cdot \frac{dn}{dT} = \frac{d\sigma}{dT} = c. \quad (3.4.5)$$

As defined in (??) the wavelength λ will be assessed along the spatial trajectory of the photon and, by virtue of the local Minkowskian character of the manifold V_4 , if we allow to pass a physical reference to another, or, equivalently, to vary locally the vector field $\underline{\gamma}$, as in special relativity λ and dT undergo a Lorentzian change for which the product must be invariant to varying local of the physical reference: therefore, we must admit that it is

$$\lambda dT \text{ absolute invariant}. \quad (3.4.6)$$

This fact leads us to define the 4-momentum of the photon with the formula

$$P^r = \frac{h}{c\lambda} \cdot \frac{dx^r}{dT} \quad (\text{h, Planck's constant}) \quad (3.4.7)$$

which, by virtue of the (??), becomes

$$P^r = \frac{h\nu}{c^2} \cdot \frac{dx^r}{dT}. \quad (3.4.8)$$

Attributed to a photon a 4-momentum, as the material particle, we can define a relative standard energy; we put exactly

$$E = -c\gamma_r P^r, \quad (3.4.9)$$

that we call the relative standard light energy.

Taking into account the (??) to (??) we can replace

$$E = -c\gamma_r \frac{h\nu dx^r}{c^2 dT} = -\frac{h\nu}{c} \gamma_r \frac{dx^r}{dT} = h\nu. \quad (3.4.10)$$

Consequently we define the relative standard mass

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2} = \frac{h}{c\lambda} \quad (3.4.11)$$

and the relative standard momentum

$$p^\rho = P^\rho = \frac{h\nu}{c^2} v^\rho = mv^\rho. \quad (3.4.12)$$

In particular, in a stationary physical reference and adapted coordinates, we can introduce, even for the free photon, the relative total energy

$$H = -cP_4 = E\gamma_4 = h\nu\sqrt{-g_{44}}. \quad (3.4.13)$$

At this point it is permissible for the photon free to take as fundamental equation of the evolution the equation [(??)]

$$DP_r = 0 \quad r = 1, 2, 3, 4. \quad (3.4.14)$$

It states, with (??), which the photon describes a geodesic of length zero in the physical reference \mathcal{S} ; thus it is noted that in a gravitational field a light beam is configured as a geodesic of V_4 ; from (??) we can also draw another remarkable result

if the physical reference is stationary. In fact it follows, in general, from (??) and (??)

$$\begin{aligned}
DP_4 &= \nabla_r P_4 dx^r = [\partial_r P_4 - (r4, s)g^{su} P_u] dx^r = \\
&= dP_4 - \frac{1}{2} (\partial_r g_{4s} + \partial_4 g_{sr} - \partial_s g_{r4}) P^s dx^r = \\
&= dP_4 - \frac{1}{2} \partial_4 g_{sr} P^s dx^r - \frac{1}{2} \partial_r g_{4s} P^s dx^r + \frac{1}{2} \partial_s g_{r4} P^s dx^r = \\
&= dP_4 - \frac{1}{2} \partial_4 g_{sr} P^s dx^r - \frac{1}{2} a \partial_r g_{4s} dx^s dx^r + \frac{1}{2} a \partial_s g_{r4} dx^s dx^r
\end{aligned}$$

that is

$$dP_4 - \frac{1}{2} \partial_4 g_{sr} P^s dx^r = 0; \quad (3.4.15)$$

if, in particular, the physical reference \mathcal{S} is stationary [(??), eqref17.4], the equation (??) reduces to

$$dP_4 = 0 \Rightarrow P_4 = \text{const.} \underline{\text{along the time line of the photon.}}$$

And then the relative total energy occurs on the condition

$$H = -cP_4 = h\nu\sqrt{-g_{44}} = \text{const.} \underline{\text{along the time line of the photon.}} \quad (3.4.16)$$

If we take into account two different positions of the photon along its time-line

and indicate them with symbols (1), (2), from (??) we deduce

$$h\nu(1)\sqrt{-g_{44}(1)} = h\nu(2)\sqrt{-g_{44}(2)} \quad (3.4.17)$$

or equivalently

$$\frac{\nu(1)}{\nu(2)} = \sqrt{\frac{g_{44}(2)}{g_{44}(1)}}. \quad (3.4.18)$$

This relation suggests that the photon, free in a gravitational field, changes its frequency along its time line, that is the spectral lines of a light beam undergoes a shift under the action of a gravitational field.

Remark 3.4.1 *At the end of this paragraph, we can say that the introduction of the relative quantities, in addition to absolute, in a general physical reference \mathcal{S} , has resulted in this reference the equation of evolution of a freely gravitating material particle expressing the equivalence between “real fields” and “apparent fields” as a principle enunciated in the manifold V_4 , an energy theorem formally similar to the Newtonian; and if the physical reference \mathcal{S} is stationary, the first integral of the total energy of the particle.*

It also allowed to follow the evolution of a free photon in a gravitational field as a particle with nonzero relative mass, nonzero 4-quantum; and in the particular case that the physical reference is stationary it allowed to detect the shift of spectral lines of a light beam under the action of a gravitational field.

3.5 Formal identification of the laws of evolution of a material particle and a photon through the use of an affine parameter

The evolution law of a free photon in a gravitational field can have the same form as the parallel law for a freely gravitating material particle

$$\frac{DP^r}{dT} = 0 \tag{3.5.1}$$

with 4-momentum P^r of norm equal to zero expressed in (??) and (??).

From (??) follows the geodesic character of the time line of the photon l . In fact, considering (??), making explicit the absolute derivation we get the following equations

$$\frac{D}{dT} \left(\frac{dx^r}{dT} \right) = \frac{d}{dT} \log \left(\frac{1}{m} \right) \frac{dx^r}{dT}. \tag{3.5.2}$$

Acting the relative standard time T as a parameter and considering

$$\varphi(T) = \frac{d}{dT} \log \left(\frac{1}{m} \right), \tag{3.5.3}$$

we can write

$$\frac{D}{dT} \left(\frac{dx^r}{dT} \right) = \varphi(T) \frac{dx^r}{dT}. \tag{3.5.4}$$

At this point, changing the parameter in this way [?]

$$du = Ke^{\int \varphi(T)dT} dT \quad (K, \text{arbitrary constant})$$

$$du = \frac{K}{m} dT \equiv \frac{Kc}{h} \lambda dT \quad (3.5.5)$$

we can give to (??) the simplified form

$$\frac{D}{du} \left(\frac{dx^r}{du} \right) = 0. \quad (3.5.6)$$

The (??) emphasizes that, interpreting a geodesic of zero length in a Einsteinian space-time as time line for a free photon, its affine parameters, invariant with respect to any change of physical reference, are obtained by means of two relative parameters, the wavelength λ of the photon and the elementary interval of relative standard time dT .

The constant factor K corresponds to the arbitrariness of the initial wavelength of several photons that can be thought of along the same geodesic of zero length. Notice how, through the use of the affine parameter and setting $K = 1$ into (??), the (??) and (??) can be written respectively

$$P^r = \frac{dx^r}{du}, \quad \frac{DP^r}{du} = 0. \quad (3.5.7)$$

It's straightforward to verify that for a material particle of constant proper mass the 4-momentum P^r and the law of motion (??) have essentially the form (??).

In fact, the 4-momentum P^r for a material particle of proper mass m_0 is

$$P^r = m_0 \frac{dx^r}{d\tau}. \quad (3.5.8)$$

Introducing into (??) the affine parameter $u = \frac{\tau}{m_0}$, we obtain

$$P^r = \frac{dx^r}{du}. \quad (3.5.9)$$

We can therefore conclude that the use of the affine parameter allows to unify in the form (??) the definition of momentum and the absolute law of motion for a material particle and a photon.

3.6 Principle of stationary action and Fermat's principle

Let us consider in a stationary space-time V_4 a system of coordinates $\{x^h\}$ adapted to the stationarity, ie such that the potentials are all independent of the coordinate x^4 . So it is identified a stationary physical reference, in which the space-manifolds of equation $x^4 = const.$ are all isometric to each other [?].

In such a physical reference we consider an arc \bar{l} of a time-line of a free material particle of proper mass constant, and sign with u an affine parameter along it and consider $\bar{P}_1 \equiv \bar{x}^h(u_1)$, $\bar{P}_2 \equiv \bar{x}^h(u_2)$ the extremes of the arc, ϵ_1 , ϵ_2 the two lines of the physical reference to which $\bar{P}_1 \equiv \bar{x}^h(u_1)$, $\bar{P}_2 \equiv \bar{x}^h(u_2)$ belong respectively. We consider a set I_l of other like-time arcs, infinitely little varied compared to \bar{l} , whose extremes orderly belong to ϵ_1 , ϵ_2 .

Let us consider the integral

$$\mathcal{F} = c \int_{T_1}^{T_2} \left(mc^2 - mv^2 - H \frac{dx^4}{cdT} \right) dT \quad (3.6.1)$$

where dT , v^2 and H are respectively defined in (??), (??) and (??) and T_1 , T_2 are the values took in the ends of each arc from the corresponding relative standard time.

Therefore the integral (??) is defined on \bar{l} and on each arc of the set I_l .

Let us evaluate the expression of the first variation that the functional (??) suffers when it change from the arc \bar{l} to any other arcs of I_l . We represent these arcs depending on the common parameter u

$$x^h = \bar{x}^h(u) + \delta x^h(u) \quad (h = 1, 2, 3, 4) \quad (3.6.2)$$

with $u_1 \leq u \leq u_2$, $\delta x^h(u)$ infinitesimal functions of class C^2 subject to the condition

$$\delta x^\rho(u_1) = \delta x^\rho(u_2) = 0 \quad (\rho = 1, 2, 3). \quad (3.6.3)$$

Taking into account that on each arc

$$-\gamma_h \frac{dx^h}{dT} = c \quad (3.6.4)$$

the (??) becomes

$$\begin{aligned}\mathcal{F} &= c \int_{u_1}^{u_2} \left\{ m_0 c^2 \left[\left(\gamma_h \frac{dx^h}{du} \right)^2 - \gamma_{hk} \frac{dx^h}{du} \frac{dx^k}{du} \right]^{1/2} - H \frac{dx^4}{du} \right\} du \equiv \\ &\equiv \int_{u_1}^{u_2} [L(x, \dot{x}) - H \dot{x}^4] du\end{aligned}\quad (3.6.5)$$

where

$$\dot{x}^h = \frac{dx^h}{du},$$

$$L(x, \dot{x}) \equiv m_0 c^2 \left[(\gamma_h \dot{x}^h)^2 - \gamma_{hk} \dot{x}^h \dot{x}^k \right]^{1/2}.$$

The first variation of \mathcal{F} is

$$\begin{aligned}\delta\mathcal{F} &= \int_{u_1}^{u_2} \delta L - \int_{u_1}^{u_2} \delta H dx^4 - \int_{u_1}^{u_2} H d(\delta x^4) = \\ &= \int_{u_1}^{u_2} \left[\frac{\partial L}{\partial x^h} \delta x^h + \frac{\partial L}{\partial \dot{x}^h} \delta \dot{x}^h \right] du - \int_{u_1}^{u_2} \delta H dx^4 - [H \delta x^4]_{u_1}^{u_2} + \int_{u_1}^{u_2} \delta x^4 dH.\end{aligned}\quad (3.6.6)$$

Since the equation

$$\int_{u_1}^{u_2} \frac{\partial L}{\partial \dot{x}^h} \delta \dot{x}^h du = \int_{u_1}^{u_2} \frac{\partial L}{\partial \dot{x}^h} d(\delta x^h) = \left[\frac{\partial L}{\partial \dot{x}^h} \delta x^h \right]_{u_1}^{u_2} - \int_{u_1}^{u_2} \frac{d}{du} \left(\frac{\partial L}{\partial \dot{x}^h} \right) \delta x^h du$$

is valid, considering (??), the (??) becomes

$$\begin{aligned}\delta\mathcal{F} &= \int_{u_1}^{u_2} \left[\frac{\partial L}{\partial x^h} - \frac{d}{du} \left(\frac{\partial L}{\partial \dot{x}^h} \right) \right] \delta x^h du + \left[\frac{\partial L}{\partial \dot{x}^4} \delta x^4 \right]_{u_1}^{u_2} + \\ &- \int_{u_1}^{u_2} \delta H dx^4 - [H \delta x^4]_{u_1}^{u_2} + \int_{u_1}^{u_2} \delta x^4 dH.\end{aligned}\quad (3.6.7)$$

where

$$\frac{\partial L}{\partial \dot{x}^4} = m_0 c^2 \left[(\gamma_h \dot{x}^h)^2 - \gamma_{hk} \dot{x}^h \dot{x}^k \right]^{-1/2} \gamma_h \dot{x}^h \gamma_4.\quad (3.6.8)$$

Let us suppose that the arc \bar{l} belong to an integral line of the Lagrangian system

$$L_h \equiv -\frac{d}{du} \left(\frac{\partial L}{\partial \dot{x}^h} \right) - \frac{\partial L}{\partial x^h} = 0 \quad (3.6.9)$$

and assume as affine parameter the proper time τ . As we have seen the relative total energy of a freely gravitating material particle (??) is constant, so it follows

$$\delta \mathcal{F} = - \int_{\tau_1}^{\tau_2} \delta H dx^4. \quad (3.6.10)$$

If then we consider only those arcs on which the relative total energy H is always equal to the same constant value, the first variation will be equal to zero. So we have the complete equivalence between the Lagrangian system (??) and the variational equation

$$\delta_H \mathcal{F} = 0. \quad (3.6.11)$$

which expresses that the isoenergetic variations of the functional (??) are identically equal to zero.

The Lagrangian system (??) is also equivalent to the variational equation

$$\delta \int_{T_1}^{T_2} L(x, \dot{x}) dT = 0$$

that is the relative expression of the equations of motion of a free material particle in a gravitational field also non-stationary. Therefore the (??) expresses another relative variational formulation of the laws of motion of a material particle.

Since it has

$$\frac{d\sigma}{dT} = v, \quad \frac{dx^4}{dT} = -\frac{c + \gamma_\rho v^\rho}{\gamma_4} \quad (3.6.12)$$

the functional (??) becomes

$$\mathcal{F} = \int_{Q_1}^{Q_2} \left[mc^3 \left(1 - \frac{v^2}{c^2} \right) - mc^2(c + \gamma_\rho v^\rho) \right] \frac{1}{v} d\sigma = -c \int_{Q_1}^{Q_2} \left(mv + mc\gamma_\rho \frac{v^\rho}{v} \right) d\sigma \quad (3.6.13)$$

where Q_1 and Q_2 represent the intersections of the time-lines ϵ_1 and ϵ_2 with any of the spatial manifold of equation $x^4 = const..$

So (??) becomes

$$\delta_H \int_{Q_1}^{Q_2} \left(mv + mc\gamma_\rho \frac{v^\rho}{v} \right) d\sigma = 0. \quad (3.6.14)$$

If we suppose that the manifold V_4 is static and with adapted coordinates, (??) becomes

$$\delta_H \int_{Q_1}^{Q_2} mvd\sigma = 0. \quad (3.6.15)$$

So in a static space-time the variational principle (??) formally coincides with the classical Maupertuis's principle.

If the space-time V_4 reduces to a pseudo-Euclidean manifold with Galilean coordinates, it would $\gamma_4 = -1$, $\gamma_\rho = 0$, from (??) we get the first integral $v = const$ and the (??) becomes

$$\delta_H \int_{Q_1}^{Q_2} d\sigma = 0 \quad (3.6.16)$$

and then represent the variational expression of the law of inertia.

Let us now draw on the functional (??) and suppose that the space-time arc of trajectory \bar{l} and every arc of the set I_l are light kind, that is

$$ds^2 = \gamma_{hk} dx^h dx^k - (\gamma_h dx^h)^2 = d\sigma^2 - c^2 dT^2 = 0 \quad (3.6.17)$$

So it follows, as it is known, $v = c$, then (??) becomes

$$\mathcal{F} = - \int_{T_1}^{T_2} H dx^4 \quad (3.6.18)$$

So, since H is constant, its variation can assume the following form

$$\delta_H \int dx^4 = 0 \quad (3.6.19)$$

From (??) we get the spatial expression

$$dx^4 = -\frac{1}{\gamma_4} \left[\gamma_h dx^h + (\gamma_{hk} dx^h dx^k)^{1/2} \right] \quad (3.6.20)$$

and (??) has the purely spatial form

$$\delta \int_{Q_1}^{Q_2} \mathcal{L}(x, \dot{x}) du \equiv \delta \int_{Q_1}^{Q_2} \frac{1}{\gamma_4} \left[\gamma_h \frac{dx^h}{du} + \left(\gamma_{hk} \frac{dx^h}{du} \frac{dx^k}{du} \right)^{1/2} \right] du = 0. \quad (3.6.21)$$

where we consider only the spatial trajectory of the photon and don't take into account its total energy.

We will show that the eurlians associated to the variational principle (??) $\left[\frac{d}{du} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\rho}\right) - \frac{\partial \mathcal{L}}{\partial x^\rho} = 0\right]$ are equal to the differential equations of the trajectory of the photon in the physical space.

In fact we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{x}^\rho} &= \frac{\gamma_\rho}{\gamma_4} + \frac{1}{\gamma_4} \left(\gamma_{hk} \frac{dx^h}{du} \frac{dx^k}{du} \right)^{1/2} \gamma_{hk} \frac{dx^k}{du} \\ \frac{\partial \mathcal{L}}{\partial x^\rho} &= \frac{\partial}{\partial x^\rho} \left(\frac{\gamma_h}{\gamma_4} \right) \frac{dx^h}{du} + \left(\gamma_{hk} \frac{dx^h}{du} \frac{dx^k}{du} \right)^{1/2} \frac{\partial}{\partial x^\rho} \left(\frac{1}{\gamma_4} \right) + \\ &\quad + \frac{1}{2\gamma_4} \left(\gamma_{hk} \frac{dx^h}{du} \frac{dx^k}{du} \right)^{-1/2} \frac{\partial}{\partial x^\rho} (\gamma_{hk}) \frac{dx^h}{du} \frac{dx^k}{du}\end{aligned}$$

and if $\bar{\lambda}$ is an arc of integral curve of the eurlians associated to the (??) and we assume the abscissa curvilinea σ as parameter instead of u , we get

$$\gamma_{hk} \frac{dx^h}{du} \frac{dx^k}{du} = 1 \tag{3.6.22}$$

Therefore the eurlians of the variational principle (??) become

$$\begin{aligned}\frac{d}{d\sigma} \left(\frac{\gamma_\rho}{\gamma_4} + \frac{1}{\gamma_4} \gamma_{hk} \frac{dx^k}{d\sigma} \right) - \frac{\partial}{\partial x^\rho} \left(\frac{\gamma_h}{\gamma_4} \right) \frac{dx^h}{d\sigma} - \frac{\partial}{\partial x^\rho} \left(\frac{1}{\gamma_4} \right) + \\ - \frac{1}{2\gamma_4} \frac{\partial}{\partial x^\rho} (\gamma_{hk}) \frac{dx^h}{d\sigma} \frac{dx^k}{d\sigma} = 0\end{aligned}$$

or putting $\partial_\rho \equiv \frac{\partial}{\partial x^\rho}$, the other following equation

$$\begin{aligned}
& \frac{1}{\gamma_4} \left[\frac{d}{d\sigma} \left(\gamma_{hk} \frac{dx^k}{d\sigma} \right) - \frac{1}{2} \partial_\rho \gamma_{hk} \frac{dx^h}{d\sigma} \frac{dx^k}{d\sigma} \right] = \\
& = -\frac{d}{d\sigma} \left(\frac{\gamma_\rho}{\gamma_4} \right) - \frac{d}{d\sigma} \left(\frac{1}{\gamma_4} \right) \gamma_{hk} \frac{dx^k}{d\sigma} + \partial_\rho \left(\frac{\gamma_h}{\gamma_4} \right) \frac{dx^h}{d\sigma} + \partial_\rho \left(\frac{1}{\gamma_4} \right).
\end{aligned} \tag{3.6.23}$$

If we consider that the physical reference and the chosen system of coordinates are adapted to the stationary of the space-time, that is

$$\partial_\rho \gamma_{hk} = \tilde{\partial}_\rho \gamma_{hk}, \quad \gamma_4 \left[\partial_\rho \left(\frac{\gamma_h}{\gamma_4} \right) - \partial_h \left(\frac{\gamma_\rho}{\gamma_4} \right) \right] = \gamma_4 \left[\tilde{\partial}_\rho \left(\frac{\gamma_h}{\gamma_4} \right) - \tilde{\partial}_h \left(\frac{\gamma_\rho}{\gamma_4} \right) \right] \equiv \tilde{\Omega}_{\rho h},$$

multiplig both of sides of (??) for γ_4 and putting $\lambda^\rho = \frac{dx^\rho}{d\sigma}$, the first member is

$$\frac{\hat{D}\lambda_\rho}{d\sigma} \equiv \frac{d}{d\sigma} (\gamma_{hk} \lambda^k) - \frac{1}{2} \tilde{\partial}_\rho \gamma_{hk} \lambda^h \lambda^k$$

and the second becomes

$$\begin{aligned}
& \gamma_4 \left[\partial_\rho \left(\frac{1}{\gamma_4} \right) - \frac{d}{d\sigma} \left(\frac{1}{\gamma_4} \right) \gamma_{hk} \lambda^k \right] + \tilde{\Omega}_{\rho h} \lambda^h = -\partial_\rho \log(-\gamma_4) + \\
& + \partial_k \log(-\gamma_4) \lambda^k \lambda_\rho + \tilde{\Omega}_{\rho h} \lambda^h = -\partial_k \log(-\gamma_4) (\delta_\rho^k - \lambda^k \lambda_\rho) + \tilde{\Omega}_{\rho h} \lambda^h.
\end{aligned}$$

We can write (??) in the following way

$$\frac{\hat{D}\lambda_\rho}{d\sigma} = -\partial_k \log(-\gamma_4) (\delta_\rho^k - \lambda^k \lambda_\rho) + \tilde{\Omega}_{\rho h} \lambda^h. \tag{3.6.24}$$

These last equations represent the equations of the spatial trajectory of a photon

in a stationary physical reference.

It is thus demonstrated that the spatial trajectories of a photon gravitating in a stationary space-time satisfy the variational principle of minimum coordinated time $\delta \int dx^4 = 0$, put even in purely spatial form (??).

This coordinated time has a physical sense because it represents the coordinated time that makes the stationarity of V_4 ; therefore, the principle (??) means essentially Fermat's principle in a stationary universe.

In conclusion the Fermat's principle can assume the following form

$$\delta_H \int_{Q_1}^{Q_2} m (c + \gamma_\rho v^\rho) d\sigma = 0. \quad (3.6.25)$$

if we consider the expression (??) of the principle of stationary action, putting $v = c$.

Chapter 4

Einsteinian gravitational field equations and their translation into a generic physical reference

4.1 Gravitational field equations in vacuum and relative formulation of conservation conditions for a chosen physical reference

4.1.1 Relative formulation of gravitational field equations in vacuum

Let $\{x^h\}$ be a system of physically admissible coordinated, and so an assigned physical frame of reference \mathcal{S} , in an einsteinian space-time manifold V_4 . Let us consider natural

projection of the contracted curvature tensor R_{jm} [?]

$$\left\{ \begin{array}{l}
 \mathcal{P}_{\Sigma\Sigma}(R_{jm}) = \tilde{R}_{jm}^* + \frac{1}{4}\tilde{K}_i^i \left(\tilde{K}_{mj} + \tilde{\Omega}_{mj} \right) - \frac{1}{2} \left(\tilde{K}_m^i + \tilde{\Omega}_m^i \right) \tilde{K}_{ij} + \\
 \qquad \qquad \qquad + \frac{1}{2}\gamma^4\partial_4 \left(\tilde{K}_{mj} + \tilde{\Omega}_{mj} \right) + \frac{1}{2}\tilde{\Omega}_j^i\tilde{\Omega}_{im} - C_j C_m - \tilde{\nabla}_m^* C_j \\
 \\
 \mathcal{P}_{\Sigma\Theta}(R_{jm}) = \gamma_m \left[\frac{1}{2} \left\{ \tilde{\nabla}_j^* \tilde{K}_i^i - \tilde{\nabla}_h^* \left(\tilde{K}_j^h + \tilde{\Omega}_j^h \right) \right\} + C_i \tilde{\Omega}_j^i \right] \\
 \\
 \mathcal{P}_{\Theta\Sigma}(R_{jm}) = \gamma_j \left[\frac{1}{2} \left\{ \tilde{\nabla}_m^* \tilde{K}_i^i - \tilde{\nabla}_i^* \left(\tilde{K}_m^i + \tilde{\Omega}_m^i \right) \right\} + C^i \tilde{\Omega}_{im} \right] \\
 \\
 \mathcal{P}_{\Theta\Theta}(R_{jm}) = \gamma_j \gamma_m \left[-\frac{1}{2}\gamma^4\partial_4 \tilde{K}_i^i - \frac{1}{4}\tilde{K}^{ij} \tilde{K}_{ij} + \right. \\
 \qquad \qquad \qquad \left. + \frac{1}{4}\tilde{\Omega}^{ij} \tilde{\Omega}_{ij} + C^i C_i + \tilde{\nabla}_i^* C^i \right]
 \end{array} \right. \tag{4.1.1}$$

and observe that the scalar curvature

$$R = g^{jm} (\mathcal{P}_{\Sigma\Sigma} + \mathcal{P}_{\Sigma\Theta} + \mathcal{P}_{\Theta\Sigma} + \mathcal{P}_{\Theta\Theta}) R_{jm}$$

as a result of the (??) takes the expression

$$\begin{aligned}
 R = \tilde{R}^* + \frac{1}{4} \left[\left(\tilde{K}_i^i \right)^2 + \tilde{K}^{ij} \tilde{K}_{ij} + \tilde{\Omega}^{ij} \tilde{\Omega}_{ij} \right] + \\
 + \gamma^4 \partial_4 \left(\tilde{K}_i^i \right) - 2 \left(\tilde{\nabla}_i^* C^i + C^i C_i \right)
 \end{aligned} \tag{4.1.2}$$

where

$$\begin{aligned} \tilde{R}^* &= g^{jm} \tilde{R}_{jm}^* = g^{jm} \left(\tilde{P}_{jm}^* - \frac{1}{2} \tilde{K}_j^r \tilde{\Omega}_{rm} \right) = \gamma^{\alpha\beta} \tilde{P}_{\alpha\beta}^* \\ &\left(\tilde{P}_{4r}^* = 0 = \tilde{K}_{4r} = \tilde{\Omega}_{4r} \right) \end{aligned} \quad (4.1.3)$$

$$\tilde{P}_{\alpha\beta}^* = -\tilde{\partial}_\beta \left\{ \widetilde{\alpha^\rho} \right\}^* + \tilde{\partial}_\rho \left\{ \widetilde{\beta^\alpha} \right\}^* - \left\{ \widetilde{\alpha^\rho} \right\}^* \left\{ \widetilde{\beta^\sigma} \right\}^* + \left\{ \widetilde{\beta^\alpha} \right\}^* \left\{ \widetilde{\rho^\sigma} \right\}^* .$$

At this point, let us determine the natural projections of the gravitational equations

$$G_{jm} \equiv R_{jm} - \frac{1}{2} R g_{jm} = 0 \quad (4.1.4)$$

If we put

$$\left\{ \begin{array}{l} s_{jm} = \mathcal{P}_{\Sigma\Sigma}(R_{jm}) \quad (s_{4j} = 0), \quad \mathcal{P}_{\Sigma\Theta}(R_{jm}) = S_i \gamma_m \\ S_\alpha = \frac{1}{2} \left[\tilde{\nabla}_\alpha^* \left(\tilde{K}^i \right) - \tilde{\nabla}_h^* \left(\tilde{K}_\alpha^h + \tilde{\Omega}_\alpha^h \right) \right] + C^{\beta\gamma} \tilde{\Omega}_{\beta\alpha} \quad (S_4 = 0) \\ \mathcal{I} = \frac{1}{4} \left[\left(\tilde{K}_\alpha^\alpha \right)^2 - \tilde{K}^{\alpha\beta} \tilde{K}_{\alpha\beta} + 3 \tilde{\Omega}^{\alpha\beta} \tilde{\Omega}_{\alpha\beta} \right] \end{array} \right. \quad (4.1.5)$$

we obtain the following natural projections of the gravitational equations

$$\left\{ \begin{array}{l} \mathcal{P}_{\Sigma\Sigma}(G_{jm}) = \gamma_{j\beta} (\delta_m^k + \gamma_m \gamma^k) G_k^\beta = \gamma_{j\beta} G_m^\beta + \gamma_m \gamma^4 \gamma_{j\beta} G_4^\beta = s_{jm} - \frac{1}{2} R \gamma_{jm} \\ \mathcal{P}_{\Sigma\Theta}(G_{jm}) = \mathcal{P}_{\Sigma\Theta}(R_{jm}) = -\gamma_m \gamma^4 \gamma_{j\beta} G_4^\beta = \gamma_m S_j \\ \mathcal{P}_{\Theta\Sigma}(G_{jm}) = \mathcal{P}_{\Theta\Sigma}(R_{jm}) = \gamma_j S_m \\ \mathcal{P}_{\Theta\Theta}(G_{jm}) = \gamma_4 \gamma_r G_4^r \gamma_j \gamma_m = \frac{1}{2} (\tilde{R}^* + \mathcal{I}) \gamma_j \gamma_m \end{array} \right. \quad (4.1.6)$$

So the gravitational equations (??) are equivalent to the following system of three tensorial equations:

$$\left\{ \begin{array}{l} s_{\alpha\rho} - \frac{1}{2} R \gamma_{\alpha\rho} = 0 \\ S_\alpha = 0 \\ \tilde{R}^* + \mathcal{I} = 0. \end{array} \right. \quad (4.1.7)$$

4.1.2 Relative formulation of the conservation equations of gravitational tensor

Bearing in mind the so-called “conservations conditions” of the gravitational tensor:

$$\nabla_j G_m^j \equiv 0 \quad (4.1.8)$$

let us perform the natural projection of the vector $\nabla_j G_m^j$. From (??) it follows

$$G_m^j \equiv s_m^j - \frac{1}{2} R \gamma_m^j + \gamma_m S^j + \gamma^j S_m + \frac{1}{2} (\tilde{R}^* + \mathcal{I}) \gamma^j \gamma_m \quad (4.1.9)$$

and by decomposition formulas [?] it follows

$$\begin{aligned} \nabla_j s_m^j &\equiv \tilde{\nabla}_j^* s_m^j + \frac{1}{2} (\tilde{K}_{jr} + \tilde{\Omega}_{jr}) s^{jr} \gamma_m + C_h s_m^h, \\ \nabla_j \gamma_m^j &\equiv \frac{1}{2} \tilde{K}_\alpha^\alpha \gamma_m + C_m, \\ \nabla_j \gamma_m &\equiv \frac{1}{2} (\tilde{K}_{jm} + \tilde{\Omega}_{jm}) - \gamma_j C_m, \\ \nabla_j S^j &\equiv \tilde{\nabla}_j^* S^j + C_h S^h, \\ \gamma^j \nabla_j S_m &\equiv \gamma^4 \partial_4 S_m - \frac{1}{2} (\tilde{K}_{mh} + \tilde{\Omega}_{mh}) S^h + C_h S^h \gamma_m, \\ \Delta_j (\gamma^j \gamma_m) &\equiv \frac{1}{2} \tilde{K}_\alpha^\alpha \gamma_m + C_m, \\ \gamma^j \nabla_j (\tilde{R}^* + \mathcal{I}) &\equiv \gamma^4 \partial_4 (\tilde{R}^* + \mathcal{I}). \end{aligned} \quad (4.1.10)$$

So the eqs. (??) assume the following relative formulation:

$$\left\{ \begin{array}{l} \tilde{\nabla}_j^* s_m^j + C_h s_m^h - \frac{1}{2} R C_m - \frac{1}{2} \tilde{\nabla}_m^* R + \frac{1}{2} \tilde{K}_\alpha^\alpha S_m + \\ \quad + \gamma^4 \partial_4 S_m + \tilde{\Omega}_{mh} S^h + \frac{1}{2} (\tilde{R}^* + \mathcal{I}) C_m \\ \frac{1}{2} (\tilde{K}_{jl} + \tilde{\Omega}_{jl}) s^{jl} - \frac{1}{4} R \tilde{K}_\alpha^\alpha + \tilde{\nabla}_j^* S^j + 2C_j S^j + \\ \quad + \frac{1}{4} \tilde{K}_\alpha^\alpha (\tilde{R}^* + \mathcal{I}) + \frac{1}{2} \gamma^4 \partial_4 (\tilde{R}^* + \mathcal{I}) \equiv 0. \end{array} \right. \quad (4.1.11)$$

4.1.3 Other form of the gravitational equations in the empty space

As is well known

$$G \equiv G_i^i \equiv R_i^i - \frac{1}{2} R g_i^i, \quad R_{jm} \equiv G_{jm} - \frac{1}{2} G g_{jm}$$

so the (??) are equivalent to the equations

$$R_{jm} = 0. \quad (4.1.12)$$

But it is easy to verify that the equations (??) are also equivalent to the system of tensorial equations ¹

¹It suffices to show that the (??) are a consequence of the (??).

$$\left\{ \begin{array}{l} s_{\alpha\beta} \equiv \gamma_{\alpha\rho} R_{\beta}^{\rho} + \gamma_{\beta} \gamma^4 \gamma_{\alpha\rho} R_4^{\rho} = 0 \\ S_{\alpha} \equiv -\gamma^4 \gamma_{\alpha\rho} R_4^{\rho} = -\gamma^4 (R_{\alpha 4} + \gamma_{\alpha} \gamma^4 R_{44}) = 0 \\ \frac{1}{2} (\tilde{R}^* + \mathcal{I}) \equiv \gamma^4 \gamma_r G_4^r \equiv \gamma^4 \gamma_r (R_4^r - \frac{1}{2} R g_4^r) = 0. \end{array} \right. \quad (4.1.13)$$

From (??) and (??), this system assumes the following expression:

$$\left\{ \begin{array}{l} s_{\alpha\beta} \equiv \tilde{R}_{\alpha\beta}^* + \frac{1}{4} \tilde{K}_i^i (\tilde{K}_{\beta\alpha} + \tilde{\Omega}_{\beta\alpha}) - \frac{1}{2} (\tilde{K}_{\beta}^i + \tilde{\Omega}_{\beta}^i) \tilde{K}_{i\alpha} + \\ \quad + \frac{1}{2} \gamma^4 \partial_4 \tilde{K}_{\alpha\beta} - \tilde{\nabla}_{\beta}^* C_{\alpha} + \frac{1}{2} \gamma^4 \partial_4 \tilde{\Omega}_{\beta\alpha} + \frac{1}{2} \tilde{\Omega}_{\alpha}^i \tilde{\Omega}_{i\beta} - C_{\alpha} C_{\beta} = 0 \\ S_{\beta} \equiv \frac{1}{2} \left[\tilde{\nabla}_{\beta}^* \tilde{K}_i^i - \tilde{\nabla}_{\mu}^* (\tilde{K}_{\beta}^{\mu} + \tilde{\Omega}_{\beta}^{\mu}) \right] + C^{\alpha} \tilde{\Omega}_{\alpha\beta} = 0 \\ \tilde{R}^* + \mathcal{I} \equiv \tilde{R}^* + \frac{1}{4} \left[(\tilde{K}_i^i)^2 - \tilde{K}^{\alpha\beta} \tilde{K}_{\alpha\beta} + 3 \tilde{\Omega}^{\alpha\beta} \tilde{\Omega}_{\alpha\beta} \right] = 0. \end{array} \right. \quad (4.1.14)$$

4.2 The gravitational field equations in a perfect fluid, their relative formulation, conservation conditions of the energy-momentum tensor

4.2.1 Relative formulation of the gravitational field equations in a perfect fluid

As is well known the system of gravitational field equations for a perfect fluid can be written as

$$G_{ik} \equiv R_{ik} - \frac{1}{2}Rg_{ik} = -\chi T_{ik} \quad (4.2.1)$$

If we consider the natural projections of the symmetrical energy-momentum tensor $T_{ik} \equiv [p_0g_{ik} + (\mu_0c^2 + p_0)\gamma_i\gamma_k]$ we obtain

$$\begin{aligned} \mathcal{P}_{\Sigma\Sigma}(T_{jm}) &= p_0\gamma_{jm} \\ \mathcal{P}_{\Sigma\Theta}(T_{jm}) &= \mathcal{P}_{\Theta\Sigma}(T_{jm}) = 0 \\ \mathcal{P}_{\Theta\Theta}(T_{jm}) &= \mu_0c^2\gamma_i\gamma_m \end{aligned} \quad (4.2.2)$$

By the (??), we deduce that the gravitational equations (??) are equivalent to the following system of three tensorial equations:

$$\left\{ \begin{array}{l} s_{\alpha\rho} - \frac{1}{2}R\gamma_{\alpha\rho} = -\chi p_0\gamma_{\alpha\rho} \\ S_\alpha = 0 \\ \tilde{R}^* + \mathcal{I} = -2\chi\mu_0c^2. \end{array} \right. \quad (4.2.3)$$

4.2.2 Relative formulation of the conservation equations of the energy-momentum tensor

By (??), we deduce the following “conservations equations” for the energy-momentum tensor:

$$\nabla_j T_i^j = 0 \quad (4.2.4)$$

Let us consider the natural projections of the vector $\nabla_j T_i^j$:

$$\nabla_j (p_0\gamma_i^j + \mu_0c^2\gamma_i\gamma^j) \equiv \nabla_j (p_0\gamma_i^j) + \partial_j\mu_0 \cdot \gamma_i\gamma^j c^2 + \mu_0c^2\gamma^j\nabla_j\gamma_i + \mu_0c^2\gamma_i\nabla_j\gamma^j = 0$$

that are equivalent to the natural projections of the gravitational tensor and so to its relative formulation expressed by (??).

They also give the momentum conservation equation for projection [?] [?]

$$\left(\tilde{\partial}_i p_0 + p_0 C_i \right) dS_0 = -\mu_0 c^2 C_i \cdot dS_0 \quad (4.2.5)$$

and the material energy conservation equation

$$\gamma^4 \partial_4 (\mu_0 c^2 dS_0) = -\frac{1}{2} p_0 \tilde{K}_\alpha^\alpha dS_0 \quad (4.2.6)$$

where dS_0 is the proper volume element of the fluid.

If we consider the following formulas:

$$C_i \equiv \gamma^4 (\partial_4 \gamma_i - \partial_i \gamma_4), \quad \tilde{K}_{ij} \equiv \gamma^4 \partial_4 \gamma_{ij}, \quad \gamma^4 \partial_4 (dS_0) = \frac{1}{2} \tilde{K}_\alpha^\alpha dS_0$$

the previous conservation equations assume the following form:

$$\tilde{\partial}_\alpha p_0 + (p_0 + \mu_0 c^2) \gamma^4 (\partial_4 \gamma_\alpha - \partial_\alpha \gamma_4) = 0 \quad (4.2.7)$$

$$\gamma^4 \partial_4 (\mu_0 c^2) + \frac{1}{2} (p_0 + \mu_0 c^2) \gamma^{\alpha\beta} \gamma^4 \partial_4 \gamma_{\alpha\beta} = 0. \quad (4.2.8)$$

4.2.3 A different expression of the gravitational field equations in the “perfect fluid” scheme

By (??), we have

$$G \equiv G_i^i \equiv -R = -\chi (3p_0 - \mu_0 c^2) \quad (4.2.9)$$

and deduce

$$R_{jm} = -\frac{1}{2} \chi [(\mu_0 c^2 - p_0) \gamma_{jm} + (\mu_0 c^2 + 3p_0) \gamma_j \gamma_m] \quad (4.2.10)$$

and vice versa.

If we consider the natural projections of both sides of the equation (??), we obtain

$$\left\{ \begin{array}{l} s_{\alpha\beta} = -\frac{1}{2}\chi(\mu_0 c^2 - p_0)\gamma_{\alpha\beta} \\ S_\alpha = 0 \\ \mathcal{P}_{\Theta\Theta}(R_{jm}) = -\frac{1}{2}\chi(\mu_0 c^2 + 3p_0)\gamma_i\gamma_m. \end{array} \right. \quad (4.2.11)$$

So the equations (??) are equivalent to the equations (??), and to the relative equations expressed by (??) and (??).

By (??) we can obtain the new system of equations

$$\left\{ \begin{array}{l} s_{\alpha\beta} = -\frac{1}{2}\chi(\mu_0 c^2 - p_0)\gamma_{\alpha\beta} \\ S_\alpha = 0 \\ \mathcal{P}_{\Theta\Theta}(G_{jm}) \equiv \frac{1}{2}(\tilde{R}^* + \mathcal{I})\gamma_j\gamma_m = -\chi\mu_0 c^2\gamma_j\gamma_m. \end{array} \right. \quad (4.2.12)$$

Taking into account the following formulas:

$$\left\{ \begin{array}{l} s_{\alpha\beta} \equiv \gamma_{\alpha\sigma} (\delta_{\beta}^k + \gamma_{\beta}\gamma^k) R_k^{\sigma} \equiv \gamma_{\alpha\sigma} R_{\beta}^{\sigma} + \gamma_{\alpha\sigma} \cdot \gamma_{\beta}\gamma^4 R_4^{\sigma} \\ S_{\alpha} \equiv -\gamma^4 \gamma_{\alpha k} R_4^k \equiv -\gamma^4 (R_{\alpha 4} + \gamma_{\alpha}\gamma^4 R_{44}) \\ \mathcal{P}_{\Theta\Theta}(G_{jm}) \equiv \gamma_j \gamma_m \gamma_r \gamma^4 (R_4^r - \frac{1}{2} R g_4^r) \end{array} \right. \quad (4.2.13)$$

The following identity can be deduced from (??)₂

$$\gamma_{\alpha\rho} R_4^{\rho} = 0 \Rightarrow R_4^{\rho} = 0. \quad (4.2.14)$$

So the equations (??)₁ and (??)₃ become respectively

$$s_{\alpha\beta} = \gamma_{\alpha\rho} R_{\beta}^{\rho} = -\frac{1}{2} \chi (\mu_0 c^2 - p_0) \gamma_{\alpha\beta}, \quad (4.2.15)$$

$$\frac{1}{2} (\tilde{R}^* + \mathcal{I}) \equiv \gamma_r \gamma^4 \left(R_4^r - \frac{1}{2} R g_4^r \right) = -\chi \mu_0 c^2. \quad (4.2.16)$$

Multiplying both sides of the equation (??) by $\gamma^{\sigma\alpha}$, we obtain

$$R_{\beta}^{\sigma} = -\frac{1}{2} \chi (\mu_0 c^2 - p_0) \delta_{\beta}^{\sigma}. \quad (4.2.17)$$

Substituting into the equation (??), we deduce

$$R_4^4 = \frac{1}{2} \chi (\mu_0 c^2 + 3p_0), \quad (4.2.18)$$

and considerig (??), (??), (??), we can write

$$R = \chi (3p_0 - \mu_0 c^2). \quad (4.2.19)$$

Taking into account this last identity and the equation $(??)_3$, the equation $(??)_3$ is a consequence of the equation

$$\mathcal{P}_{\Theta\Theta}(R_{jm}) = \mathcal{P}_{\Theta\Theta}(G_{jm}) + \frac{1}{2}R\mathcal{P}_{\Theta\Theta}(g_{jm}). \quad (4.2.20)$$

We can then choose the equations

$$\left\{ \begin{array}{l} s_{\alpha\beta} = -\frac{1}{2}\chi(\mu_0 c^2 - p_0)\gamma_{\alpha\beta} \\ S_\alpha = 0 \\ \tilde{R}^* + \mathcal{I} = -2\chi\mu_0 c^2 \end{array} \right. \quad (4.2.21)$$

as other relative expression of the gravitational field equations in the “perfect fluid” scheme.

4.2.4 Relative formulation of the gravitational field equations and conservation conditions of the momentum in the pure matter scheme

In the pure matter scheme the pressure tensor is zero, so the system $(??)$ becomes

$$\left\{ \begin{array}{l} s_{jm} = -\frac{1}{2}\chi\mu_0c^2\gamma_{jm} \\ S_\alpha = 0 \\ \tilde{R}^* + \mathcal{I} = -2\chi\mu_0c^2 \end{array} \right. \quad (4.2.22)$$

and the momentum conservation equation (??) becomes

$$C_\alpha \equiv \gamma^r (\nabla_r \gamma_\alpha - \nabla_\alpha \gamma_r) \equiv \gamma^4 (\partial_4 \gamma_\alpha - \partial_\alpha \gamma_4) = 0 \quad (4.2.23)$$

and we can say that the current lines of pure matter are geodesic of the metric universe. Finally, the material energy conservation equation (??) becomes

$$\gamma^4 \partial_4 (\mu_0 c^2) = -\frac{1}{2} \mu_0 c^2 \tilde{K}_\rho^\rho. \quad (4.2.24)$$

4.3 The Maxwell-Einstein equations in the empty space

4.3.1 Relative formulation of the Maxwell equations

Let V_4 be the riemannian manifold of normal hyperbolic type with signature $(+++ -)$ and $\gamma(x)$ the field of unitary vectors tangent to the congruence of the world lines of the particles oriented towards the future.

The Maxwell equations are in the empty space:

$$F_j \equiv \nabla_i F_j^i = 0, \quad (4.3.1)$$

$$\mathcal{F}_j \equiv \nabla_i \mathcal{F}_j^i = 0, \quad (4.3.2)$$

where F_{ij} is the antisymmetrical electromagnetic field tensor and \mathcal{F}_{ij} is the odd dual antisymmetrical tensor of F_{ij}

$$\mathcal{F}_{ij} = (*F)_{ij} = \frac{1}{2} \eta_{ijkl} F^{lm} \quad (4.3.3)$$

where $\eta_{ijkl} = \sqrt{g} \epsilon_{ijkl}$ is the antisymmetrical Ricci tensor.

Now we refer to the natural decomposition of F_{ij} [?]:

$$F_{ij} = H_{ij} - E_i \gamma_j + \gamma_i E_j \quad (4.3.4)$$

where $H_{ij} \equiv \gamma_{ir} \gamma_{js} F^{rs}$ is the magnetic field tensor and $E_i \equiv \gamma_{ir} \gamma_s F^{rs}$ is the electric field vector.

Let us introduce the magnetic field spatial vector

$$\mathcal{H}_\alpha = (*H)_\alpha = \frac{1}{2} \tilde{\eta}_{\alpha\rho\tau} H^{\rho\tau} \quad (\mathcal{H}_4 \equiv 0) \quad (4.3.5)$$

where $\tilde{\eta}_{\alpha\rho\tau}$ is the antisymmetrical Ricci tensor of the three-dimensional spatial platform Σ_x .

By naturally decomposing equations (??) and (??) we obtain the following relative Maxwell equations:

$$\begin{aligned} \mathcal{P}_\Sigma(F_j) \equiv \tilde{F}_j &\equiv 2\gamma^{is}\eta_{j sr}\tilde{\nabla}_i^*\mathcal{H}^r + 2\gamma_{jr}\tilde{\eta}^{r h\alpha}C_h\mathcal{H}_\alpha - \gamma^4\partial_4 E_j + \tilde{K}_{ij}E^i - \frac{1}{2}\tilde{K}_i^i E_j = 0 \quad (\tilde{F}_4 \equiv 0) \\ \mathcal{P}_\Theta(F_j) \equiv F_\theta\gamma_j &\equiv \left(\tilde{\nabla}_i^*E^i - \frac{4}{c}\omega_\alpha\mathcal{H}^\alpha\right)\gamma_j = 0 \end{aligned} \quad (4.3.6)$$

$$\begin{aligned} \mathcal{P}_\Sigma(\mathcal{F}_j) \equiv \tilde{\mathcal{F}}_j &\equiv -2\gamma^{is}\eta_{j sr}\tilde{\nabla}_i^*E^r - 2\gamma_{jr}\tilde{\eta}^{r h\alpha}C_hE_\alpha - \gamma^4\partial_4\mathcal{H}_j + \tilde{K}_{ij}\mathcal{H}^i - \frac{1}{2}\tilde{K}_i^i\mathcal{H}_j = 0 \quad (\tilde{F}_4 \equiv 0) \\ \mathcal{P}_\Theta(\mathcal{F}_j) \equiv \mathcal{F}_\theta\gamma_j &\equiv \left(\tilde{\nabla}_i^*\mathcal{H}^i - \frac{4}{c}\omega_\alpha E^\alpha\right)\gamma_j = 0 \end{aligned} \quad (4.3.7)$$

where $\omega_\alpha = \frac{c}{4}\tilde{\eta}^{\alpha\beta\gamma}\tilde{\Omega}_{\beta\gamma}$ is the local angular velocity of the frame.

On the other hand, as is known, the vectors F_j and \mathcal{F}_j , first members of the Maxwell equations, have zero divergence for the antisymmetry of the electromagnetic field tensor F_{ij} and its dual tensor \mathcal{F}_{ij} ; so it follows the following conditions:

$$\begin{aligned} \nabla_i F^i &= \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}F^i) = \frac{1}{2\sqrt{-g}}\partial_i[\sqrt{-g}\nabla_r(\eta^{irmn}\mathcal{F}_{mn})] = \\ &= \frac{1}{2\sqrt{-g}}\epsilon^{irmn}\partial_i\partial_r\mathcal{F}_{mn} = 0 \end{aligned} \quad (4.3.8)$$

$$\nabla_i \mathcal{F}^i = 0. \quad (4.3.9)$$

and their relative formulations ²:

$$\begin{aligned}\nabla_i F^i &\equiv g^{ik} \nabla_i (\tilde{F}_k + F_\theta \gamma_k) \\ &\equiv \gamma^{\alpha\beta} \tilde{\partial}_\alpha \tilde{F}_\beta - \gamma^4 \partial_4 F_\theta - \gamma^{\alpha\beta} \equiv \left\{ \widetilde{\alpha\beta}^\sigma \right\} * \tilde{F}_\sigma + C_\alpha \tilde{F}^\alpha + \frac{1}{2} F_\theta \tilde{K}_\alpha^\alpha = 0\end{aligned}\quad (4.3.10)$$

$$\begin{aligned}\nabla_i \mathcal{F}^i &\equiv g^{ik} \nabla_i (\tilde{\mathcal{F}}_k + \mathcal{F}_\theta \gamma_k) \equiv \\ &\equiv \gamma^{\alpha\beta} \tilde{\partial}_\alpha \tilde{\mathcal{F}}_\beta - \gamma^4 \partial_4 \mathcal{F}_\theta - \gamma^{\alpha\beta} \left\{ \widetilde{\alpha\beta}^\sigma \right\} * \tilde{\mathcal{F}}_\sigma + C_\alpha \mathcal{F}^\alpha + \frac{1}{2} \tilde{\mathcal{F}}_\theta \tilde{K}_\alpha^\alpha = 0\end{aligned}\quad (4.3.11)$$

4.3.2 Relative formulation of the gravitational field equations in a pure electromagnetic field

Let us consider the Einstein equations in a pure electromagnetic field:

$$G_{ij} \equiv R_{ij} - \frac{1}{2} R g_{ij} = -\chi T_{ij} \quad (4.3.12)$$

where $T_{ij} \equiv g^{ls} F_{il} F_{js} - \frac{1}{4} g_{ij} F_{lm} F^{lm}$ is the Maxwell symmetrical energy-momentum tensor for which

$$T \equiv g^{ij} T_{ij} = 0. \quad (4.3.13)$$

2

$$\nabla_i \tilde{F}_j \equiv \tilde{\partial}_i \tilde{F}_j - \left\{ \widetilde{ij}^h \right\} * \tilde{F}_h + \frac{1}{2} (\tilde{K}_{ir} + \tilde{\Omega}_{ir}) \tilde{F}^r \gamma_j + \left[\frac{1}{2} (\tilde{K}_{jh} + \tilde{\Omega}_{jh}) \tilde{F}^h - \gamma^4 \partial_4 \tilde{F}_j \right] - C_h \tilde{F}^h \gamma_i \gamma_j,$$

$$\nabla_i (F_\theta \gamma_j) \equiv \frac{1}{2} F_\theta (\tilde{K}_{ij} + \tilde{\Omega}_{ij}) \tilde{F}^h + \tilde{\partial}_i F_\theta \cdot \gamma_j - F_\theta \gamma_j \cdot C_j + \gamma^4 \partial_4 F_\theta \cdot \gamma_i \gamma_j.$$

We give the following definitions [?]:

$$\begin{aligned}
 \tilde{P}_{ij} = \tilde{T}_{ij} = \gamma_{ir}\gamma_{js}T^{rs} & \quad \text{relative standard density of momentum current (Poynting tensor)} \\
 \tilde{P}_i = -c\gamma_{ij}\gamma_rT^{jr} & \quad \text{relative standard density of energy current (Poynting vector)} \\
 h = \tilde{T} = \gamma_r\gamma_sT^{rs} & \quad \text{relative standard density of energy}
 \end{aligned}
 \tag{4.3.14}$$

Taking into account the natural projections of the symmetrical tensors G_{ij} and R_{ij} (see (??) and (??)) and these definitions, the gravitational equations (??) are equivalent to the following system of three tensorial equations relative to the rest reference \mathcal{S} :

$$\left\{ \begin{array}{l}
 s_{\alpha\rho} - \frac{1}{2}R\gamma_{\alpha\rho} = -\chi\tilde{P}_{\alpha\rho} \quad (\tilde{P}_{4j} \equiv 0) \\
 S_\alpha = -\frac{1}{c}\chi\tilde{P}_\alpha \quad (\tilde{P}_4 \equiv 0) \\
 \tilde{R}^* + \mathcal{I} = -2\chi h.
 \end{array} \right.
 \tag{4.3.15}$$

On the other hand, by (??) and (??)

$$G \equiv G_i^i \equiv -R = -\chi g^{ij}T_{ij} = 0
 \tag{4.3.16}$$

so the relative form of gravitational equations (??) can be expressed by the following system:

$$\left\{ \begin{array}{l} s_{\alpha\rho} = -\chi\tilde{P}_{\alpha\rho} \quad (\tilde{P}_{4j} \equiv 0) \\ S_{\alpha} = -\frac{1}{c}\chi\tilde{P}_{\alpha} \quad (\tilde{P}_4 \equiv 0) \\ \tilde{R}^* + \mathcal{I} = -2\chi h. \end{array} \right. \quad (4.3.17)$$

4.4 Gravitational field equations, their relative form and conservation conditions for the energy-momentum tensor in a charged perfect fluid

4.4.1 Relative formulation of the gravitational field equations in a charged perfect fluid

As is well known, the gravitational equations for a charged perfect fluid, without induction, in a four-dimensional riemannian manifold V_4 are:

$$G_{ik} \equiv R_{ik} - \frac{1}{2}Rg_{ik} = -\chi T_{ik} \equiv -\chi [p_0g_{ik} + (\mu_0c^2 + p_0)\gamma_i\gamma_k + \tau_{ik}] \quad (4.4.1)$$

where

$$\tau_{ik} = F_{il}F_k^l - \frac{1}{4}g_{ik}F_{lm}F^{lm} \quad (4.4.2)$$

is the electromagnetic energy-tensor.

In this case, the Maxwell equations can be written as:

$$F_k \equiv \nabla_i F_k^i = J_k, \quad \mathcal{F}_k \equiv \nabla_i \mathcal{F}_k^i = 0, \quad (4.4.3)$$

where J_k is the world current density vector and, with vanishing conductivity, is:

$$J^k = \frac{\rho^*}{c} U^k \quad (4.4.4)$$

with ρ^* proper charged density and U^k 4-velocity of the fluid.

By (??) and (??), if $\det \|\gamma_{ir}\| \neq 0$, there is a biunivocal correspondence between the electromagnetic field tensor F^{rs} and the two spatial vectorial fields E_i and \mathcal{H}_i .

Besides:

$$\begin{aligned} \tilde{\tau}_{\alpha\beta} &= \gamma_{\alpha\delta} \gamma_{\beta\nu} \tau^{\delta\nu} = F_{\alpha l} F^{\nu l} \gamma_{\beta\nu} - \frac{1}{4} \gamma_{\alpha\beta} F_{lm} F^{lm}; \\ \tilde{\tau}_\alpha &= -\gamma_{\alpha\nu} \gamma_s F^{\nu s} = -E_\alpha; \\ \tilde{\tau} &= \gamma_r \gamma_s \tau^{rs} = \gamma^r \gamma^s F_{rl} F_s^l + \frac{1}{4} F_{lm} F^{lm}. \end{aligned}$$

If we consider the natural projections of the symmetrical energy-momentum tensor $T_{ik} \equiv [p_0 g_{ik} + (\mu_0 c^2 + p_0) \gamma_i \gamma_k + \tau_{ik}]$ we obtain

$$\left\{ \begin{array}{l} \mathcal{P}_{\Sigma\Sigma}(T_{ik}) \equiv \tilde{P}_{ik} = p_0 \gamma_{ik} + \tilde{\tau}_{ik}; \\ \mathcal{P}_{\Sigma\Theta}(T_{ik}) \equiv \frac{1}{c} \tilde{P}_i \gamma_k = \tilde{\tau}_i \gamma_k; \\ \mathcal{P}_{\Theta\Sigma}(T_{ik}) = \tilde{\tau}_k \gamma_i; \\ \mathcal{P}_{\Theta\Theta}(T_{ik}) \equiv h \gamma_i \gamma_k = (\mu_0 c^2 + \tilde{\tau}) \gamma_i \gamma_k \end{array} \right. \quad (4.4.5)$$

where, in this case, the relative standard density of momentum current (Poynting tensor) \tilde{P}_{ik} , the relative standard density of energy current (Poynting vector) \tilde{P}_i and the relative standard density of energy h can be expressed by the following formulas:

$$\left\{ \begin{array}{l} \tilde{P}_{ik} \equiv \tilde{T}_{ik} \equiv p_0 \gamma_{ik} + \tilde{\tau}_{ik} \\ \tilde{P}_i \equiv c \tilde{\tau}_i \\ h \equiv \tilde{T} \equiv p_0 c^2 + \tilde{\tau} \end{array} \right. \quad (4.4.6)$$

Taking into account the natural projections of the symmetrical tensors G_{ij} and R_{ij} (see (??) and (??)) and (??), the gravitational equations (??) are equivalent to the following system of three tensorial equations relative to the rest reference \mathcal{S} :

$$\left\{ \begin{array}{l} s_{\alpha\rho} - \frac{1}{2} R \gamma_{\alpha\rho} = -\chi \tilde{P}_{\alpha\rho} \\ S_\alpha = -\frac{1}{c} \chi \tilde{P}_\alpha \\ \tilde{R}^* + \mathcal{I} = -2\chi h. \end{array} \right. \quad (4.4.7)$$

If we now consider the natural decomposition of the world current density vector J_k :

$$J_k = \mathcal{P}_\Sigma(J_k) + \mathcal{P}_\Theta(J_k) = -\gamma_k \gamma_r J^r = \rho_* \gamma_k \quad (4.4.8)$$

the Maxwell equations (??) are equivalent to the following relative Maxwell equations:

$$\left\{ \begin{array}{l}
 \tilde{F}_\beta \equiv 2\gamma^{is}\eta_{\beta sr}\tilde{\nabla}_i^*\mathcal{H}^r + 2\gamma_{\beta r}\tilde{\eta}^{rh\alpha}C_h\mathcal{H}_\alpha - \gamma^4\partial_4E_\beta + \tilde{K}_{i\beta}E^i - \frac{1}{2}\tilde{K}_i^iE_\beta = 0 \\
 F_\theta \equiv \tilde{\nabla}_i^*E^i - \frac{4}{c}\omega_\alpha\mathcal{H}^\alpha = \rho_* \\
 \tilde{\mathcal{F}}_\beta \equiv -2\gamma^{is}\eta_{\beta sr}\tilde{\nabla}_i^*E^r - 2\gamma_{\beta r}\tilde{\eta}^{rh\alpha}C_hE_\alpha - \gamma^4\partial_4\mathcal{H}_\beta + \tilde{K}_{\alpha\beta}\mathcal{H}^\alpha - \frac{1}{2}\tilde{K}_\alpha^\alpha\mathcal{H}_\beta = 0 \\
 \mathcal{F}_\theta \equiv \tilde{\nabla}_\alpha^*\mathcal{H}^\alpha + \frac{4}{c}\omega_\alpha E^\alpha = 0.
 \end{array} \right. \tag{4.4.9}$$

4.4.2 Relative formulation of the conservation conditions of the energy-momentum tensor

Now we consider the conservation equations and we provide their relative form.

In this regard, we consider the differential identity of the Einstein tensor

$$\nabla_j G_i^j \equiv 0. \tag{4.4.10}$$

As a consequence of which, the energy tensor, appearing in the second member of the equation (??), must satisfy the condition

$$\nabla_j T_i^j \equiv \nabla_j (p_0\gamma_i^j + \mu_0c^2\gamma_i\gamma^j) + \nabla_j\tau_i^j = 0. \tag{4.4.11}$$

Taking into account the Maxwell equations, we should be noted that the electromagnetic energy-tensor verifies the equation

$$\nabla_j \tau_i^j = F_{li} J^l \quad (4.4.12)$$

so it follows the conservation equation

$$\nabla_j (p_0 \gamma_i^j + \mu_0 c^2 \gamma_i \gamma^j) + F_{li} J^l = 0. \quad (4.4.13)$$

By naturally projecting, the identity (??) assumes the relative formulation expressed by (??). By naturally projecting (??) and considering $F_{li} J^l = -\rho_* E_i$, we can write the relative conservation equations:

$$\begin{cases} \tilde{\partial}_j p_0 + (p_0 + \mu_0 c^2) C_i - \rho_* E_i = 0 \\ \gamma^4 \partial_4 (\mu_0 c^2) + \frac{1}{2} (p_0 + \mu_0 c^2) \tilde{K}_\alpha^\alpha = 0 \end{cases} \quad (4.4.14)$$

Finally, the relative formulation of the conditions $\nabla_i F^i = 0$ and $\nabla_i \mathcal{F}^i = 0$ is expressed by the system consists of (??) and (??).

4.4.3 A different expression of the Einstein field equations

Taking into account (??), the equations (??) become

$$R_{ik} = -\frac{1}{2} \chi [(\mu_0 c^2 - p_0) \gamma_{ik} + (\mu_0 c^2 + 3p_0) \gamma_i \gamma_k + 2\tau_{ik}] \quad (4.4.15)$$

and their relative form is

$$\left\{ \begin{array}{l} s_{\alpha\beta} = -\frac{1}{2}\chi [(\mu_0 c^2 - p_0) \gamma_{\alpha\beta} + 2\tilde{\tau}_{\alpha\beta}] \\ S_\alpha = -\chi\tilde{\tau}_\alpha \\ \mathcal{P}_{\Theta\Theta}(R_{ik}) = -\frac{1}{2}\chi [(\mu_0 c^2 + 3p_0) \gamma_i\gamma_k + 2\tilde{\tau}\gamma_i\gamma_k]. \end{array} \right. \quad (4.4.16)$$

This system is equivalent to the system (??).

By (??) and (??), the equation

$$\mathcal{P}_{\Theta\Theta}(G_{jm}) \equiv \frac{1}{2} (\tilde{R}^* + \mathcal{I}) \gamma_j\gamma_m = -\chi (\mu_0 c^2 + \tilde{\tau}) \gamma_j\gamma_m$$

can replace the equation (??)₃ and we can choose the equations

$$\left\{ \begin{array}{l} s_{\alpha\beta} = -\frac{1}{2}\chi [(\mu_0 c^2 - p_0) \gamma_{\alpha\beta} + 2\tilde{\tau}_{\alpha\beta}] \\ S_\alpha = -\chi\tilde{\tau}_\alpha \\ \tilde{R}^* + \mathcal{I} = -2\chi (\mu_0 c^2 + \tilde{\tau}) \end{array} \right. \quad (4.4.17)$$

as a different relative expression of the Einstein equations.

4.4.4 Summary of the fundamental equations

At the conclusion of the our analysis, we should rewrite the fundamental equations for the relativistic study of a charged perfect fluid in the absence of induction and conductivity. In a physical reference in which the time-lines are the stream-lines of

the fluid, the fundamental equations consist of (??) and (??), and the conservation equations (??):

$$\left\{ \begin{array}{l} s_{\alpha\beta} = -\frac{1}{2}\chi [(\mu_0 c^2 - p_0) \gamma_{\alpha\beta} + 2\tilde{\tau}_{\alpha\beta}] \\ S_\alpha = -\chi \tilde{\tau}_\alpha \\ \tilde{R}^* + \mathcal{I} = -2\chi (\mu_0 c^2 + \tilde{\tau}) \\ \tilde{F}_\beta \equiv 2\gamma^{is} \eta_{\beta sr} \tilde{\nabla}_i^* \mathcal{H}^r + 2\gamma_{\beta r} \tilde{\eta}^{rh\alpha} C_h \mathcal{H}_\alpha - \gamma^4 \partial_4 E_\beta + \tilde{K}_{i\beta} E^i - \frac{1}{2} \tilde{K}_i^i E_\beta = 0 \\ F_\theta \equiv \tilde{\nabla}_i^* E^i - \frac{4}{c} \omega_\alpha \mathcal{H}^\alpha = \rho_* \\ \tilde{\mathcal{F}}_\beta \equiv -2\gamma^{is} \eta_{\beta sr} \tilde{\nabla}_i^* E^r - 2\gamma_{\beta r} \tilde{\eta}^{rh\alpha} C_h E_\alpha - \gamma^4 \partial_4 \mathcal{H}_\beta + \tilde{K}_{\alpha\beta} \mathcal{H}^\alpha - \frac{1}{2} \tilde{K}_\alpha^\alpha \mathcal{H}_\beta = 0 \\ \mathcal{F}_\theta \equiv \tilde{\nabla}_\alpha^* \mathcal{H}^\alpha + \frac{4}{c} \omega_\alpha E^\alpha = 0 \\ \tilde{\partial}_j p_0 + (p_0 + \mu_0 c^2) C_i - \rho_* E_i = 0 \\ \gamma^4 \partial_4 (\mu_0 c^2) + \frac{1}{2} (p_0 + \mu_0 c^2) \tilde{K}_\alpha^\alpha = 0 \end{array} \right. \quad (4.4.18)$$

4.5 The gravitational field equations in a “continuum system subjected to reversible transformations”. Their relative formulation

4.5.1 The deformation tensor

As is known, the space-time ambient of the gravitational and electromagnetic phenomena is a 4-dimensional normal hyperbolic Riemannian manifold V_4 , where x^h are local coordinates or eulerian coordinates and $+++-$ its signature.

Let us suppose that the Riemannian structure of the variety V_4 is determined by the regular evolution, in a its region, of a continuous material system \mathcal{S} , which we denote by \mathcal{C} the general configuration and by \mathcal{C}^* a well-defined reference configuration. Let \mathcal{C} be the variable configuration with equation

$$x^4 = const \tag{4.5.1}$$

and let \mathcal{C}^* be the reference configuration with equation

$$x^4 = 0 . \tag{4.5.2}$$

Let us introduce in V_4 a second local coordinate system, material coordinates or lagrangian coordinates y^k , with $y^4 = x^4$ and y^1, y^2, y^3 coordinates of the generic particle $P^* \in \mathcal{C}^*$.

Every material point $P \in \mathcal{C}$ has the eulerian coordinates

$$P \equiv (x^1, x^2, x^3, x^4 = y^4) , \tag{4.5.3}$$

and the material coordinates

$$P \equiv (y^1, y^2, y^3, y^4) , \tag{4.5.4}$$

since a homeomorphism exists among the 3-manifolds \mathcal{C}^* and every \mathcal{C} ; consequently the relation

$$det. \left\| \frac{\partial x^h}{\partial y^k} \right\| \neq 0 \tag{4.5.5}$$

is locally satisfied.

Let us now consider the stream lines of the material points P of the system \mathcal{S} as the time-like congruence $\{L\}$ that we will employ, from now in the future, as the physical frame of reference, and let the vector \mathbf{u} be tangent to the stream lines, set towards the future, with the norm

$$\|\mathbf{u}\| = -1$$

and the contravariant eulerian components

$$u^\rho(x) = \frac{\partial_4 x^\rho}{\sqrt{-\|\partial_4 P\|}} \quad (\rho = 1, 2, 3) \quad , \quad u^4(x) = \frac{1}{\sqrt{-\|\partial_4 P\|}} \quad ,$$

where ∂_4 denotes the partial derivatives with respect to y^4 ($\partial_4 \equiv \frac{\partial}{\partial y^4}$).

Besides we put

$$\left\{ \begin{array}{l} {}^*g_{hk}(y) = g_{rs}(x) \frac{\partial x^r}{\partial y^h} \frac{\partial x^s}{\partial y^k} \\ {}^*u^\rho(y) = u^k(x) \frac{\partial y^\rho}{\partial x^k} = \frac{\delta_4^\rho}{\sqrt{-\|\partial_4 P\|}} = 0 \\ {}^*u^4(y) = u^k(x) \frac{\partial y^4}{\partial x^k} = \frac{1}{\sqrt{-\|\partial_4 P\|}} \end{array} \right. \quad (4.5.6)$$

to indicate the image of the tensor fields $g_{hk}(x)$, $u^k(x)$ on the fixed reference configuration \mathcal{C}^* . Moreover we introduce the vector space T_P tangent, at every point P , to V_4 as product of two orthogonal subspaces Σ_P , Θ_P :

$$T = \Theta_P \times \Sigma_P \quad (4.5.7)$$

where Θ_P is 1-dimensional like-time space, tangent to the stream line for P , while Σ_P a 3-dimensional space orthogonal to Θ_P at P [?].

The tensor field

$$\gamma_{hr} \equiv \mathcal{P}_{\Sigma\Sigma}(g_{hr}) = g_{hr} + u_h u_r \quad (4.5.8)$$

obtained by means two projections on the 3-space Σ_P , is the metric tensor of Σ_P or the space projector, or the space metric tensor; in addition

$${}^* \gamma_{hr} (y) = {}^* g_{hr} (y) + {}^* u_h {}^* u_r \quad ({}^* \gamma_{4r} \equiv 0) \quad (4.5.9)$$

are its lagrangian covariant components.

Let us remark that the lagrangian coordinates y^k satisfy the following conditions:

$$y^\rho = \text{const.}, \quad y^4 = \text{variable on every stream line}; \quad (4.5.10)$$

therefore we will say the lagrangian coordinates are adapted to the congruence $\{L\}$.

After that we will pose

$$m_{hr}(y^1, y^2, y^3) \equiv {}^* \gamma_{hr} (y^1, y^2, y^3, 0); \quad (4.5.11)$$

namely m_{hr} is the metric tensor in the reference configuration \mathcal{C}^* . That put before, we will call *local deformation tensor* ${}^* \varepsilon_{hr} (y)$ of the system \mathcal{S} the symmetric space tensor [?]

$${}^* \varepsilon_{hr} = \frac{1}{2} \left({}^* \gamma_{hr} - \delta_h^\alpha \delta_r^\beta m_{\alpha\beta} \right). \quad (4.5.12)$$

Let us point out that the congruence $\{L\}$ is the family of trajectories of a one-parameter transformation group of which the vector field \mathbf{u} is the generator field. According to this idea, by the operator $\mathcal{L}_{\mathbf{u}}$ (Lie derivative) we have:

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathbf{u}} g_{hr} = \nabla_h u_r + \nabla_r u_h \equiv K_{hr} \quad (\text{Killing tensor}) \\ \mathcal{L}_{\mathbf{u}} \gamma_{hr} = \mathcal{P}_{\Sigma\Sigma}(K_{hr}) \equiv \tilde{K}_{hr} \quad (\text{the rate of deformation tensor}) \\ \mathcal{L}_{\mathbf{u}} \varepsilon_{hr} = \frac{1}{2} \tilde{K}_{hr} \end{array} \right. \quad (4.5.13)$$

and analogously, as image on \mathcal{C}^* ,

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathbf{u}}^* g_{hr} = \nabla_h^* u_r^* + \nabla_r^* u_h^* \\ \mathcal{L}_{\mathbf{u}}^* \gamma_{hr} = u^{*4} \partial_4^* \gamma_{hr}^* \\ \mathcal{L}_{\mathbf{u}}^* \varepsilon_{hr} = u^{*4} \partial_4^* \varepsilon_{hr} = \frac{1}{2} \mathcal{L}_{\mathbf{u}}^* \gamma_{hr}^* . \end{array} \right. \quad (4.5.14)$$

4.5.2 Conservation Law of Pure Matter. Stress Tensor.

Let us denote by μ the pure matter density in \mathcal{C} and by dV the proper volume of its generic element $C \in \mathcal{C}$; by μ^* the pure matter density of \mathcal{S} in the reference configuration \mathcal{C}^* and by dV^* the proper volume of its generic element C^* corresponding to C .

The conservation law of pure matter has the following lagrangian form:

$$\mu dV = \mu^* dV^*. \quad (4.5.15)$$

By the formulas

$$dV = \sqrt{\gamma^*} dy^1 dy^2 dy^3, \quad dV^* = \sqrt{m} dy^1 dy^2 dy^3, \quad (4.5.16)$$

where

$$m = \det \| m_{\rho\sigma} \|, \quad \gamma^* = \det \| \gamma_{\rho\sigma}^* \|$$

the equation (??) is equivalent to

$$\mu \sqrt{\frac{\gamma^*}{m}} = \mu^*. \quad (4.5.17)$$

The conservation law of pure matter is then to take the following eulerian form:

$$\nabla_h (\mu u^h) = 0. \quad (4.5.18)$$

Now consider the specific effort ϕ_n and limit ourselves to the case where the explicit actions on the boundary ∂C of C of the elements of \mathcal{S} neighbour of C are subject to the following condition

$$\lim_{C \rightarrow P} \frac{1}{C} \int_{\partial C} \phi_n d\partial C = \text{finite value} \neq 0. \quad (4.5.19)$$

Let us choose as element C of \mathcal{S} , an infinitesimal tetrahedron of Σ_P with one vertex in P and three edges in P parallel to the vectors \tilde{e}_ρ ; we denote by $n \equiv \tilde{n}^\alpha \tilde{e}_\alpha$ a like-space vector, of norm 1, orthogonal to the 2-face of the tetrahedron opposite to the vertex P , by ϕ_n the specific effort related to that 2-face, and by $\phi^{(1)}, \phi^{(2)}, \phi^{(3)}$

respectively the specific efforts related to the 2-faces of the tetrahedron whose joints are those identified by the pair $(\tilde{e}_2, \tilde{e}_3)$, $(\tilde{e}_3, \tilde{e}_1)$, $(\tilde{e}_1, \tilde{e}_2)$. If we place also

$$\tilde{e}^\rho = \gamma^{*\rho\tau} \tilde{e}_\tau, \quad \tilde{e}^\rho = \sqrt{\|\tilde{e}^\rho\|} = \sqrt{\gamma^{*\rho\rho}}, \quad \rho, \tau = 1, 2, 3 \quad (4.5.20)$$

$$\phi^\rho = \phi^{(\rho)} \tilde{e}^\rho, \quad (4.5.21)$$

by (??) we obtain the following Cauchy formula

$$\phi_n(P) = \phi^\rho \tilde{n}_\rho, \quad \rho = 1, 2, 3. \quad (4.5.22)$$

We can define at every point $P \in \mathcal{C}$ the lagrangian components of the stress tensor

$$Y^{\rho\tau} = \phi^\rho \cdot \tilde{e}_\tau. \quad \rho, \tau = 1, 2, 3 \quad (4.5.23)$$

This tensor is a tensor in Σ_P for which we postulate the symmetry relations

$$Y^{\rho\tau} = Y^{\tau\rho}, \quad \rho, \tau = 1, 2, 3. \quad (4.5.24)$$

We can think of stress as a purely spatial symmetric tensor of V_4 . At this purpose we introduce a purely spatial symmetric tensor Y^{rs} ($r, s = 1, 2, 3, 4$) related to the physical reference $\{L\}$. This definition requires to the tensor Y^{rs} the conditions

$$Y^{rs} u_s^* = 0 \quad (4.5.25)$$

that is

$$Y^{\rho 4} = -Y^{\rho\tau} \frac{u_\tau^*}{u_4^*}, \quad Y^{44} = -Y^{4\tau} \frac{u_\tau^*}{u_4^*} = Y^{\rho\tau} \frac{u_\rho^* u_\tau^*}{(u_4^*)^2}. \quad (4.5.26)$$

These relations show that the components of stress Y^{rs} can be expressed with the six distinct components $Y^{\rho\tau}$. That fact is also highlighted by the formula

$$Y^{hk} = \left(\delta_\rho^h - \delta_4^h \frac{u_\rho^*}{u_4^*} \right) \left(\delta_\tau^k - \delta_4^k \frac{u_\tau^*}{u_4^*} \right) Y^{\rho\tau}. \quad (4.5.27)$$

Finally, we can define at every point $P \in \mathcal{C}$ the eulerian components of the stress tensor

$$X^{rs}(x) = Y^{hk}(y) \frac{\partial x^r}{\partial y^h} \frac{\partial x^s}{\partial y^k}. \quad (4.5.28)$$

4.5.3 Reversible systems. Energy-momentum tensor

We will consider only reversible systems, that are the systems for which it is possible to define a state function s , the specific entropy of the system \mathcal{S} , satisfying the condition

$$ds = \frac{\partial q}{T},$$

where ∂q is the heat absorbed in the algebraic sense by the unity of mass for an infinitesimal transformation of \mathcal{S} , and T is the absolute temperature of that mass.

As we know, the equation

$$\frac{\partial q}{T} = \mathcal{L}_{\mathbf{u}s} \cdot dy^4, \quad (4.5.29)$$

where s is the entropy density, characterizes the locally reversible processes for which it is possible to define the free energy density

$$\mathcal{F} = u - E s T \tag{4.5.30}$$

where u is the internal energy density and E the mechanical equivalent of heat. We can note that the free energy density \mathcal{F} satisfies the equation

$$\mu \overset{*4}{u} \partial_4 \mathcal{F} = -Y^{\rho\tau} \overset{*4}{u} \partial_4 \overset{*}{\varepsilon}_{\rho\sigma} - E \mu s \overset{*4}{u} \partial_4 T \tag{4.5.31}$$

consequence of the first and second principle of thermodynamics. Consequently, we can define a proper energy density of pure matter

$$\mu c^2 ,$$

and its thermodynamic proper energy density

$$\mu w ;$$

where w represents \mathcal{F} in the isothermal transformations and u represents \mathcal{F} in the isentropic transformations.

It follows that we must attribute to \mathcal{S} the proper mass density

$$\mu \left(1 + \frac{w}{c^2} \right). \tag{4.5.32}$$

If we add the hypothesis that there is no exchange of heat among contiguous

elements of the system \mathcal{S} , the energy–momentum tensor of \mathcal{S} has the eulerian contravariant components

$$T^{hk} = c^2 \mu \left(1 + \frac{w}{c^2}\right) u^h u^k + X^{hk} \quad (4.5.33)$$

or the equivalent lagrangian components

$${}^*T^{hk} = c^2 \mu \left(1 + \frac{w}{c^2}\right) {}^*u^h {}^*u^k + Y^{hk} . \quad (4.5.34)$$

4.5.4 Conservation conditions of the energy-momentum tensor and their relative formulations.

Let us remember that the energy-momentum tensor must satisfy the condition

$$\nabla_k^* T^{*hk} = 0 \quad (4.5.35)$$

that is, by (??)

$$\nabla_k^* \left[c^2 \mu \left(1 + \frac{w}{c^2}\right) {}^*u^h {}^*u^k + Y^{hk} \right] = 0. \quad (4.5.36)$$

It follows from obvious transformations and (??)

$$\nabla_k^* T^{*hk} = (c^2 + w) \mu {}^*u^k \nabla_k^* {}^*u^h + \mu {}^*u^h {}^*u^k \nabla_k^* w + \nabla_k^* Y^{hk} = 0. \quad (4.5.37)$$

We will evaluate the local spatial and temporal-projection of this conservation equation. At this purpose, we put

$${}^*S_r = \gamma_{hk}^* \nabla_k^* T^{*hk}, \quad \dot{N}_r = -u_r^* u_h^* \nabla_k^* T^{*hk} \quad (4.5.38)$$

and we interest in the equation

$${}^*S_r = 0 \quad \left({}^*S_4 \equiv 0 \right). \quad (4.5.39)$$

By the condition

$$\gamma_{rh}^* u^{*h} = 0 \quad (4.5.40)$$

we deduce

$${}^*S_r \equiv \mu (c^2 + w) u^{*k} \gamma_{rh}^* \nabla_k^* u^{*h} + \gamma_{rh}^* \nabla_k^* Y^{hk} = 0. \quad (4.5.41)$$

that is the spatial projection of the conservation conditions of the energy-momentum tensor.

Now we consider its temporal projection that is the equation

$$\dot{N}_r \equiv -u_r^* u_h^* \nabla_k^* T^{*hk} = 0. \quad (4.5.42)$$

By the conditions

$$\begin{cases} \nabla_k \left(\overset{*}{u}^h \overset{*}{u}_h \right) = 2 \overset{*}{u}_h \nabla_k \overset{*}{u}^h = 0 \\ \overset{*}{u}_h \nabla_k Y^{hk} = -\frac{1}{2} Y^{hk} \left(\nabla_h \overset{*}{u}_k + \nabla_k \overset{*}{u}_h \right) = -\frac{1}{2} Y^{hk} \tilde{K}_{hk} \end{cases} \quad (4.5.43)$$

the equation (??) becomes

$$\mu \overset{*}{u}^k \nabla_k w = -\frac{1}{2} Y^{hk} \tilde{K}_{hk} = -\frac{1}{2} Y^{hk} \mathcal{L}_u \overset{*}{\gamma}_{hk} = \mu \overset{*}{u}^4 \partial_4 w. \quad (4.5.44)$$

4.5.5 First and second undefined vectorial equation of a continuous system

Taking into account the identities [?], the equation (??)

$$\begin{cases} \nabla_k u^h = \frac{1}{2} (K_k^h + \Omega_k^h), & C_r \equiv u^h \nabla_h u_r \quad (C_4 \equiv 0) \\ u^k \gamma_{rh} K_k^h = C_r, & u^k \gamma_{rh} \Omega_k^h = C_r \\ \overset{*}{\gamma}_{\tau h} \overset{*}{\nabla}_k Y^{hk} = \tilde{\nabla}_k Y_r^k + \overset{*}{C}_\tau Y_r^\tau \end{cases} \quad (4.5.45)$$

the equation (??) becomes

$$\overset{*}{S}_\rho \equiv \mu \left(1 + \frac{w}{c^2} \right) c^2 \overset{*}{C}_\rho + \overset{*}{C}_\tau Y_\rho^\tau + \tilde{\nabla}_\tau Y_\rho^\tau = 0 \quad (S_4 \equiv 0). \quad (4.5.46)$$

If we remember (??) and (??), we can rewrite (??) in the following expression:

$$\mu \left(1 + \frac{w}{c^2} \right) A_\rho = -\overset{*}{C}_\tau Y_\rho^\tau - \tilde{\nabla}_\tau Y_\rho^\tau. \quad (4.5.47)$$

where the space vector \mathbf{A} is the 4-absolute acceleration of an infinitesimal volume of \mathcal{S} at the point P and C is the curvature vector of the stream line for P and $\tilde{\nabla}_\tau$ is

the covariant derivative with respect the space metric tensor γ_{hk} .

If now we consider the integral equation:

$$-\int_C \left[\mu \left(1 + \frac{w}{c^2} \right) \overset{*}{A}{}^\rho \tilde{e}_\rho + \overset{*}{C}_\tau Y^{\rho\tau} \tilde{e}_\rho \right] dC + \int_{\partial C} \phi_n d\partial C = 0, \quad (4.5.48)$$

by the Cauchy formula (??) and divergence theorem, it follows the lagrangian expression of the equation (??). In fact, being

$$\begin{aligned} \int_{\partial C} \phi_n d\partial C &= \int_{\partial C} \phi^\rho \tilde{n}_\rho d\partial C = \\ &= - \int_C \frac{1}{\sqrt{\overset{*}{\gamma}}} \frac{\partial}{\partial y^\rho} \left(\sqrt{\overset{*}{\gamma}} \phi^{(\rho)} \right) dC = \int_C \tilde{\nabla}_\rho Y^{\rho\tau} \cdot \tilde{e}_\tau dC, \end{aligned} \quad (4.5.49)$$

the equation (??) becomes

$$-\int_C \left[\mu \left(1 + \frac{w}{c^2} \right) \overset{*}{A}{}^\rho + \overset{*}{C}_\tau Y_\rho^\tau + \tilde{\nabla}_\rho Y_\rho^\tau \right] \tilde{e}_\rho dC = 0. \quad (4.5.50)$$

This equation is verified $\forall C \in \mathcal{C}$, so it follows the local equation

$$\mu \left(1 + \frac{w}{c^2} \right) \overset{*}{A}_\rho = - \overset{*}{C}_\tau Y_\rho^\tau - \tilde{\nabla}_\tau Y^{\rho\tau}. \quad (4.5.51)$$

that is the lagrangian equation (??). For this reason the equation (??), or the equivalent (??), is called *the main equation (or the first undefined equation) of the relativistic mechanics of continuous*, and the equation (??) is denoted as *the first cardinal equation of the relativistic mechanics of continuous*. Besides, the symmetry relations (??) are called *the second undefined vectorial equation of the relativistic*

mechanics of continuous, that can also be written in the following form

$$Y^{\rho\tau} \tilde{e}_\rho \wedge \tilde{e}_\tau = 0. \tag{4.5.52}$$

4.5.6 Symbolic equation of relativistic mechanics of a reversible continuous system

Taking into account the main equation (??) and the boundary conditions:

$$Y^{h\tau} \tilde{n}_\tau = f^h(Q) \quad \forall Q \in \partial\mathcal{C} \tag{4.5.53}$$

with $\mathbf{f}(Q)$ is the spatial projection of the 4-force density $\mathbf{F}(Q)$ applying at every point Q of $\partial\mathcal{C}$, we introduce the vectorial fields

$$\begin{cases} \mathbf{V}(P) \equiv -\mu \left(1 + \frac{w}{c^2}\right) c^2 \overset{*}{C}^\rho \tilde{e}_\rho - \tilde{\nabla}_\sigma Y^{\rho\sigma} \tilde{e}_\rho - \overset{*}{C}^\lambda Y^{\lambda\rho} \tilde{e}_\rho & \forall P \in \mathcal{C} \\ \mathbf{W}(Q) \equiv \mathbf{f} - \Phi^\rho \tilde{n}_\rho & \forall Q \in \partial\mathcal{C} \end{cases} \tag{4.5.54}$$

and considering the functional

$$J(\mathbf{z}) \equiv \int_{\mathcal{C}} \mathbf{V} \cdot \mathbf{z} dC + \int_{\partial\mathcal{C}} \mathbf{W} \cdot \mathbf{z} d\partial C \tag{4.5.55}$$

where \mathbf{z} is an arbitrary regular vectorial field defined in the configuration \mathcal{C} , C is a field interior to \mathcal{C} . It is possible to demonstrate that $J(\mathbf{z}) = 0 \quad \forall \mathbf{z}$; vice versa if $J(\mathbf{z}) = 0 \quad \forall \mathbf{z}$ then

$$\begin{cases} \mathbf{V}(P) = 0 & \forall P \in \mathcal{C} \\ \mathbf{W}(Q) = 0 & \forall Q \in \partial\mathcal{C}. \end{cases} \quad (4.5.56)$$

This is easy to demonstrate proceeding by contradiction and considering the continuity of the vectorial fields $\mathbf{V}(P)$ and $\mathbf{W}(Q)$ in their respective domain [?]. If we substitute the corresponding expressions (??) in the equation (??) in \mathbf{V} and \mathbf{W} and take into account the identities [?]

$$\begin{cases} \tilde{\partial}_\rho \log \sqrt{\gamma^*} = \tilde{\Gamma}_{\rho\sigma}^\sigma, & \tilde{\partial}_\rho \tilde{e}_\tau = \tilde{\Gamma}_{\rho\tau}^\lambda \cdot \tilde{e}_\lambda \\ \frac{1}{\sqrt{\gamma^*}} \tilde{\partial}_\rho \left(\sqrt{\gamma^*} \Phi^\rho \right) = \tilde{\nabla}_\rho Y^{\rho\tau} \cdot \tilde{e}_\tau \end{cases} \quad (4.5.57)$$

the equation

$$J(\mathbf{z}) = 0 \quad (4.5.58)$$

takes the form

$$\begin{aligned} J(\mathbf{z}) \equiv & - \int_C \mu \left(1 + \frac{w}{c^2} \right) \mathbf{A} \cdot \mathbf{z} d\mathcal{C} - \int_C \frac{1}{\sqrt{\gamma^*}} \tilde{\partial}_\rho \left(\sqrt{\gamma^*} \Phi^\rho \right) \cdot \mathbf{z} d\mathcal{C} + \\ & - \int_C \tilde{C}_\lambda^* Y^{\lambda\rho} \tilde{z}_\rho d\mathcal{C} + \int_{\partial\mathcal{C}} \mathbf{f} \cdot \mathbf{z} d\partial\mathcal{C} - \int_C \Phi^\rho \tilde{n}_\rho \cdot \mathbf{z} d\partial\mathcal{C} = 0. \end{aligned} \quad (4.5.59)$$

Taking into account the following transformation [?]

$$\begin{aligned} \int_C \frac{1}{\sqrt{\gamma^*}} \tilde{\partial}_\rho \left(\sqrt{\gamma^*} \Phi^\rho \right) \cdot \mathbf{z} d\mathcal{C} &= \int_C \frac{1}{\sqrt{\gamma^*}} \tilde{\partial}_\rho \left(\sqrt{\gamma^*} \Phi^\rho \cdot \mathbf{z} \right) d\mathcal{C} - \int_C \Phi^\rho \cdot \tilde{\partial}_\rho \mathbf{z} \cdot d\mathcal{C} = \\ &= - \int_{\partial C} \Phi^\rho \cdot \mathbf{z} \cdot \tilde{n}_\rho \mathbf{z} \cdot d\partial\mathcal{C} - \int_C \Phi^\rho \cdot \tilde{\partial}_\rho \mathbf{z} \cdot d\mathcal{C}, \end{aligned}$$

the equation (??) becomes

$$\begin{aligned} J(\mathbf{z}) \equiv - \int_C \mu \left(1 + \frac{w}{c^2} \right) \mathbf{A} \cdot \mathbf{z} d\mathcal{C} + \int_C \Phi^\rho \cdot \tilde{\partial}_\rho \mathbf{z} \cdot d\mathcal{C} + \\ - \int_C \overset{*}{C}_\lambda Y^{\lambda\rho} \tilde{z}_\rho d\mathcal{C} + \int_{\partial C} \mathbf{f} \cdot \mathbf{z} d\partial\mathcal{C} = 0. \end{aligned} \quad (4.5.60)$$

Now we choose as vectorial field \mathbf{z} an arbitrary infinitesimal displacement

$$\mathbf{z} \equiv \mathbf{u} dy^4 + \tilde{d}P = \overset{*}{u} \partial_4 P dy^4 + dy^\tau \tilde{\mathbf{e}}_\tau \quad (4.5.61)$$

and we note that from the scalar equation (??) takes. We obtain from the symmetry of the stress tensor

$$\int_C \Phi^\rho \cdot \tilde{\partial}_\rho \mathbf{z} \cdot d\mathcal{C} = \int_C Y^{\rho\lambda} \tilde{e}_\lambda \cdot \tilde{\partial}_\rho \mathbf{z} \cdot d\mathcal{C} = \frac{1}{2} \int_C Y^{\rho\lambda} \left(\tilde{e}_\rho \cdot \tilde{\partial}_\lambda \mathbf{z} + \tilde{e}_\lambda \cdot \tilde{\partial}_\rho \mathbf{z} \right) d\mathcal{C}; \quad (4.5.62)$$

Substituting the expression (??) in \mathbf{z} , we obtain

$$\begin{aligned} Y^{\rho\sigma} \left(\tilde{e}_\rho \cdot \tilde{\partial}_\sigma \mathbf{z} + \tilde{e}_\sigma \cdot \tilde{\partial}_\rho \mathbf{z} \right) = \\ = Y^{\rho\sigma} \left[\tilde{e}_\rho \cdot \overset{*}{u} \tilde{\partial}_\sigma \partial_4 P + \tilde{e}_\sigma \cdot \overset{*}{u} \tilde{\partial}_\rho \partial_4 P \right] dy^4 + Y^{\rho\sigma} \left[\tilde{e}_\rho \cdot \tilde{\partial}_\sigma \tilde{e}_\tau + \tilde{e}_\sigma \cdot \tilde{\partial}_\rho \tilde{e}_\tau \right] dy^\tau. \end{aligned} \quad (4.5.63)$$

because the identities

$$\tilde{\partial}_\sigma \partial_4 P \cdot \tilde{e}_\rho = \partial_4 \tilde{\partial}_\sigma P \cdot \tilde{e}_\rho, \quad (4.5.64)$$

are valid, the expression (??) becomes

$$Y^{\rho\sigma} \left(\tilde{e}_\rho \cdot \tilde{\partial}_\sigma \mathbf{z} + \tilde{e}_\sigma \cdot \tilde{\partial}_\rho \mathbf{z} \right) == 2 \left(Y^{\rho\sigma} u^{*4} \partial_4 \tilde{\varepsilon}_{\rho\sigma}^* \cdot dy^4 + Y_\lambda^\rho \tilde{\Gamma}_{\rho\tau}^\lambda dy^\tau \right); \quad (4.5.65)$$

so the equation (??) takes the form

$$\begin{aligned} & - \int_C \mu \left(1 + \frac{w}{c^2} \right) \mathbf{A} \cdot \tilde{d}P d\mathcal{C} + dy^4 \int_C Y^{\rho\sigma} u^{*4} \partial_4 \tilde{\varepsilon}_{\rho\sigma}^* d\mathcal{C} + \int_C Y_\lambda^\rho \tilde{\Gamma}_{\rho\tau}^\lambda dy^\tau d\mathcal{C} + \\ & - \int_C Y_\lambda^\rho C_{\rho}^* dy^\lambda d\mathcal{C} + \int_{\partial C} \mathbf{f} \cdot \tilde{d}P = 0 \end{aligned} \quad (4.5.66)$$

where every integral has a physical meaning into the frame of reference:

$$\partial L^{(m)} \equiv - \int_C \mu \left(1 + \frac{w}{c^2} \right) \mathbf{A} \cdot \tilde{d}P d\mathcal{C} \quad \textit{infinitesimal work of the forces of inertia}; \quad (4.5.67)$$

$$\partial L^{(f)} \equiv \int_{\partial C} \mathbf{f} \cdot \tilde{d}P d\mathcal{C} \quad \textit{infinitesimal work of the surface forces}; \quad (4.5.68)$$

$$\partial L^{(i)} \equiv dy^4 \int_C Y^{\rho\sigma} \tilde{u}^4 \partial_4 \tilde{\varepsilon}_{\rho\sigma}^* dC \quad \text{infinitesimal work of the interior forces;} \quad (4.5.69)$$

$$\partial L^{(c)} \equiv \int_C Y_\lambda^\rho \tilde{\Gamma}_{\rho\tau}^\lambda dy^\tau dC \quad \text{infinitesimal complementary work;} \quad (4.5.70)$$

$$\partial L^{(r)} \equiv - \int_C Y_\lambda^\rho C_\rho^* dy^\lambda dC \quad \text{infinitesimal work of the interaction forces.} \quad (4.5.71)$$

Consequently, the equation (??) can be written as

$$\partial L^{(m)} + \partial L^{(i)} + \partial L^{(c)} + \partial L^{(r)} + \partial L^{(f)} = 0. \quad (4.5.72)$$

We will call this equation *symbolic equation of relativistic mechanics of the continuous systems*.

4.5.7 Constitutive equations

Taking into account the equation (??), we can translate the first law of thermodynamics for the generic element C of \mathcal{C} in the following equation:

$$E\mu\partial q \cdot dC = \mu dy^4 \mathcal{L}_{\mathbf{u}} \mathbf{u} \cdot dC + Y^{\rho\tau} \mathcal{L}_{\mathbf{u}} \tilde{\varepsilon}_{\rho\tau}^* dy^4 dC \quad (4.5.73)$$

where ∂q indicates the amount of heat absorbed from the mass of the body in

the transition from one configuration to an infinitely near, $\mathcal{L}_{\mathbf{u}} \mathbf{u} \cdot dy^4$ expresses the corresponding change in internal energy and

$$\partial l^{(i)} \equiv Y^{\rho\tau} \mathcal{L}_{\mathbf{u}} \varepsilon_{\rho\tau}^* dy^4$$

the corresponding work of deformation of the interior forces per unit volume. As we know, in the locally reversible processes the equation

$$\frac{\partial q}{T} = \mathcal{L}_{\mathbf{u}} s \cdot dy^4; \quad (4.5.74)$$

where s is the entropy density, characterizes the locally reversible processes for which it is possible to define the free energy density

$$\mathcal{F} = u - E s T$$

where u is the internal energy density and E the mechanical equivalent of heat. It is possible to deduce from (??) the equation

$$\mu^* u^* \partial_4 \mathcal{F} = -Y^{\rho\tau} u^* \partial_4 \varepsilon_{\rho\sigma}^* - E \mu s^* u^* \partial_4 T \quad (4.5.75)$$

that, supposed known the free energy \mathcal{F} , can be rewritten in the following form

$$\mu \left[\frac{\partial \mathcal{F}}{\partial \varepsilon_{\rho\tau}^*} u^* \partial_4 \varepsilon_{\rho\sigma}^* + \frac{\partial \mathcal{F}}{\partial T} u^* \partial_4 T \right] = -Y^{\rho\tau} u^* \partial_4 \varepsilon_{\rho\sigma}^* - E \mu s^* u^* \partial_4 T \quad (4.5.76)$$

and allows us to deduce the relations

$$Y^{\rho\tau} = -\mu \frac{\partial \mathcal{F}}{\partial \varepsilon_{\rho\tau}^*}, \quad s = -\frac{1}{E} \frac{\partial \mathcal{F}}{\partial T}. \quad (4.5.77)$$

As in the classical case, the thermodynamic potential \mathcal{F} must verify the Helmholtz's postulate to have physical sense, and must therefore satisfy the limitation

$$\frac{\partial^2 \mathcal{F}}{\partial^2 T} < 0. \quad (4.5.78)$$

If, on the contrary, we assume known the internal energy density u , from (??) and (??) we deduce the equation

$${}^4 u \partial_4 u \cdot dy^4 = -\frac{1}{\mu} Y^{\rho\tau} {}^4 u \partial_4 \varepsilon_{\rho\tau}^* \cdot dy^4 + E T {}^4 u \partial_4 s \cdot dy^4 \quad (4.5.79)$$

and consequently the relations

$$Y^{\rho\tau} = -\mu \frac{\partial u}{\partial \varepsilon_{\rho\tau}^*}, \quad T = \frac{1}{E} \frac{\partial u}{\partial s}. \quad (4.5.80)$$

As in the classical case, the internal energy density u must satisfy the limitation

$$\frac{\partial^2 u}{\partial^2 s} > 0. \quad (4.5.81)$$

At this point, substituting in the equation (??) the corresponding expressions of $Y^{\rho\tau}$ deduced from (??) and (??), we obtain the constitutive equations for a reversible system \mathcal{S} :

$$Y^{hk} = -\mu \left(\delta_{\rho}^h - \delta_4^h \frac{{}^* u_{\rho}}{u_4} \right) \left(\delta_{\tau}^k - \delta_4^k \frac{{}^* u_{\tau}}{u_4} \right) \frac{\partial \mathcal{F}}{\partial \varepsilon_{\rho\tau}^*} \quad (\text{isothermal processes}) \quad (4.5.82)$$

$$Y^{hk} = -\mu \left(\delta_\rho^h - \delta_4^h \frac{u_\rho}{u_4} \right) \left(\delta_\tau^k - \delta_4^k \frac{u_\tau}{u_4} \right) \frac{\partial u}{\partial \varepsilon_{\rho\tau}^*} \quad (\text{isoentropic processes}) \quad (4.5.83)$$

4.5.8 Further analysis of the relative formulation of the conservation conditions of the energy-momentum tensor

The constitutive equations for a reversible system \mathcal{S} allow us to complete the analysis of equation (??) that we rewrite

$$\mu u^{*4} \partial_4 w = -\frac{1}{2} Y^{\rho\tau} \mathcal{L}_u \gamma_{\rho\tau}^* = -Y^{\rho\tau} u^{*4} \partial_4 \varepsilon_{\rho\tau}^* \quad (4.5.84)$$

In fact, substituting in (??) the expressions

$$Y^{\rho\tau} = -\mu \frac{\partial w}{\partial \varepsilon_{\rho\tau}^*} \quad (4.5.85)$$

where w , proper thermodynamic energy density, coincides with \mathcal{F} in the isothermal processes, with u in the isoentropic processes, it follows the identity

$$\mu u^{*4} \partial_4 w = \mu u^{*4} \frac{\partial w}{\partial \varepsilon_{\rho\tau}^*} \cdot \frac{\partial \varepsilon_{\rho\tau}^*}{\partial y^4} = \mu u^{*4} \frac{\partial w}{\partial y^4}. \quad (4.5.86)$$

Therefore, the temporal projection (??) in the case of reversible processes for continuous systems reduces to an identity.

Chapter 5

The Cauchy problem for the Einsteinian gravitational field equations and its relative formulation in an assigned physical reference

5.1 Gravitational field - empty space

5.1.1 Analysis for the relative system of equations

The gravitational equations (??) constitute a system of ten equations in the ten unknown functions $\gamma_i, \gamma_{\alpha\beta}(= \gamma_{\beta\alpha})$; but only six scalar equations are independent,

because the boundary conditions (??) give four differential identities among the ten unknown functions.

We must determine which equations are appropriate to choose to solve the Cauchy problem (??) , so we assign to the manifold V_4 , which is to be a differentiable manifold, a hypersurface $\bar{\Sigma}$ and a coordinate system $\{x^h\}$ for which $x^4 = 0$ on $\bar{\Sigma}$. Let us suppose that in a 4-dimentional neighborhood $W \subset \bar{\Sigma}$ there are a field of symmetric double tensors $\gamma_{ik}(\gamma_{4r} = 0)$ and a field of covariant vectors γ_i such that, assigned to V_4 the metric tensor $g_{ik} = \gamma_{ik} - \gamma_i\gamma_k$, the tensorial equation (??)₁ is valid. So we have

$$\begin{aligned} \gamma^{\alpha\beta} s_{\alpha\beta} = s_\alpha^\alpha = \tilde{R}^* + \frac{1}{4} (\tilde{K}_\alpha^\alpha)^2 - \frac{1}{2} \tilde{K}^{\alpha\beta} \tilde{K}_{\alpha\beta} + \\ + \frac{1}{2} \gamma^{\alpha\beta} \gamma^4 \partial_4 (\tilde{K}_{\beta\alpha} + \tilde{\Omega}_{\beta\alpha}) - C^\alpha C_\alpha - \tilde{\nabla}_\alpha^* C^\alpha + \frac{1}{2} \tilde{\Omega}^{\alpha\beta} \tilde{\Omega}_{\alpha\beta} = 0 \end{aligned}$$

so, taking into account (??), (??)₃ and the following easily verified identity

$$\tilde{K}^{\alpha\beta} = -\gamma^4 \partial_4 \gamma^{\alpha\beta} \tag{5.1.1}$$

we obtain

$$R = -(\tilde{R}^* + \mathcal{I}). \tag{5.1.2}$$

Besides, it follows from (??)

$$\begin{cases} \gamma^4 \partial_4 S_\alpha = -\frac{1}{2} \tilde{\nabla}_\alpha^* (\tilde{R}^* + \mathcal{I}) - \frac{1}{2} \tilde{K}_\beta^\beta S_\alpha - \tilde{\Omega}_{h\alpha} S^h - (\tilde{R}^* + \mathcal{I}) C_\alpha \\ \frac{1}{2} \gamma^4 \partial_4 (\tilde{R}^* + \mathcal{I}) = -\tilde{\nabla}_\alpha^* S^\alpha - 2C_\alpha S^\alpha - \frac{1}{2} \tilde{K}_\alpha^\alpha (\tilde{R}^* + \mathcal{I}). \end{cases} \tag{5.1.3}$$

The equations (??), interpreted as partial differential equations in the unknown functions S_α , $\tilde{R}^* + \mathcal{I}$, can be transformed into normal form with respect to the variable x^4 .

In fact, if we put

$$H_\alpha(x) \equiv -\frac{1}{2}\partial_\alpha \left(\tilde{R}^* + \mathcal{I} \right) - \frac{1}{2}\tilde{K}_\beta^\beta S_\alpha + \tilde{\Omega}_{\alpha h} S^h - \left(\tilde{R}^* + \mathcal{I} \right) C_\alpha$$

$$N(x) \equiv -\gamma^{\alpha\beta}\partial_\alpha S_\beta + \gamma^{\alpha\beta} \left\{ \begin{matrix} \tilde{\rho} \\ \alpha\beta \end{matrix} \right\}^* S_\rho - 2C_\alpha S^\alpha - \frac{1}{2}\tilde{K}_\alpha^\alpha \left(\tilde{R}^* + \mathcal{I} \right)$$

$$D(x) = \frac{\gamma^{\alpha\beta}\gamma_\alpha H_\beta + N(x)}{1 - \gamma^{\alpha\beta}\gamma_\alpha\gamma_\beta}$$

the (??) assume the following form:

$$\begin{cases} \gamma^4\partial_4 S_\alpha = -\gamma_\alpha D + H_\alpha \\ \gamma^4\partial_4 \left(\tilde{R}^* + \mathcal{I} \right) = 2D \end{cases} \quad (5.1.4)$$

5.1.2 The initial condition problem

The problem of initial conditions consists in obtaining potentials $g_{\alpha\beta}$ and their first derivatives on an initial space-like manifold V_3 . At this purpose, let us observe that the equations (??) constitute a system of four partial differential equations of first order, linear and homogeneous in the four unknown functions S_α , $\tilde{R}^* + \mathcal{I}$, of normal form with respect to the variable x^4 . They are satisfied in the neighborhood $W \subset \bar{\Sigma}$ of course the zero solution

$$S_\alpha = 0, \quad \tilde{R}^* + \mathcal{I} = 0$$

and we know that this is the only one which vanishes on the hypersurface $\bar{\Sigma}$ if the coefficients have continuous first derivatives [?].

We have thus established that if in a 4-dimensional neighborhood $W \subset \bar{\Sigma}$ exist two tensor fields $\gamma_{ik}(\gamma_{4r} = 0)$ and γ_i which satisfy the equation tensor (??)₁ and also the (??)₂ and (??)₃ on the hypersurface $\bar{\Sigma}$, in the same neighborhood they represent a solution of the entire system (??).

Furthermore, if we assume that the vector field γ satisfies on $\bar{\Sigma}$ the condition

$$\gamma_\alpha(x^\mu, 0) = 0 \tag{5.1.5}$$

the (??)₂ and (??)₃ assume the following form on $\bar{\Sigma}$

$$\begin{cases} \tilde{\nabla}_\beta^* (\tilde{K}_\mu^\mu) = \tilde{\nabla}_\mu^* \tilde{K}_\beta^\mu \\ \tilde{R}^* = -\frac{1}{4} \left[(\tilde{K}_\mu^\mu)^2 - \tilde{K}^{\alpha\beta} \tilde{K}_{\alpha\beta} \right]. \end{cases} \tag{5.1.6}$$

The (??), in which the transverse partial derivatives are replaced by ordinary derivatives, express a differential link between the tensors $\gamma^i, \gamma_{ik}, \gamma^4 \partial_4 \gamma_{ik}$.

Therefore, we conclude, according to Lichnerowicz, that the study of the Cauchy problem for the system (??) can be done by solving first the problem of initial conditions (??), and then taking into account the results obtained by performing the integration of the system (??)₁ (problem of evolution).

5.1.3 The problem of the restricted evolution

At this point, we are able to enunciate the intrinsic formulation of the evolution problem for the gravitational equations (??).

In a differentiable manifold V_4 , we assign:

- 1) a field of contravariant vectors γ^i ;
- 2) a field of covariant vectors η_i , which verify $\eta_i \gamma^i = -1$;
- 3) a hypersurface $\bar{\Sigma}$ to which the field γ^i is not tangent at any point.

Let us choose a bounded 3-dimensional set D in $\bar{\Sigma}$, a generic point $Q \in D$ and the 3-dimensional vectorial space $\bar{\pi}_Q$ tangent to $\bar{\Sigma}$ in Q . In D we assign two fields of double symmetric tensors $\bar{\gamma}_{ij}$, $\bar{\varphi}_{ij}$ belonging to the product space $\bar{\pi}_Q \otimes \bar{\pi}_Q$ for each point Q and there verifying the conditions

$$\bar{\gamma}_{ij} \gamma^j = 0, \quad \bar{\varphi}_{ij} \gamma^j = 0 \quad \text{in } \bar{\Sigma} \quad (5.1.7)$$

and

$$\bar{\gamma}_{ij} \bar{\xi}^i \bar{\xi}^j > 0 \quad \text{with } \bar{\xi}^r \in \bar{\pi}_Q. \quad (5.1.8)$$

The fields $\bar{\gamma}_{ij}(Q)$ and $\bar{\varphi}_{ij}(Q)$ are called ‘‘Cauchy data’’.

In these hypotheses we choose in V_4 a system of coordinates x^i adapted to the vector field γ^i and to the hypersurface $\bar{\Sigma}$ ($x^\alpha = \text{const.}$, $x^4 = \text{var.}$, and $x^4 = 0$ in $\bar{\Sigma}$).

Our aim is to determinate, in a 4-dimensional neighborhood $W \subset \bar{\Sigma}$, a double symmetric tensor field γ_{ij} subjected to the conditions

$$\gamma_{ij} \gamma^j = 0 \quad \forall Q \in \bar{\Sigma}, \quad (5.1.9)$$

and from $(??)_1$ satisfying the equations

$$\left\{ \begin{aligned}
 & (\gamma^4)^2 [\partial_4 \partial_4 \gamma_{\alpha\beta} - \eta_\alpha \eta_\beta \gamma^{\rho\sigma} \partial_4 \partial_4 \gamma_{\rho\sigma} + \gamma^{\rho\sigma} \eta_\rho (\eta_\alpha \partial_4 \partial_4 \gamma_{\beta\sigma} + \eta_\beta \partial_4 \partial_4 \gamma_{\alpha\sigma} - \eta_\sigma \partial_4 \partial_4 \gamma_{\alpha\beta})] = \\
 & = \gamma^{\rho\sigma} [\partial_\rho \partial_\sigma \gamma_{\alpha\beta} + \partial_\alpha \partial_\beta \gamma_{\rho\sigma} - \partial_\rho \partial_\beta \gamma_{\alpha\sigma} - \partial_\rho \partial_\alpha \gamma_{\beta\sigma} + \eta_\rho \gamma^4 (\partial_\sigma \partial_4 \gamma_{\alpha\beta} - \partial_\beta \partial_4 \gamma_{\alpha\sigma} - \partial_\alpha \partial_4 \gamma_{\beta\sigma})] + \\
 & + \gamma^{\rho\sigma} [\eta_\alpha \gamma^4 (\partial_\beta \partial_4 \gamma_{\rho\sigma} - \partial_\rho \partial_4 \gamma_{\beta\sigma}) + \eta_\beta \gamma^4 (\partial_\alpha \partial_4 \gamma_{\rho\sigma} - \partial_\rho \partial_4 \gamma_{\alpha\sigma})] + \\
 & + h_{\alpha\beta} (\eta_i, \gamma^i, \partial_r \eta_i, \partial_s \gamma^j, \partial_s \partial_r \eta_i, \gamma_{ij}, \partial_r \gamma_{ij})
 \end{aligned} \right. \tag{5.1.10}$$

(with $h_{\alpha\beta}$ rational functions of the indicated arguments) and verifying the conditions in $\bar{\Sigma}$

$$\begin{aligned}
 \gamma_{ij} &= \bar{\gamma}_{ij} \\
 \gamma^4 \partial_4 \gamma_{ij} &= \bar{\varphi}_{ij}
 \end{aligned} \tag{5.1.11}$$

being in D

$$\left\{ \begin{aligned}
 & \tilde{\nabla}_\beta^* (\bar{\varphi}_\mu^\mu) = \tilde{\nabla}_\mu^* (\bar{\varphi}_\beta^\mu) \\
 & \tilde{R}^* = -\frac{1}{4} [(\bar{\varphi}_\mu^\mu)^2 - \bar{\varphi}^{\alpha\beta} \bar{\varphi}_{\alpha\beta}]
 \end{aligned} \right. \tag{5.1.12}$$

where in (??) the tensor field $\tilde{K}_{\alpha\beta}$ is replaced by the tensor field $\bar{\varphi}_{\alpha\beta}$.

5.1.4 Existence and uniqueness of solution for the restricted evolution problem

The system (??) consists of six partial differential equations in the six unknown functions $\gamma_{\alpha\beta}$ and, since by hypothesis $\eta_4 \neq 0$, can be written in normal form with respect to the variable x^4 .

If we assume analytical all the data of the problem, by virtue of Cauchy-Kowalesky's theorem, it admits a unique solution that satisfies the conditions (??) (??) on $\bar{\Sigma}$.

When the Cauchy data are not analytical, the problem of evolution can still be solved uniquely.

In fact, for arbitrary field of covariant vectors η_i in V_4 , we can choose the first three components η_α equal to zero.

So the system (??) becomes

$$\begin{aligned} -\gamma^{\rho\sigma}\partial_\rho\partial_\sigma\gamma_{\alpha\beta} + (\gamma^4)^2\partial_4\partial_4\gamma_{\alpha\beta} + \gamma^{\rho\sigma}\partial_\alpha(\partial_\rho\gamma_{\beta\sigma} - \frac{1}{2}\partial_\beta\gamma_{\sigma\rho}) + \\ + \gamma^{\rho\sigma}\partial_\beta(\partial_\sigma\gamma_{\alpha\rho} - \frac{1}{2}\partial_\alpha\gamma_{\rho\sigma}) - h_{\alpha\beta} = 0. \end{aligned} \quad (5.1.13)$$

In this case there are the equalities

$$\gamma^{\rho\sigma}\gamma_{\sigma\beta} = \delta_\beta^\rho, \quad \frac{1}{2}\gamma^{\rho\sigma}\partial_\beta\gamma_{\rho\sigma} = \left\{ \begin{matrix} \tilde{\nu} \\ \beta\nu \end{matrix} \right\}^* = \partial_\beta \log \sqrt{\gamma} \quad (\gamma = \det \|\gamma_{\alpha\beta}\|) \quad (5.1.14)$$

so (??) assume the following form

$$-\gamma^{\rho\sigma}\partial_\rho\partial_\sigma\gamma_{\alpha\beta} + (\gamma^4)^2\partial_4\partial_4\gamma_{\alpha\beta} - (\gamma_{\beta\sigma}\partial_\alpha u^\sigma + \gamma_{\alpha\rho}\partial_\beta u^\rho) - a_{\alpha\beta} = 0. \quad (5.1.15)$$

where $u^\sigma \equiv \frac{1}{\sqrt{\gamma}}\partial_\rho(\sqrt{\gamma} \cdot \gamma^{\rho\sigma})$ and $a_{\alpha\beta}$ are functions that depend on only the first derivatives of $\gamma_{\mu\nu}$.

The system (??) has the same structure as that considered by Y. Bruhat [?][?], that we can write as follows

$$R_{jm} \equiv -\frac{1}{2}g^{rs}\partial_r\partial_s g_{jm} - \frac{1}{2}(g_{mn}\partial_j u^n + g_{jn}\partial_m u^n) - H_{jm} = 0 \quad (5.1.16)$$

where $u^n \equiv \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g} \cdot g^{nr})$ and H_{jm} are polynomials in g_{rs} , g^{rs} and their first derivatives.

Taking into account the Bruhat's results, the tensorial field γ_{ik} that satisfies the problem of evolution is uniquely determined in V_4 and so the metric tensor $g_{ik} = \gamma_{ik} - \eta_i\eta_k$ with the field of covariant vectors η_i previously chosen. It is immediate to verify that the field of covariant vectors η_i provides a representation of the covariant vector field γ .

5.2 Gravitational field - perfect fluid

5.2.1 Conditions to uniquely solve the Cauchy problem

The analysis of the fundamental equations (??), (??), (??) shows that the equations (??)₂, (??)₃ give the conditions for compatibility of Cauchy, and the other equations (??), (??), (??)₁ reflect properly the problem of evolution.

The conditions to uniquely solve the Cauchy problem for equations (??), (??), (??) near a hypersurface $\bar{\Sigma}$ by local equation $x^4 = 0$, are:

- a) must be tangent in any of its points, or the corresponding elementary cone or a current-line of fluid;
- b) should not be a wave hypersurface hydrodynamic.

Being Q any point, if we call $\bar{\pi}_Q$ the vector space tangent to $\bar{\Sigma}$ in Q , just that we have on $\bar{\Sigma}$

$$(\gamma_{ik} - \gamma_i \gamma_k) \bar{\xi}^i \bar{\xi}^k > 0 \quad \forall \bar{\xi}^r \in \bar{\pi}_Q; \quad \gamma_i \gamma^i = \gamma_4 \gamma^4 = -1 \quad (5.2.1)$$

the condition a) is verified.

For the condition b), giving to the equation (??) the following form [?]

$$\frac{1}{2} \gamma^{\alpha\beta} \gamma^4 \partial_4 \gamma_{\alpha\beta} = \frac{1}{2} \tilde{K}_i^i = \nabla_i \gamma^i, \quad (5.2.2)$$

the system of equations (??) (??) can be written as

$$\begin{cases} \gamma^{j\alpha} \tilde{\partial}_\alpha p_0 + (p_0 + \mu_0 c^2) \gamma^r \nabla_r \gamma^i = 0 \\ \gamma^4 \partial_4 (\mu_0 c^2) + (p_0 + \mu_0 c^2) \nabla_i \gamma^i = 0 \end{cases} \quad (5.2.3)$$

where $p_0 = p_0(\mu_0)$ is the state equation of the fluid.

By the conditions of compatibility of Hugoniot applied to the longitudinal derivative of p_0 , to the transvers gradient of p_0 and to the tensor $\nabla_i \gamma^i$,

$$\Delta(\gamma^4 \partial_4 \mu_0) = \lambda^{(\mu)} \gamma^4 \partial_4 z; \quad \Delta(\tilde{\partial}_\alpha p_0) = \frac{dp_0}{d\mu_0} \lambda^{(\mu)} \tilde{\partial}_\alpha z; \quad \Delta(\nabla_i \gamma^j) = \lambda^{(j)} \partial_i z \quad (5.2.4)$$

where $\lambda^{(\mu)}$, $\lambda^{(i)}$ are the characteristic parameters of discontinuities and

$$z(x^1, x^2, x^3, x^4) = \text{const.}$$

the equation of $\bar{\Sigma}$, we deduce from (??) the characteristic equation

$$(c\gamma^4 \partial_4 z)^2 = \frac{dp_0}{d\mu_0} \gamma^{\alpha\beta} \tilde{\partial}_\alpha z \cdot \tilde{\partial}_\beta z. \quad (5.2.5)$$

So for $z = x^4$, we obtain the condition

$$\gamma^{\alpha\beta} \gamma_\alpha \gamma_\beta - (c\gamma^4)^2 \cdot \frac{dp_0}{d\mu_0} \neq 0 \quad (5.2.6)$$

which must be satisfied on $\bar{\Sigma}$ by the given conditions of Cauchy, to exclude that $\bar{\Sigma}$, of local equations $x^4 = 0$ is a wave hypersurface hydrodynamic.

5.2.2 Analysis of the fundamental equations

Taking into account the differential relations (??) and (??) from which we deduce the equation

$$\nabla_j (G_m^j + \chi T_m^j) = 0; \quad (5.2.7)$$

and the equations (??), (??) and (??), this equation gives rise to the following equations related to a physical reference S :

$$\begin{aligned} \tilde{\nabla}_j^* s_m^j + C_h s_m^h - \left(\frac{1}{2}R - \chi p_0\right) C_m - \tilde{\partial}_m \left(\frac{1}{2}R - \chi p_0\right) + \\ + \frac{1}{2} S_m \tilde{K}_\alpha^\alpha + \gamma^4 \partial_4 S_m + \tilde{\Omega}_{jm} S^j + \left[\frac{1}{2} \left(\tilde{R}^* + \mathcal{I}\right) + \chi \mu_0 c^2\right] C_m = 0, \end{aligned} \quad (5.2.8)$$

$$\begin{aligned} \frac{1}{2} \left(\tilde{K}_{jr} + \tilde{\Omega}_{jr}\right) s^{jr} \gamma_m - \left(\frac{1}{2}R - \chi p_0\right) \cdot \frac{1}{2} \tilde{K}_\alpha^\alpha \gamma_m + \left(\tilde{\nabla}_j^* S^j + 2C_h S^h\right) \gamma_m + \\ + \left[\frac{1}{2} \left(\tilde{R}^* + \mathcal{I}\right) \cdot \frac{1}{2} \tilde{K}_\alpha^\alpha \gamma_m + \gamma^4 \partial_4 \left[\frac{1}{2} \left(\tilde{R}^* + \mathcal{I}\right) + \chi \mu_0 c^2\right] \gamma_m = 0. \end{aligned} \quad (5.2.9)$$

Let us consider a differentiable manifold V_4 and a field of contravariant vectors $\gamma^i(x)$, which is no point Q are tangent to the hypersurface $\bar{\Sigma}$, and a field of covariant vectors $\gamma_i(x)$, two scalar functions $\mu_0(x)$, $p_0(x)$, a field of covariant symmetric tensors $\gamma_{ik}(x)$ such that, given the metric tensor of V_4 $g_{ik} = \gamma_{ik} - \gamma_i \gamma_k$, the following equations are verified in the same neighborhood of $\bar{\Sigma}$:

$$\left\{ \begin{array}{l} \gamma^i \gamma_i = -1 \\ s_{jm} \gamma^m = 0 \\ s_{\alpha\beta} = -\frac{1}{2} \chi (\mu_0 c^2 - p_0) \gamma_{\alpha\beta} \\ \tilde{\partial}_\alpha p_0 + (p_0 + \mu_0 c^2) \gamma^4 (\partial_4 \gamma_\alpha - \partial_\alpha \gamma_4) = 0 \\ \gamma^4 \partial_4 (\mu_0 c^2) + \frac{1}{2} (p_0 + \mu_0 c^2) \gamma^{\alpha\beta} \gamma^4 \partial_4 \gamma_{\alpha\beta} = 0 \end{array} \right. \quad (5.2.10)$$

and on $\bar{\Sigma}$ the conditions

$$(S_\alpha)_{\bar{\Sigma}} = 0, \quad \left(\tilde{R}^* + \mathcal{I} + 2\chi \mu_0 c^2 \right)_{\bar{\Sigma}} = 0. \quad (5.2.11)$$

It can be shown as if we admit the existence of the tensor and scalar functions in the same neighborhood of $\bar{\Sigma}$, then the condition (??) are verified also in $\bar{\Sigma}$, so

$$S_\alpha = 0, \quad \tilde{R}^* + \mathcal{I} + 2\chi \mu_0 c^2 = 0. \quad (5.2.12)$$

To this end, it should be stated first that the assumptions

- a) the last two equations (??) are nothing more than the translation of the equation (??) in the form related to the physical reference determined by the field of vectors γ^i ;
- b) the field of vectors γ_i is the covariant representation of the field of vectors γ^i ;
- c) for [?], we have

$$\begin{aligned} \gamma^{\alpha\beta} s_{\alpha\beta} \equiv \tilde{R}^* + \frac{1}{4} \left(\tilde{K}_\alpha^\alpha \right)^2 - \frac{1}{2} \tilde{K}^{\alpha\rho} \tilde{K}_{\alpha\rho} + \frac{1}{2} \gamma^{\alpha\beta} \gamma^4 \partial_4 \left(\tilde{K}_{\alpha\beta} + \tilde{\Omega}_{\beta\alpha} \right) + \\ - C^\beta C_\beta - \tilde{\nabla}_\beta^* C^\beta + \frac{1}{2} \tilde{\Omega}^{i\beta} \tilde{\Omega}_{i\beta} = -\frac{3}{2} \chi (\mu_0 c^2 - p_0), \end{aligned}$$

and, consequently, by virtue of the identities

$$R \equiv \tilde{R}^* + \frac{1}{4} \left[\left(\tilde{K}_\alpha^\alpha \right)^2 + \tilde{K}^{\alpha\beta} \tilde{K}_{\alpha\beta} + \tilde{\Omega}^{\alpha\beta} \tilde{\Omega}_{\alpha\beta} \right] + \gamma^4 \partial_4 \left(\tilde{K}_\alpha^\alpha \right) - 2 \left(\tilde{\nabla}_\alpha^* C^\alpha + C^\rho C_\rho \right), \quad (5.2.13)$$

$$\tilde{K}^{\alpha\beta} \equiv -\gamma^4 \partial_4 \gamma^{\alpha\beta} \quad (5.2.14)$$

we deduce

$$R = - \left[\tilde{R}^* + \mathcal{I} + 3\chi (\mu_0 c^2 - p_0) \right]. \quad (5.2.15)$$

That said, the identities (??), (??) become

$$\begin{cases} \gamma^4 \partial_4 S_\alpha = -\frac{1}{2} \tilde{\partial}_\alpha \left[\tilde{R}^* + \mathcal{I} + 2\chi \mu_0 c^2 \right] - \left[\tilde{R}^* + \mathcal{I} + 2\chi \mu_0 c^2 \right] C_\alpha - \frac{1}{2} \tilde{K}_\beta^\beta S_\alpha - \tilde{\Omega}_\alpha^\beta S_\beta \\ \gamma^4 \partial_4 \left[\tilde{R}^* + \mathcal{I} + 2\chi \mu_0 c^2 \right] = - \left[\tilde{R}^* + \mathcal{I} + 2\chi \mu_0 c^2 \right] \tilde{K}_\rho^\rho - 2 \left(\tilde{\nabla}_\rho^* S^\rho + 2C_\rho S^\rho \right). \end{cases} \quad (5.2.16)$$

The linear partial differential equations in the unknown functions S_α , $\tilde{R}^* + \mathcal{I} + 2\chi \mu_0 c^2$ can be written in normal form with respect to the variable x^4 , a condition that has [see (??)]

$$\gamma^{\alpha\beta} \gamma_\alpha \gamma_\beta \neq 1 \quad (5.2.17)$$

and admit the unique solution

$$S_\alpha \equiv 0, \quad \tilde{R}^* + \mathcal{I} + 2\chi\mu_0c^2 \equiv 0$$

which becomes null on the hypersurface $\bar{\Sigma}$.

So, it is clear that the Cauchy problem for a perfect fluid can be split into two different problems:

- 1) the initial data problem;
- 2) and, after solving the initial data problem, the restricted evolution problem.

5.2.3 Formulation of initial data problem

Let us suppose given, in the differentiable four-dimensional manifold V_4 :

- a) a part of regular hypersurface $\bar{\Sigma}$;
- b) a three dimensional domain D on $\bar{\Sigma}$;
- c) $\forall Q \in D$ two scalar functions $\bar{\mu}_0(Q)$, $p_0(\bar{\mu}_0)$ and a field of covariant vectors $\bar{\gamma}_i(Q)$;
- d) a part of hypersurface $\bar{\Sigma}$ that has a field of contravariant vectors γ^i , that are tangent to $\bar{\Sigma}$ and which verify the condition

$$\bar{\gamma}^i \bar{\gamma}_i = -1 \quad [\bar{\gamma}^i = (\gamma^i)_{\bar{\Sigma}}]. \quad (5.2.18)$$

Let us consider the vectorial space $\bar{\pi}_Q$ tangent to $\bar{\Sigma}$ in Q , and choose a system of coordinates (x^i) adapted to the field of vectors $\gamma^i(x)$ and to the hypersurface $\bar{\Sigma}$. Our

aim is to determine, in $\bar{\Sigma}$, two symmetric tensors $\bar{\gamma}_{ij}$, $\bar{\varphi}_{ij}$ (that represent respectively the metric tensor chosen for $\bar{\Sigma}$ and the prefixed determination of the deformation tensor $\tilde{K}_{\alpha\beta}$), subjected to the conditions [see (??), (??)]

$$\bar{\gamma}_{ij}\bar{\gamma}^j = 0, \quad \bar{\varphi}_{ij}\bar{\gamma}^j = 0 \quad (5.2.19)$$

$$(\bar{\gamma}_{ij} - \bar{\gamma}_i\bar{\gamma}_j)\bar{\xi}^i\bar{\xi}^j > 0, \quad \forall \bar{\xi}^r \in \bar{\pi}_Q \quad (5.2.20)$$

$$\bar{\gamma}^{\alpha\beta}\bar{\gamma}_\alpha\bar{\gamma}_\beta \neq (c\bar{\gamma}^4)^2 \cdot \frac{d\bar{\mu}_0}{dp_0}, \quad \bar{\gamma}^{\alpha\beta}\bar{\gamma}_\alpha\bar{\gamma}_\beta \neq 1 \quad (5.2.21)$$

and satisfying the equations [?]

$$\begin{cases} \frac{1}{2} \left[\tilde{\nabla}_\alpha^* (\bar{\varphi}_\rho^\rho) - \tilde{\nabla}_\sigma^* (\bar{\varphi}_\alpha^\sigma + \bar{\omega}_\alpha^\sigma) \right] + \bar{C}^\beta \bar{\omega}_{\beta\alpha} = 0 \\ \bar{\gamma}^{\alpha\beta} \left(\tilde{P}_{\alpha\beta}^* \right)_{\bar{\Sigma}} + \frac{1}{4} \left[(\bar{\varphi}_\alpha^\alpha)^2 - \bar{\varphi}^{\alpha\beta} \bar{\varphi}_{\alpha\beta} + 3\bar{\omega}^{\alpha\beta} \bar{\omega}_{\alpha\beta} \right] = -2\chi\bar{\mu}_0 c^2. \end{cases} \quad (5.2.22)$$

The vector $\bar{C}_\alpha \in \bar{\pi}_Q$ and the tensor $\bar{\omega}_{\alpha\beta} \in \bar{\pi}_Q \otimes \bar{\pi}_Q$ are evaluated through the formula

$$\bar{C}_\alpha = -\frac{p'_0(\bar{\mu}_0)}{p_0(\bar{\mu}_0) + \bar{\mu}_0 c^2} \partial_\alpha \bar{\mu}_0 + \frac{p'_0(\bar{\mu}_0)}{2c^2} \cdot \bar{\gamma}^{\rho\sigma} \bar{\varphi}_{\rho\sigma} \cdot \bar{\gamma}_\alpha \quad \left(p'_0 = \frac{dp_0}{d\bar{\mu}_0} \right) \quad (5.2.23)$$

$$\bar{\omega}_{\alpha\beta} = \partial_\alpha \bar{\gamma}_\beta - \partial_\beta \bar{\gamma}_\alpha + \frac{p'_0(\bar{\mu}_0)}{p_0(\bar{\mu}_0) + \bar{\mu}_0 c^2} (\bar{\gamma}_\beta \partial_\alpha \bar{\mu}_0 - \bar{\gamma}_\alpha \partial_\beta \bar{\mu}_0),$$

the scalar $\bar{\gamma}^{\alpha\beta} \left(\tilde{P}_{\alpha\beta}^* \right)_{\bar{\Sigma}}$ can be deduced if we put in

$$\tilde{P}_{\alpha\beta}^* \equiv -\tilde{\partial}_\beta \tilde{\Gamma}_{\alpha\rho}^{*\rho} + \tilde{\partial}_\rho \tilde{\Gamma}_{\beta\alpha}^{*\rho} - \tilde{\Gamma}_{\alpha\rho}^{*\sigma} \tilde{\Gamma}_{\beta\sigma}^{*\rho} + \tilde{\Gamma}_{\beta\alpha}^{*\sigma} \tilde{\Gamma}_{\rho\sigma}^{*\rho} \quad , \quad (P_{4r} \equiv 0)$$

the functions

$$(\gamma^4 \partial_4 \gamma_{\alpha\beta})_{\bar{\Sigma}} = \bar{\varphi}_{\alpha\beta}, \quad (\partial_4 \gamma_\alpha)_{\bar{\Sigma}} = \partial_\alpha \bar{\gamma}_4 + \frac{\bar{\gamma}_4 p'_0(\bar{\mu}_0)}{p_0(\bar{\mu}_0) + \bar{\mu}_0 c^2} [\partial_\alpha \bar{\mu}_0 + \bar{\gamma}_\alpha (\gamma^4 \partial_4 \mu_0)_{\bar{\Sigma}}] \quad (5.2.24)$$

evaluating the values $\partial_4 \partial_4 \gamma_{\rho\sigma}$ on $\bar{\Sigma}$.

5.2.4 Formulation of restricted evolution problem

Assuming to have solved the problem of initial data, we propose to determine, in a neighborhood of $\bar{\Sigma}$, a field of covariant vectors $\gamma_i(x)$ and a double symmetric tensor field γ_{ij} satisfying

$$\gamma^i \gamma_i = -1, \quad \gamma_{ij} \gamma^j = 0 \quad (5.2.25)$$

and a scalar function $\mu_0(x)$ verifying (??), (??) and (??)₁ in $\bar{\Sigma}$ and the conditions

$$(\bar{\mu}_0)_{\bar{\Sigma}} = \bar{\mu}_0, \quad (\gamma_i)_{\bar{\Sigma}} = \bar{\gamma}_i, \quad (\gamma_{ij})_{\bar{\Sigma}} = \bar{\gamma}_{ij}, \quad (\gamma^4 \partial_4 \gamma_{ij})_{\bar{\Sigma}} = \bar{\varphi}_{ij}. \quad (5.2.26)$$

For the problem so formulated, it exists, in physical sense, an unique solution, which verifies the initial data [?].

Hence, supposed known this solution and assigned, for the space-time V_4 , the metric

tensor $g_{ik} = \gamma_{ik} - \gamma_i \gamma_k$, the vector field γ_i identifies a well-defined physical reference system S .

5.3 Gravitational field - pure matter

5.3.1 Fundamental equations

Let us consider the state equation $p_0 = p_0(\mu_0)$. The tensor components s_{ik} , not identically zero, have the form

$$s_{\alpha\beta} \equiv a_{\alpha\beta}^{(rs\rho\sigma)} \partial_r \partial_s \gamma_{\rho\sigma} + b_{\alpha\beta}^{(rsi)} \partial_r \partial_s \gamma_i + p_{\alpha\beta}(\gamma_{\rho\sigma}, \partial_r \gamma_i, \gamma_i, \partial_4 \gamma_{\rho\sigma}) \quad (s_{4i} \equiv 0) \quad (5.3.1)$$

where the coefficients $a_{\alpha\beta}^{(rs\rho\sigma)}$ and $b_{\alpha\beta}^{(rsi)}$ are polynomials in the functions $\gamma_i, \gamma_{\rho\sigma}$ and the terms $p_{\alpha\beta}$ are polynomials in the functions $\gamma_{\rho\sigma}, \partial_r \gamma_i, \gamma_i, \partial_4 \gamma_{\rho\sigma}$.

The equation 1 are solvable with respect to $\partial_4 \partial_4 \gamma_{\rho\sigma}$ in a neighborhood of the hypersurface $\bar{\Sigma}$, provided that the conditions , are satisfied, and all the fundamental equations (??), and can be written:

$$\left\{ \begin{array}{l} \partial_4 \partial_4 \gamma_{\rho\sigma} = L_{\rho\sigma} (\partial_\rho \partial_4 \gamma_{\alpha\beta}, \partial_\rho \partial_4 \gamma_i, \mu_0) + N_{\rho\sigma} (\partial_4 \gamma_{\alpha\beta}, \partial_r \gamma_i) \\ \partial_4 \partial_4 \gamma_\alpha = \partial_4 \partial_\alpha \gamma_4 \\ \partial_4 \mu_0 = -\mu_0 \gamma^{\rho\sigma} \partial_4 \gamma_{\rho\sigma} \end{array} \right. \quad (5.3.2)$$

where $L_{\rho\sigma}$ and $N_{\rho\sigma}$ are polynomials, some linear, other non-linear functions in the

brackets, which are coefficients of the functions of $\gamma_i, \gamma_{\rho\sigma}, \gamma^i, \gamma^{\rho\sigma}$.

These equations suggest an intrinsic formulation of the problem of initial data and the restricted problem of evolution.

5.3.2 Formulation of initial data problem

Let us suppose given, in the differentiable four-dimensional manifold V_4 :

- a) a part of regular hypersurface $\bar{\Sigma}$;
- b) a three dimensional domain D on $\bar{\Sigma}$;
- c) $\forall Q \in D$ two scalar functions $\bar{\mu}_0(Q), p_0(\bar{\mu}_0)$ and a field of covariant vectors $\bar{\gamma}_i(Q)$;
- d) a part of hypersurface $\bar{\Sigma}$ that has a field of contravariant vectors γ^i , that are tangent to $\bar{\Sigma}$, and a field of covariant vectors γ_i which verify the condition

$$\gamma^i \gamma_i = -1, \quad \gamma^i (\partial_i \gamma_r - \partial_r \gamma_i) = 0, \quad (5.3.3)$$

and on $\bar{\Sigma}$

$$(\gamma_i)_{\bar{\Sigma}} = \bar{\gamma}_i. \quad (5.3.4)$$

Let us consider the vectorial space $\bar{\pi}_Q$ tangent to $\bar{\Sigma}$ in Q , and choose a system of coordinates (x^i) adapted to the field of vectors $\gamma^i(x)$ and to the hypersurface $\bar{\Sigma}$. Our aim is to determine, in $\bar{\Sigma}$, two symmetric tensors $\bar{\gamma}_{ij}, \bar{\varphi}_{ij}$ subjected to the conditions

$$\bar{\gamma}_{ij} \bar{\gamma}^j = 0, \quad \bar{\varphi}_{ij} \bar{\gamma}^j = 0 \quad (5.3.5)$$

$$(\bar{\gamma}_{ij} - \bar{\gamma}_i \bar{\gamma}_j) \bar{\xi}^i \bar{\xi}^j > 0, \quad \forall \bar{\xi}^r \in \bar{\pi}_Q \quad (5.3.6)$$

and satisfying the equations

$$\left\{ \begin{array}{l} \tilde{\nabla}_\alpha^* (\bar{\varphi}_\rho) - \tilde{\nabla}_\sigma^* (\bar{\varphi}_\alpha^\sigma + \bar{\omega}_\alpha^\sigma) = 0 \\ \bar{\gamma}^{\alpha\beta} \left(\tilde{P}_{\alpha\beta}^* \right)_{\bar{\Sigma}} + \frac{1}{4} [(\bar{\varphi}_\alpha^\alpha)^2 - \bar{\varphi}^{\alpha\beta} \bar{\varphi}_{\alpha\beta} + 3\bar{\omega}^{\alpha\beta} \bar{\omega}_{\alpha\beta}] = -2\chi \bar{\mu}_0 c^2. \end{array} \right. \quad (5.3.7)$$

5.3.3 Formulation of restricted evolution problem

Assuming to have solved the problem of initial data, we propose to determine, in a neighborhood of $\bar{\Sigma}$, a scalar function $\mu_0(x)$ and a double symmetric tensor field γ_{ij} satisfying

$$\gamma_{ij} \gamma^j = 0 \quad (5.3.8)$$

the following equations

$$\left\{ \begin{array}{l} \partial_4 \partial_4 \gamma_{\rho\sigma} = L_{\rho\sigma} (\partial_\rho \partial_4 \gamma_{\alpha\beta}, \mu_0) + N_{\rho\sigma} (\partial_4 \gamma_{\alpha\beta}) \\ \partial_4 \mu_0 = -\frac{1}{2} \mu_0 \gamma^{\rho\sigma} \partial_4 \gamma_{\rho\sigma} \end{array} \right. \quad (5.3.9)$$

and the following conditions on $\bar{\Sigma}$

$$(\bar{\mu}_0)_{\bar{\Sigma}} = \bar{\mu}_0, \quad (\gamma_{ij})_{\bar{\Sigma}} = \bar{\gamma}_{ij}, \quad (\gamma^4 \partial_4 \gamma_{ij})_{\bar{\Sigma}} = \bar{\varphi}_{ij}. \quad (5.3.10)$$

The problem so formulated has one and only one solution [?].

Chapter 6

Einsteinian gravitational field equations in the case of spherical symmetry

6.1 Definition of a manifold with spherical spatial symmetry

We are interested in giving the intrinsic definition of space-time manifold with spatial spherical symmetry [?]. For this reason, let be a Riemannian three-dimensional manifold V_3 , with positive definite metric $d\sigma^2$, and we consider in a its point O the tangent space, having an arbitrary orthonormal basis $(e_1, e_2, , e_3)$. Each of the ∞^2 geodesics outgoing from O , called geodesic rays, can be identified by two parameters: eg. θ , colatitude with respect to the polar axis e_3 , and φ , longitude, counted positively

around e_3 from e_1 . Then assigned a convex neighborhood U of O , P is the generic point of U and ρ its distance (geodesic) from O ; so it is introduced a system of polar coordinates ρ, θ, φ . Recall also that, denoting by S_2 the generic closed surface of equation $\rho = \text{const.}$, the geodesic rays, of equation $\rho = \text{var.}$, are normally intersected by surfaces S_2 .

There, we say that the Riemannian manifold V_3 has (in U) spherical structure around O - and this point will be called the *center of symmetry* of the manifold - when it occurs the following circumstance: the Riemannian metric $d\bar{\sigma}^2$, induced on each surface S_2 by the metric of V_3 , is the same metric of a sphere of Euclidean space,

$$d\bar{\sigma}^2 = a^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2)$$

whose radius a is a positive function, a priori any, of ρ , which satisfies the condition $a(0) = 0$.

It follows that the metric of more general three-dimensional Riemannian manifold, with spherical symmetry around a its point O , in polar coordinates takes the form

$$d\sigma^2 = d\rho^2 + a^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2). \quad (6.1.1)$$

Let us consider a 4-dimensional Riemannian manifold V_4 , with normal hyperbolic metric with signature $+ + + -$. We will say that this manifold is locally equipped with spatial spherical symmetry, around a its point O with a line of universe l , if it admits, in a neighborhood of l , a physical reference S for which the following conditions are verified:

- I. the reference S admits the space-time trajectory l of the point O between the

lines of the universe;

II. the reference S is irrotational, which implies the normal character of the congruence Γ that identifies it;

III. the spatial sections V_3 normal to the lines of S have spherical symmetry around O ;

IV. the bijective correspondence that the time-lines of S lead between any two sections V_3 leads center on center, geodesic rays on geodesic rays and O -spheres on O -spheres.

So, let us assume a time parameter t , constant on each of the V_3 ; the generic point P of V_4 can be identified by three spatial coordinates ρ, θ, φ in V_3 and by the value assumed by the parameter t on the same section.

This reference S and the coordinate system ρ, θ, φ, t associated with it are said adapted to the spherical character of V_4 . Thus, in adapted coordinates, the metric of a manifold V_4 , with spatial spherical symmetry, takes the form

$$ds^2 = A(r, t)dr^2 + R^2(r, t)(d\theta^2 + \sin^2 \theta d\varphi^2) - C(r, t)c^2 dt^2 \quad (6.1.2)$$

where A and C are positive functions of r and t .

6.2 Physical characteristics of the class of the frames of reference associated to Levi-Civita's curvature coordinates and gaussian polar coordinates

Let us consider the evolution of a perfect fluid which produces a spherically symmetric 4-manifold V_4 , by a state equation $p_0 = p_0(\mu_0)$.

As we saw in Chapter 2, the metric of V_4 can be written

$$ds^2 = e^{2\lambda(r,t)} dr^2 + Y^2(r,t) d\Omega^2 - e^{2v(r,t)} dt^2 \quad (6.2.1)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$.

and, therefore, Einstein's field equations, (??), and the conservation equations, (??) and (??), reduce to the following systems¹:

¹Gravitational coordinates $\chi = 1$, $c = 1$ will be used in the following.

$$\left\{ \begin{array}{l} \dot{Y}' - \dot{Y}v' - Y'\dot{\lambda} = 0 \\ \frac{2}{e^{2v}}\dot{\lambda}\dot{Y} + \frac{1}{Y^2}\left(1 + \frac{\dot{Y}^2}{e^{2v}}\right) - \frac{1}{e^{2\lambda}}\left[2\frac{Y''}{Y} + \left(\frac{Y'}{Y}\right)^2 - 2\lambda'\frac{Y'}{Y}\right] = \mu_0 \\ \frac{1}{e^{2\lambda}}\left[-\left(\frac{Y'}{Y}\right)^2 - 2\frac{Y'}{Y}v'\right] + \frac{1}{Y^2} + \frac{1}{e^{2v}}\left(2\frac{\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y} - 2\frac{\dot{Y}}{Y}\dot{v}\right) = -p_0 \\ \frac{1}{e^{2\lambda}}\left[-\frac{Y''}{Y} - v'' - (v')^2 - \frac{Y'}{Y}v' + \lambda'\frac{Y'}{Y} + \lambda'v'\right] + \\ + \frac{1}{e^{2v}}\left[\frac{\ddot{Y}}{Y} + \ddot{\lambda} + \left(\dot{\lambda}\right)^2 + \frac{\dot{Y}}{Y}\dot{\lambda} - \dot{v}\frac{\dot{Y}}{Y} - \dot{\lambda}\dot{v}\right] = -p_0 \end{array} \right. \quad (6.2.2)$$

$$\left\{ \begin{array}{l} \partial_\alpha p_0 + (p_0 + \mu_0)v' = 0 \\ \partial_4 \mu_0 + (p_0 + \mu_0)\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right) = 0, \end{array} \right. \quad (6.2.3)$$

where ' stays for ∂_r ; $\dot{}$ stays for ∂_t . Eq.(6.2.1) may be also expressed in two equivalent forms:

$$v' = \frac{e^\lambda}{\dot{Y}}\partial_t\left(\frac{Y'}{e^\lambda}\right) \text{ or } \dot{\lambda} = \frac{e^v}{Y'}\partial_r\left(\frac{\dot{Y}}{e^v}\right); \quad (6.2.4)$$

Introducing (??) into (??) we obtain equivalently:

$$\left\{ \begin{array}{l} \frac{1}{Y'Y^2} \partial_r \left[Y \left(1 + \frac{\dot{Y}^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) \right] = \mu_0 \\ \frac{1}{\dot{Y}Y^2} \partial_t \left[Y \left(1 + \frac{\dot{Y}^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) \right] = -p_0 \\ \frac{1}{2Y Y'} \partial_r \left\{ \frac{1}{\dot{Y}} \partial_t \left[Y \left(1 + \frac{\dot{Y}^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) \right] \right\} + \\ + \frac{1}{2Y'} \left[\frac{1}{\dot{Y}} \partial_t \left(\frac{\dot{Y}^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) - \frac{1}{Y'} \partial_r \left(\frac{\dot{Y}^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) \right] v' = -p_0 \end{array} \right. \quad (6.2.5)$$

Levi-Civita's curvature coordinates

Let us analyze the physical properties of the evolution of a perfect fluid, whose stream lines coincide with the class of reference associated to a system of curvature coordinates R_c . We can observe that the motions, whose stream lines coincide with those of R_c , are static. Indeed, in this case we have $\dot{Y} = \dot{\lambda} = 0$ in view of (??)₁. Conversely, r is comoving with the rigid motions of a perfect fluid, in view of $\dot{Y} = 0$.

Furthermore, if we take into account an equation of state of the kind $p_0 = p_0(\mu_0)$, we have that v' is time-independent, for (??)₁; and that the same holds for v , provided a suitable change of coordinates, internal to R_c , is made. Consequently, both R_c and its comoving motions are static. Thus we can say:

The rigid (static) motions of a perfect fluid (of a perfect fluid with a state equation $p_0 = p_0(\mu_0)$) are the ones and only the ones for which the associated comoving systems admit curvature coordinates.

We shall not investigate consequences of eqs. (??), (??), in the static case, since

there is no reason for splitting the initial data and the restricted evolution problems, where unknowns do not depend on time. We note only that eqs. (??)₁ and (??)₂ are, in this case, satisfied identically. No restriction on the equation of state follows, therefore, from the assumption of curvature comoving coordinates.

Gaussian polar coordinates

Let us now consider some physical properties of the evolution of a perfect fluid, whose stream lines coincide with the class of reference associated to a system of gaussian polar coordinates \bar{R}_p . The following first integral is deduced from (??)₂:

$$\frac{\dot{Y}}{e^v} = G(t) \tag{6.2.6}$$

where the function $G(t)$ can be chosen freely by choosing the scale of t . The term on the left hand side is the derivation of the distance with respect to proper time, i.e. what is usually interpreted as the velocity of the particles, because it is the velocity of the material O-spheres having radius ρ .

From the physical point of view, a first consequence of using \bar{R}_p comoving frames, in the case of perfect fluid, is that the above velocities are time independent. Further consequences can be discussed by separating the initial data and the restricted evolution problems.

6.2.1 Exact solutions in polar gaussian coordinates

The breaking of Cauchy problem in two different problems, the initial data problem and the restricted evolution problem affords an undoubted advantage experimented in more detail into three articles [?], [?] and [?]. In this section, we examine the results obtained with the use of polar gaussian coordinates.

So, we start to consider the initial condition problem and to take a regular hypersurface Σ , having eq. $t = \bar{t}_o$. The problem of initial data, as it is known [?], consists in the choice on Σ of the spatial metric tensor and of the deformation tensor satisfying $(??)_{2,3}$.

Since, in the case of spherical symmetry, the only non-vanishing components of γ_{ij} and \tilde{K}_{ij} may be evaluated respectively by the scalar function $v(r, t)$ and the scalar functions $\lambda(r, t)$, $Y(r, t)$, the initial data problem can be enunciated in this way:

a) state equation $p_0 = p_0(\mu_0)$.

We search for seven scalar functions² \bar{v} , \bar{Y} , $\bar{\lambda}$, $\bar{\chi}$, $\bar{\psi}$, $\bar{\mu}_o$, $\bar{p}_o = p_o(\bar{\mu}_o)$ satisfying the following equations on Σ :

$$\left\{ \begin{array}{l} \psi(r) = \frac{e^v}{Y'} \partial_r \left(\frac{\chi}{e^v} \right) \\ \frac{1}{Y'Y^2} \partial_r \left[Y \left(1 + \frac{\chi^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) \right] = \bar{\mu}_o \end{array} \right. \quad (6.2.7)$$

with conditions

$$\psi(r) = \dot{\lambda}(r, 0) \quad , \quad \chi(r) = \dot{Y}(r, 0) \quad , \quad p'_0 = \frac{dp_0}{d\mu_0} \neq 0 \quad (6.2.8)$$

²We put $\chi = \dot{Y}$ and $\psi = \dot{\lambda}$

The differential system (??) is obtained by eqs.(??)₂ and (??)₁; the relation ensuring that Σ is not a characteristic hypersurface.

Besides γ_{ij} and \tilde{K}_{ij} , initial data determine also $\tilde{\Omega}_{ij}$ and C_i on Σ . Indeed, $\tilde{\Omega}_{ij} = 0$, because of the orthogonal form of the metric and C_i can be so expressed:

$$C_i = -\frac{p'(\bar{\mu}_0)}{p_0(\bar{\mu}_0) + \bar{\mu}_0} \partial_1 \bar{\mu}_0 \delta_i^1 \quad (6.2.9)$$

where δ_i^j is the Kronecker symbol. Therefore, all the geometrical objects describing the first order differential properties of stream lines are assigned on Σ .

b) state equation $p_0 = 0$.

The initial data problem, in the case of dust ($p_0 = 0$), has been discussed by Laserra [?]. Analogous results hold under the assumption $p_0 = \text{const}$. In consequence, we just briefly recall that, in this case, it is possible the use of proper time as temporal coordinate (hence $v = 0$). So only five scalar functions $\bar{\lambda}$, \bar{Y} , $\bar{\chi}$, $\bar{\psi}$, $\bar{\mu}_0$ satisfying eqs.(??), with vanishing v , have to be chosen.

The solution of the initial data problem requires the solution of the two non linear equations (??). However the problem can be solved in general, because it is possible to remove the only non linear term Y' appearing in eq. (??)₂, adequately choosing the radial coordinate. Indeed, if we choose the intrinsic radius of the O -spheres³ of Σ as radial coordinate (which is always possible through an internal change of coordinates⁴

³We call O -spheres, the spheres of Σ having center at the symmetry center of V_4 and an assigned geodetic radius ρ

⁴An internal change is of the following type [?]: $r' = r'(r)$; $\theta' = \theta$; $\varphi' = \varphi$; $t' = t'(t)$

we obtain that $Y(r, 0) = r$ on Σ and, in general, only on Σ . Then eq. (??)₂ yields (see also [?]):

$$\partial_r \left[r \left(1 + \frac{\bar{\chi}^2}{e^{2\bar{v}}} - \frac{1}{e^{2\bar{\lambda}}} \right) \right] = r^2 \mu_0 \quad (6.2.10)$$

the integration of which gives

$$r \left(1 + \frac{\bar{\chi}^2}{e^{2\bar{v}}} - \frac{1}{e^{2\bar{\lambda}}} \right) = m(r) \quad (6.2.11)$$

where:

$$m(r) \equiv \int_{r_0}^r s^2 \mu_0(s) ds. \quad (6.2.12)$$

Eq. (??) is no longer differential but algebraic. Let $e^{\bar{\lambda}}$, r , $e^{\bar{v}}$, $\bar{\chi}$, $\bar{\mu}_0$ be a solution. The choice of the initial data can then be completed by the following function:

$$\bar{\psi}(r) = e^{\bar{v}} \partial_r \left(\frac{\bar{\chi}}{e^{\bar{v}}} \right). \quad (6.2.13)$$

So, we consider state equation $p_0 = p_0(\mu_0)$.

The spatial metric tensor, the deformation tensor and the density, coincident on Σ with the initial data, and satisfying eqs.(??)₁, (??) and (??) during the evolution, have to be determined. Four scalar functions are then to be found:

$$\lambda(r, t), \quad Y(r, t), \quad v(r, t), \quad \mu_0(r, t). \quad (6.2.14)$$

They must be solution of (??)_{2,3} and (??) and (??) in a suitable neighbourhood of Σ , and must also satisfy the conditions posed on Σ by the initial data:

$$\left\{ \begin{array}{l} (\lambda)_{\Sigma} = \bar{\lambda}, (Y)_{\Sigma} = \bar{Y} \\ (v)_{\Sigma} = \bar{v}, (\mu_0)_{\Sigma} = \bar{\mu}_0 \\ (\psi)_{\Sigma} = \bar{\psi}, (\chi)_{\Sigma} = \bar{\chi}. \end{array} \right. \quad (6.2.15)$$

(For the case $p_0 = 0$ see [?]).

It is not possible, without further hypotheses, to obtain an analytic expression for the solution of the restricted evolution problem. Nevertheless, the compact form (??) assumed by eqs. (??) points out more clearly the structure of the problem and allows to choose more adequately in which order the equations should be integrated. In this respect, the key equation is the one we may get by comparing eqs. (??)_{1,2}, namely:

$$\partial_t (\mu_0 Y' Y^2) = -\partial_r (p_0 \dot{Y} Y^2) \quad (6.2.16)$$

Three remarks about this equation are in order:

Remark 6.2.1 *As we have already seen, eq. (??) comes from eqs. (??)_{1,2}. On the other hand, eqs. (??) and (??)_{1,2} yield:*

$$\partial_t \left\{ \mu_0 Y' Y^2 - \partial_r \left[Y \left(1 + \frac{\dot{Y}^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) \right] \right\} = 0 \quad (6.2.17)$$

It is sufficient, therefore, that eq. (??)₁ is initially satisfied in order to be sure that it

will be satisfied at every instant t .

Remark 6.2.2 Eq. (??) also follows from the conservation equations on account of eq. (??).

Remark 6.2.3 Eq. (??)₃ follows from the conservation equations because of (??)_{1,2} and viceversa.

That being considered, for the restricted evolution problem the following equations can be chosen:

$$\left\{ \begin{array}{l} v' = \frac{e^\lambda}{Y} \partial_t \left(\frac{Y'}{e^\lambda} \right) \text{ or } \dot{\lambda} = \frac{e^v}{Y'} \partial_r \left(\frac{\dot{Y}}{e^v} \right) \\ \frac{1}{Y Y^2} \partial_t \left[Y \left(1 + \frac{\dot{Y}^2}{e^{2v}} - \frac{(Y')^2}{e^{2\lambda}} \right) \right] = -p_0 \\ \partial_r p_0 + (p_0 + \mu_0) v' = 0 \\ \partial_t \mu_0 + (p_0 + \mu_0) \left(\dot{\lambda} + 2 \frac{\dot{Y}}{Y} \right) = 0 . \end{array} \right. \tag{6.2.18}$$

In fact, eq. (??) follows from (??)_{1,3,4} as stated by remark 2. Eq. (??)₂, therefore, if initially satisfied, will be satisfied at every instant t , as stated by remark 1. Eq. (??)₃ follows from eqs. (??), as stated by remark 3.

For this problem, an existence and uniqueness theorem is proved to be valid, under the only assumption of differentiable data (see [?] and in particular [?]).

Initial data problem

In order to solve the initial data problem, we begin with the solution of (??)₂ in curvature coordinates:

$$e^{2\lambda} = \frac{1}{1 - \frac{2m}{r} + \frac{\chi^2}{e^{2v}}}; \quad Y = r; \quad e^{2v} = h^2(r) \quad (6.2.19)$$

where the function $h(r)$ depends only on r . With the help of the transformation from curvature to gaussian polar coordinates [?]:

$$\rho = \int_{r_0}^r e^{\lambda(s)} ds \quad (6.2.20)$$

we easily obtain the solution:

$$e^{\bar{\lambda}} = 1; \quad \bar{Y} = r_\rho; \quad e^{\bar{v}} = g(\rho) \quad (6.2.21)$$

where the function g is the transform of h in (??). Because of (??), the choice of the initial data is completed by:

$$\bar{\psi} = 0 \quad ; \quad \bar{\chi} = g(\rho)G(0) \quad (6.2.22)$$

In two interesting cases (??) can be integrated easily:

I) $\frac{2m}{r} = \text{const}$ (euclidean initial hypersurface);

II) $\mu_0 = \text{const}$ (uniform density).

In the latter case, we have $r = h \sin(\rho)$ (assuming $1 + \frac{\bar{x}}{e^{2v}} = h$ and $\frac{\mu_0}{3} = 1$). In the former case, we have simply $r = \tau \rho$; i.e. r and ρ coincide except for the const. τ that can be always chosen equal to unity by choosing the radial unity.

Restricted evolution problem

From now, we will use r in place of ρ for indicating the geodetic radius. We consider the system of equations which may be obtained in equivalent way from (??) (see sect.3) by substituting (??)₃ for (??)₃. We are going to solve the problem under the auxiliary assumption that r and t are separable in Y :

$$Y(r, t) = x(r) y(t) \quad (6.2.23)$$

Such a restriction will be discussed at the hand of the present section.

From (??)₂, namely from the first integral (??), we have: $e^v = x(r) f(t)$ ($f(t)$ can be chosen freely by choosing the scale of t). Elimination of p_0 from (??)₂ and (??)₃ gives:

$$\frac{1}{2y^3 \dot{y}} \partial_t \left[y^2 \left(1 + \frac{\dot{y}^2}{f^2} - x'^2 y^2 \right) \right] = \frac{x}{x'} \partial_r (x')^2 \quad (6.2.24)$$

Separation of variables gives the two following differential equations:

$$\left\{ \begin{array}{l} \partial_t \left[y^2 \left(1 + \frac{\dot{y}^2}{f^2} - \frac{c}{2} y^2 \right) \right] = 0 \\ \partial_r (x^2 x'^2) = \frac{c}{2} \partial_r x^2 \end{array} \right. \quad (6.2.25)$$

where c is an integration constant. By integration, the equations (??) give immediately

$$\begin{cases} y^2 \left(1 + \frac{\dot{y}^2}{f^2} - \frac{c}{2}y^2 \right) = d \\ x^2 x'^2 = \frac{c}{2}x^2 + k \end{cases} \quad (6.2.26)$$

where d and k are constants, and k is equal to zero in view of the regularity condition at the center $Y(0, t) = 0$ [?].

The constant c may be different from zero or equal to zero.

Let us first consider the case $c \neq 0$.

$c/2$ can be made equal to unity, by adapting the scale of t . Hence, from eq. (??)₂ we obtain $x(r) = r$ and $Y(r, t) = ry(t)$. In consequence, every space like hypersurface generated by the fluid during its evolution is euclidean. Then, from eq. (??)₁ we obtain the following family of solutions:

$$\sqrt{z^2 - z + \bar{d}} + z - \frac{1}{2} = \bar{d}e^{[\pm 2 \int f dt]} \quad (6.2.27)$$

where $z = y^2$ and \bar{d} is a constant. The solution obtained by Wesson in [?]:

$$\sqrt{z^2 - z + \bar{d}} + z - \frac{1}{2} = \bar{d}t^{\pm 2} \quad (6.2.28)$$

and the solution obtained by Gutman in [?] (mentioned also in [?]):

$$z = \frac{1}{2} + Ae^{+t} + Be^{-t} \quad (6.2.29)$$

where A and B are const., are subclasses of the above family. Indeed, (??) and (??) may be obtained by putting into (??) respectively $f = 1/t$, $f = 1/2$. Such solutions are not physically different from each other, as they differ only in the choice of the time scale factor $f(t)$.

Let us now calculate p_0 and μ_0 . We obtain from (??)₂:

$$p_0 = \frac{N(t)}{r^2} \quad (6.2.30)$$

where $N(t) \equiv -\frac{\partial_t \left[y \left(1 + \frac{y^2}{f^2} - y^2 \right) \right]}{y\dot{y}}$

and from (??)₃:

$$\mu_0 = \frac{N(t)}{r^2} \quad (6.2.31)$$

Our original assumptions then imply the following state equation: $p_0 = \mu_0$; i.e. a special case of the γ -law that is of interest in cosmology .

Let us now consider the case $c = 0$.

From (??)₂ we have $x' = 0$: hence $v' = 0$ and, because of (??)₃, $p_0 = \text{const.}$

An exact solution can be obtained also in this case. Indeed, putting $c = 0$ into (??)₁, we obtain the following differential equation:

$$y^2 \left(1 + \frac{\dot{y}^2}{f^2} \right) = d \quad (6.2.32)$$

that is integrated by:

$$y = \pm d^{1/25} \quad y^2 = d - \left[\pm \int f dt + \bar{d} \right]^2 \quad (6.2.33)$$

⁵This solution is static.

with \bar{d} is constant. Then μ_0 can be computed by (??)₄:

$$\mu_0 = \bar{\mu}_0 \quad \text{or} \quad p_0 + \mu_0 = \frac{F(r)}{d - \left(\pm \int f dt + \bar{d} \right)^2} \quad (6.2.34)$$

where the integration function $F(r)$ is fixed by the initial data.

As regards the solution formerly obtained in the case $c = 0$, let us make two remarks.

Remark 6.2.4 *Wesson has obtained the family of solutions (??) under the following assumptions.*

(a) *In the first place he assumes the validity of the so called Dimensional Cosmological Principle*

$$\mu_0 Y^2 = \eta(t/t_0); \quad \frac{M}{Y} = P(t/t_0); \quad Y = rS(t/t_0) \quad (6.2.35)$$

where η , P , S are dimensionless arbitrary functions and M is defined by:

$$M \equiv Y \left(1 + \frac{\dot{Y}^2}{e^{2v}} - \frac{Y'^2}{e^{2\lambda}} \right) [?] \quad (6.2.36)$$

(a') *In the second place he considers the state equation $p_0 = \mu_0$ and adequately chooses the scale of time and part of the initial data.*

The same family (??) has been obtained by us, as a particular case of (??), under the following other assumptions:

(b) *The use of gaussian polar comoving coordinates and the separation of r and t in Y .*

It is not difficult to prove that the assumptions (a) and (b) are equivalent (this is the reason why the solution (??) has been obtained both by Wesson's and by our method).

Indeed, if the assumptions (b) hold, we have that $x(r) = r$ (as above proved) and that the ratio M/Y , which is expressed by:

$$\frac{M}{Y} = 1 + \frac{\dot{y}^2}{f^2} - y^2 \quad (6.2.37)$$

does not depend on the radius. The assumptions (a) are then satisfied (we recall that eqs. (??)_{1,2} are equivalent).

Conversely, if the assumptions (a) hold, we have that $e^v = rf(t)$, because of eqs.(??)_{2,3}. Thus, the ratio \dot{Y}/e^v does not depend on the radius and comoving gaussian polar coordinates, in view of (??)₂, can be constructed. The assumptions (b) are then satisfied.

We realize, by virtue of these considerations, that the further assumptions (a') may be omitted also in the method followed by Wesson for obtaining the solution (??).

Remark 6.2.5 *Two kinds of hypotheses, as shown in the preceding remark, allow for the obtainement of the family of solutions (??) the hypothesis of comoving gaussian polar coordinates or the hypothesis of $p_0 = \mu_0$ state equation.*

Concerning the first, we have discussed comoving gaussian polar coordinates by separating variables r and t . This separation of variables is a particular case of another that may be proved to be valid in the most general case, namely $Y = Q(v)h(r)$ with $v = f_1(r)t + f_2(r)$ and $\mu_0 = \mu_0(v)$. With regard to the second one, the $p_0 = \mu_0$

equation of state has been considered under only one restrictive assumption which, as we are going to prove, consists in the choice of gaussian polar comoving coordinates, or equivalently, M/Y independent of r .

In fact, choosing gaussian polar coordinates for $p_0 = \mu_0$ fluid and using conservation laws, we have:

$$\frac{Y^4}{e^{2v}} = \frac{\bar{L}(r)}{\bar{H}(t)} \quad (6.2.38)$$

where $\bar{L}(r)$ and $\bar{H}(t)$ are integration functions. Then, eliminating e^v with the aid of (??), we obtain:

$$\frac{\dot{Y}}{Y^2} = H(t)L(r) \quad (6.2.39)$$

where $L(r)$ and $H(t)$ need not be specified. Integration of (??) gives that r and t have to be separated in Y . So all assumptions (b) hold and, as we have above seen, M/Y does not depend on the radius.

Otherwise, choosing M/Y independent of r , comparing (??)_{1,2}, (with $p_0 = \mu_0$) and putting M/Y in evidence, we obtain the following differential equations:

$$\frac{Y}{Y'} \left(\frac{M}{Y} \right)' = - \left[\left(\frac{\dot{M}}{Y} \right) \frac{Y}{\dot{Y}} + 2 \frac{M}{Y} \right] \quad (6.2.40)$$

Being $\left(\frac{M}{Y} \right)' = 0$, an easy integration gives $M/Y = g(r)/Y^2$. This means that r and t have to be separated in Y : hence $\left(\frac{M}{Y} \right)' = 0$ implies $x(r) = r$. In this way, holding all assumptions (a), comoving gaussian polar coordinates can be constructed.

Further families of solutions can be obtained without the above assumption on M/Y , both with Y again of the type $Y = x(r)y(t)$.

The separation of variables $Y = x(r)y(t)$ is indeed a necessary but not a sufficient

condition in order that M/Y is only time dependent (in the case of $p_0 = \mu_0$ state equation). On this respect, comparing again eqs. (??)_{1,2}, we obtain the following first order partial differential equation:

$$U(r) \left(\frac{M}{Y} \right)' + V(t) \left(\frac{\dot{M}}{Y} \right) = \frac{M}{Y} \quad (6.2.41)$$

The integration of the latter gives for M/Y the form:

$M/Y = g_1(r)g_2(v)$ with $v = \alpha(t) + \beta(r)$, which is not a function dependent only on time. The investigation of the latter form for M/Y might hopefully give further results.

6.3 Physical frames of reference associated to isotropic coordinates

In this section, we study the evolution of a perfect fluid, whose stream lines coincide with the class of reference associated to a system of isotropic coordinates. Through the separate invariant formulation of the initial data problem and the restricted evolution problem, we analyze shear free motion which $p = p(\mu)$ or $p = 0$. All the possible solutions of the restricted evolution problem are determined by specifying all the admissible choice of the data in the case of shear free. Furthermore, if the characteristics of the motions are initially valid, they are valid as time evolves.

6.3.1 General zero shear consequences with no state equation

The Einstein's equations and conservations equations (??), written in the case of spherical symmetry using isotropic coordinates, become[?]

$$\left\{ \begin{array}{l} \frac{\dot{Y}}{e^v} = Y e^{f(t)} \\ \mu = 3e^{2f(t)} - e^{-2\lambda} \left(2\lambda'' + \lambda'^2 + 4\frac{\lambda'}{r} \right) \\ p = \dot{\lambda}^{-1} e^{-3\lambda} \partial_t \left[e^\lambda \left(\lambda'^2 + 2\frac{\lambda'}{r} \right) - e^{3\lambda+2f} \right] \\ p = \dot{\lambda}^{-1} e^{-3\lambda} \partial_t \left[e^\lambda \left(\lambda'' + \frac{\lambda'}{r} \right) - e^{3\lambda+2f} \right] \\ p' = -(\mu + p) \frac{\dot{\lambda}}{\lambda} \\ \dot{\mu} = -3(\mu + p) \dot{\lambda} \end{array} \right. \quad (6.3.1)$$

Next we consider other three useful equations:

$$\left\{ \begin{array}{l} e^\lambda (\lambda'' - \lambda'^2 - \lambda'/r) = -\tilde{F} \\ \dot{\lambda}'' - \left(2\lambda' + \frac{1}{r} \right) \dot{\lambda}' - \dot{\lambda} e^{-\lambda} \tilde{F} = 0 \\ \mu' = 2/r e^{-3\lambda} \left[\tilde{F}' r + 3\tilde{F} \right] \end{array} \right. \quad (6.3.2)$$

where $\tilde{F} = \tilde{F}(r)$.

The first is obtained by eliminating p from equation (??)_{3,4}; the second is obtained by differentiating Eq.(??)₁ with respect to t ; the third is finally obtained by differentiating equation (??)₂ with respect to r and eliminating λ'' and λ''' with the aid of equation (??)₁.

Eqs.(??)₂ and (??), on account of the transformations $L = e^{-\lambda}$, $x = r^2$, $F(x) = \tilde{F}/4x$, may assume the following more tractable forms:

$$\left\{ \begin{array}{l} \mu(x) = 3e^{2f(t)} + 8xLL_{xx} + 12LL_x - 12xL_x^2 \\ L_{xx} = L^2F(x) \\ \dot{L}_{xx} = 2L\dot{L}F(x) \\ \mu_x = 4L^3(2xF_x + 5F) \end{array} \right. \tag{6.3.3}$$

where $F(x)$ is a geometrical invariant, which measures the geometrical field energy [?].

Hereafter we will consider the initial data and the restricted evolution problem. By using comoving isotropic coordinates, the special simpler form of Eqs.(??) gives another more suitable choice of the equations, which we are going to discuss.

Initial data problem

The equations of the initial data problem are (??)_{1,2}, but, in this case, also the other equations have a spatial character in the sense that there are not derivatives of the initial data with respect to t , and the function F depends only on the radius. In consequence, all the equations at our disposal have to be taken into account in the choice of the initial data. It should be noted that, in this case, the initial data can be determined by the following functions:

$$\left\{ \begin{array}{ll} \lambda(r, \bar{t}) \stackrel{def}{=} \bar{\lambda}(r) & ; \quad \dot{\lambda}(r, \bar{t}) \stackrel{def}{=} \bar{\psi} \\ \mu(r, \bar{t}) \stackrel{def}{=} \bar{\mu}(r) & ; \quad p(r, \bar{t}) \stackrel{def}{=} \bar{p}(r) \\ \dot{v}(r, \bar{t}) \stackrel{def}{=} \bar{\gamma}(r) & ; \quad \dot{p}(r, \bar{t}) \stackrel{def}{=} \bar{\pi}(r) \end{array} \right. \quad (6.3.4)$$

the other initial data being determined by the condition of zero shear (i.e. by $Y = e^{\lambda}r$, $\dot{Y}/Y = \dot{\lambda}$, and Eqs. (??)₁₋₆). We are going to show that only one quantity can be freely chosen where comoving isotropic coordinates are introduced. This quantity may be either $\bar{\lambda}$, or $\bar{\mu}$, or $\bar{\psi}$, or $F(r)$.

In order to prove this, we start by assigning $\bar{\lambda}$ (supposed twice differentiable with respect to r and one with respected to t). Then, because of $Y = e^{\lambda}r$, the spatial line-element is initially assigned. Next F may be evaluated by (??)₃: it is a second order ordinary differential equation which uniquely determines the value of $\bar{\psi}$ (provided $\bar{\psi}$, $\bar{\psi}'$, are assigned at $r = r_0$, in particular at the origin). The remaining quantities μ , v , p can be next respectively evaluated by Eqs. (??)₁, (??)₁ (??)₃.

The set of the initial data is thus entirely determined (except $\bar{\gamma}$ and $\bar{\pi}$ because of the lack of a state equation).

That being stated, it is possible to see that the choice either of $\bar{\mu}$, or of $\bar{\psi}$, or of F , uniquely individuates the value of $\bar{\lambda}$. In fact, if we choose either the density $\bar{\mu}$, or the gravitational field energy F , or the "velocity" of the particles $\frac{\dot{Y}}{e^v}$, $\bar{\lambda}$ may be uniquely determined respectively by Eqs. (??)₁, (??)₂, (??)₁ (provided $\bar{\lambda}$, $\bar{\lambda}'$ and $\bar{\psi}$, $\bar{\psi}'$ are assigned at the center).

The previous considerations allow us to conclude:

Proposition 6.3.1 *Where considering comoving isotropic coordinates, the initial data (except $\bar{\gamma}$, $\bar{\pi}$) may be entirely determined by assigning in equivalent way, on the initial hypersurface Σ , either the spatial metric, or the density, or the gravitational field energy, or the "velocity" of the particles.*

Restricted evolution problem

In this case it is possible to choose Eqs. (??)₁₋₄, (??)₂ as the equations of the restricted evolution problem.

It is known that the key equation of the problem is (??)₂ [?].

In [?] Stephany, improving a previous result [?], proved that the following choices of $F(x)$:

$$F = x^n, F = e^x, F = x^{-15/7}, F = (ax^2 + 2bx + c)^{-5/2}$$

enables us either to reduce Eq. (??)₂ to a first order differential equation to obtain exact solutions.

On account of the above considerations on the problem of the initial data, we may

realize that these special choices of $F(x)$ give rise to a complete assignement of the data. But such restrictions in assigning the initial data have not generally physical meaning. Hence, in the following, we confine ourselves to the study of those cases in which an equation of state is valid.

Into the present section, we are going to discuss more in detail only fluid distributions with initially uniform density.

On this respect, let us firstly calculate the value of $F(x)$. Choosing $f(\bar{t})$ so that $\bar{\mu} - 3e^{f(\bar{t})} = 0$ (which is always possible), it is easy to see that Eq. (??)₂ admits the following solution: $e^\lambda = \left(-\frac{c}{r} + k\right)$ where c and k are integration constants. Substituting the latter solution into (??)₂ we obtain $F = (cx)^{-5/2}$. Hence, from the Eq. (??)₄ we obtain $\mu_x = 0$: i.e. the density is uniform also during the entire evolution. Observing also that the solution $e^\lambda = \left(-\frac{c}{r} + k\right)$ is regular at the center only if $c = 0$ (namely only if the initial space-like hypersurface is euclidean), we conclude:

Proposition 6.3.2 *If a shear free spherically symmetric distribution in General Relativity is uniform at an initial time ($\bar{\mu} = \text{const}$), then the density is uniform also during the entire evolution of the system. In particular, if the above distribution is also regular at the center, then the initial space-like hypersurface and every subsequent ones are euclidean.*

6.3.2 Shear free fluid distributions with one-parameter state equation

Let us consider the density not uniform, by means of conservation equations and of the condition of functional dependence for μ and p we obtain [?]:

$$\mu(v) \quad L = u(v)l(x) \quad v = t + G(x) , \quad (6.3.5)$$

where a t -coordinate normalization has been used. Then inserting (??) into (??)_{2,4} we have:

$$\begin{cases} u_v \frac{1}{l} (G_x l^2)_x + u l_{xx} + u_{vv} l G_x^2 = u^2 l^2 F \\ u_v G_x = 4u^3 l^3 (2x F_x + 5F) \end{cases} \quad (6.3.6)$$

Separating the variables, we get the following equations:

$$\begin{cases} \frac{1}{l} (G_x l^2)_x = c_1 l^2 F \\ l_{xx} = c_2 l^2 F \\ l G_x^2 = c_3 l^2 F \\ 2x F_x + 5F = c_4 G_x l^{-3} \\ c_1 u_v + c_2 u + c_3 u_{vv} = u^2 \\ u_v = c_4 u^3 , \end{cases} \quad (6.3.7)$$

where c_1, c_2, c_3, c_4 are separation constants. These equations are valid also in the case of uniform density. In fact, in this case, using conservation equations, we see that the simple separation of variables holds: $L = u(t)l(x)$ (see [?]), which may be considered as a particular case of Eqs.(??) for $G_x = 0$. Further, we obtain $F = cx^{-5/2}$

from Eq. (??)₄. Thus the Eqs. (??), for $c_1 = 0$, are valid also in this case.

That being stated, it is possible to prove also that $(G_x l^2)_x = 0$ (see Appendix). Thus Eq. (??)₁ can be replaced by the following equation:

$$G_x l^2 = \alpha \quad (6.3.8)$$

where α is an integration constant.

We shall now try to find all the admissible determinations of the initial data, by finding all the admissible determination of F . On this respect, we eliminate G_x and l from Eq. (??)₄ with the use of (??)₃,(??). As a result we obtain the following linear ordinary differential equations:

$$2xF_x + \lambda F = 0 \quad ; \quad \lambda \equiv 5 - \frac{c_4 c_3}{\alpha} \quad (6.3.9)$$

which is easily integrated by $F = \beta x^{-\lambda/2}$, where β is an integration constant. The admissible choices of the constant λ may be now determined by comparing two expression of l_{xx} .

The first, which comes from Eq. (??)₃ eliminating $G_x = \alpha/l^2$ and $F = \beta x^{-\lambda/2}$:

$$l_{xx} = h_1 \frac{\lambda}{10} \left(\frac{\lambda}{10} - 1 \right) x^{\frac{\lambda}{10}-2} \quad (6.3.10)$$

where $h_1 = \left(\frac{\alpha^2}{c_3 \beta} \right)^{1/5}$.

The second, which comes from Eqs. (??)_{2,3} eliminating G_x as above:

$$l_{xx} = h_2 \frac{1}{l^3}, \quad (6.3.11)$$

where $h_2 = \frac{c_2 \alpha^2}{c_3}$.

Comparing (??) and (??) we obtain

$$h_1^4 \frac{\lambda}{10} \left(\frac{\lambda}{10} - 1 \right) x^{\frac{4}{10} \lambda - 2} = h_2 . \quad (6.3.12)$$

First we suppose $h_1 \neq 0$.

The admissible determinations of the constant λ are

$$\lambda = 0 \vee \lambda = 10 \text{ if } h_2 = 0$$

$$\lambda = 5 \quad \text{if } h_2 \neq 0$$

With these determinations of λ we have the following determinations of F ⁶

$$F = \beta \quad , \quad F = \beta x^{-5/2} .$$

Now we suppose $h_1 = 0$, we have $\alpha = 0$ and, consequently, $G_x = 0$. Therefore, because of the Eq. (??)₄ with $\mu_x = 0$, F can only assume the already considered determinations $F = \beta$, with $\beta = 0$ or $\beta \neq 0$, and $F = x^{-5/2}$.

Initial data problem.

Case $F = 0$

The solution of the Eq. (??)₂ is $L = \bar{B}(bx + 1)$, where \bar{B} and b are integration constants, and the density is uniform because of (??)₄. Likewise, the solution of the

⁶We disregard $F = \beta x^{-5}$ because it is the transform of $F = \beta$ in $r = 1/\bar{r}$.

Eq. (??)₃ is $\dot{L} = Cx + D$, where C and D are integration constants. This is consistent with the equation of state $p = p(\mu)$ (yielding $p = p(t)$ and $\lambda' = 0$ because of (??)₅) only if $C = 0$.

The initial data may be in conclusion chosen as follows:

$$e^{\bar{\lambda}} = \frac{1}{\bar{B}(bx+1)} \quad ; \quad \bar{\psi} = \frac{-D}{\bar{B}(bx+1)} \quad ; \quad \bar{\mu} = \bar{\mu}_0, \quad (6.3.13)$$

where $\bar{\mu}_0 = \bar{\mu}(0)$.

These data represent the assignment of a homogeneous and isotropic line-element; and in particular, for $b = 0$, the assignment of a spatially euclidean line-element.

Case $F = \beta$ ($\beta \neq 0$)

We may assume $\beta = 6$ by adapting the scale of the radius. In this case, therefore, the solution of the Eq. (??)₂ is $L(x) = P(x + \delta)$, where P is the Weierstrass elliptic function with invariants 0 e γ . The integration constants δ , γ are determined by the values of L and L_x at the center: $P(\delta) = L(0)$, $\gamma = 4L^3(0) - P_x^2(\delta)$.

In addition, if we choose $\bar{\psi}$ satisfying Eq.(??)₄, in which L is substituted by $P(x + \delta)$, and if we calculate $\bar{\mu}$ by means of (??)₁, we get the following set of data:

$$e^{\bar{\lambda}} = \frac{1}{P(x+\delta)} \quad ; \quad \bar{\psi} \quad ; \quad \bar{\mu} = 3e^{2f(0)} + 12(PP_x + \gamma x) \quad (6.3.14)$$

Case $F = x^{-5/2}$

There comes from the Eq.(??)₄ that the density is uniform.

We shall show below, considering the restricted evolution problem, that the solution

corresponding to the choice $F = x^{-5/2}$ is static. It is therefore useless to specify the set of the initial data.

Restricted evolution problem

Case $F = 0$

The unique solution of the restricted evolution problem, corresponding to the above mentioned data, may be obtained by solving Eq.(??)₂ as time evolves. We obtain $L = B(t)(bx + 1)$ (also taking into account that $p = p(t)$ gives $\dot{\lambda}' = 0$). In accordance with this solution, we generally obtain the well know Friedmann–Robertson–Walker model; in particular, if $c = 0$ (which correspondes to the case of initially euclidean space-like hypersurface) we obtain the Einstein De Sitter model. The homogeneity and isotropy of the line-element, and, in particular, the euclidean character of the spatial metric, hence, are characteristics preserved during the entire evolution of the system. The function $B(t)$ being still disposable, in this case there are no restrinctions, as it is well know, on the choice of the state equation.

Case $F = \beta$, $\beta \neq 0$.

The unique solution determined by the admissible initial data is the Wyman solution [?]. Indeed evaluating l , G_x with the help of Eqs.(??)₃ and (??), we see that $l = const$ and $G_x = const$. Next, adequately choosing the scale of r and t , we can assume $l = 1$, $G(x) = x/6$, $v = t + x/6$. Finally, calculating $u(v)$ with the help of Eq.(??)₅ (with $c_1 = c_2 = 0$), we obtain $L = P(v + \delta)$, where P is the above mentioned Weierstrass elliptic function with invariants $0, \gamma$ (the integration constants coinciding

with the ones of the initial data problem). Thus, we obtain the solution $L = P(x + \delta)$. This solution always allows for the existence of a state equation of the type $p = p(\mu)$ ⁷ [?].

It is to be pointed up that the equation of the type $p = p(\mu)$ is subject to a condition of consistence which can be easily derived as follows. Define a function $\sigma(\mu)$ through the following equation:

$$\frac{d\sigma}{d\mu} = \frac{\sigma}{p + \mu} \quad (6.3.15)$$

Next insert the following consequence of the conservation laws:

$$-\frac{u_v}{u} = \frac{\sigma}{p + \mu} \quad ; \quad u = \sigma^{1/3} l(x) h(x) \quad (6.3.16)$$

We can write (??)₅ in the following form:

$$\left(\frac{u_v}{u}\right)_v - \left(\frac{u_v}{u}\right)^2 = 6u \quad (6.3.17)$$

In view of (??) and for a suitable choice of the integration function $l(x)h(x)$, we finally obtain the consistency condition:

$$3\sigma \frac{d^2\sigma}{d^2\mu} + \left(\frac{d\sigma}{d\mu}\right)^2 = 6\sigma^{1/3} \quad (6.3.18)$$

⁷ Let us briefly recall that, in order to prove this, it is sufficient to prove that the density, where $L = P$, can always be given in the functional form $\mu(r, t) = \mu(v)$. This form, in fact, in view of Eq.(??)₆, is not only necessary but also sufficient for the existence of an equation of state $p = p(\mu)$.

Case $F = x^{-5/2}$.

In this case the solution is static. In fact, because of Eqs. (??)₄ and (??)₅ (with $c_1 = c_3 = 0$) we have $u = \text{const}$ and $G_x = 0$. Then λ e Y do not depend on time. Also the function v does not depend on time, up to a change of the t-scala, since the (??)₂ causes v to assume the following form: $v = f_1(r) + f_2(r)$.

In this case, therefore, a state equation trivially exists.

Resuming these results, we can say:

Proposition 6.3.3 *Where considering shear free spherically symmetric fluid distributions in General Relativity in which $p = p(\mu)$, the only two non static solutions allowed by the admissible initial data are either FRW or Wyman models.*

With regard to this proposition let us make two remarks.

Remark 6.3.4 *It has been proved by Mansouri⁸ [?] that, under the assumption of the Proposition 4., the space times in which there is a comoving space-like hypersurface $x = x_b$ of zero pressure are either static or FRW models. This result may also be more coincisely proved by using the method followed till now.*

In fact, if we write the Eq.(??)₂ at the boundary $x = x_b$ and if we take into account that $\mu(x_b, t)$ does not depend on time we have either $u(t) = \text{const.}$ or $\sigma_b = l_b = 0$ at x_b . In the first case, it follows $\dot{\lambda} = \dot{Y} = \dot{v} = 0$ (the latter $\dot{v} = 0$ up to a change of the t-scale, as shown above). In the second case, it follows $G_x = 0$, that is to say $\mu_x = 0$. Hence, from (??)₄ we obtain either $F = cx^{-5/2}$ or $F = 0$: i.e. necessarily FRW models, because the former choice of F given static models.

⁸A more transparent proof has been given by Glass [?].

Remark 6.3.5 *The results exposed into this section have been obtained under the assumption of shear vanishing during the entire evolution of the system. But, on account of the unicity theorem recalled in the introduction, the same result can be proved to be valid also under the less restrictive assumption of shear vanishing only at an initial time. This may be done in a sense that we are going to precise.*

On this purpose, let us assume that the shear vanished initially. Next, let us introduce on Σ a set of isotropic coordinates (which is always possible through an internal change of coordinates). Thus all equations considered for the problem of the initial data are still valid if we assume (with an additional hypothesis) the validity of the condition $(??)_2$ on the “velocity” of the particles. As a consequence, if on Σ the shear vanishes and Eq. $(??)_2$ is assumed to be valid, the admissible initial data coincide with those already considered in the case of shear vanishing during the entire evolution. Hence, because of the unicity theorem, also the solutions of the restricted evolution problem must coincide with those already considered.

Finally, taking into account that the Eq. $(??)_2$ is identically satisfied in the case of uniform density ($\dot{\lambda}' = F = 0$), we conclude:

Proposition 6.3.6 *During the entire evolution of a spherically symmetric system in which $p = p(\mu)$, the shear vanishes if and only if at an initial time the shear vanishes and the “velocity” of the particles satisfy the condition $(??)_2$. In particular, this condition can be omitted in the case of initially uniform density.*

Let us conclude with an application of such proposition to the cosmological context.

We have shown that only two dynamical models are consistent with the assumption

of zero shear: i.e. the FRW and Wyman ones. But the latter seems to be unrealistic since it is possible to check, by direct inspection, that it is consistent with the equations of state generally considered in cosmology: $p = \gamma\mu$ (γ -law) and $p = k\mu^h$ (polytropic law) only if $\gamma = -1/6$ or $k = 0$. In the first case, we obtain $p < 0$, which seems must be excluded for physical reasons. In the second case, we obtain $p = \mu = \text{const}$, which gives static models ⁹.

These considerations and propositions ?? and ?? allows us to say:

Proposition 6.3.7 *With regard to centrally symmetric systems in which $p = p(\mu)$ it is sufficient to assume the validity of the Hubble's law at an initial time to ensure the validity of the law, and the uniformity of the density, during the entire evolution.*

6.3.3 Zero shear consequences in the case of state equation

$$p = 0.$$

Initial data problem.

From the first conservation equation and $p = 0$ we have: $\dot{\lambda}' = 0$.

The Eq. (??)₂ then gives either $F = 0$ or $\dot{\lambda} = 0$. The initial data, consequently, have to be chosen in the following way (see also the case $F = 0$ of the precedent section).

Case $F = 0$.

$$e^{\bar{\lambda}} = \frac{1}{\bar{B}(bx + 1)} \quad ; \quad \bar{\psi} = \frac{-D}{\bar{B}(bx + 1)} \quad ; \quad \bar{\mu} = \bar{\mu}_0 \quad (6.3.19)$$

⁹Concerning the unrealistic character of the Wyman models see also [?]

Case $\dot{\lambda} = 0$.

We are showing below, dealing with the restricted evolution problem, that in this case the solution is static.

Restricted evolution problem.

The solutions uniquely determined by the admissible initial data may be easily specified as follows.

Case $F = 0$.

Analogously to the case $F = 0$ considered in the precedent section, the FRW solution is the only admissible one as time evolves.

Case $\dot{\lambda} = 0$. The first conservation equation gives $v' = 0$. Hence, as above, the solution is static.

With this in mind, a reasoning similar to that made at the hand of the precedent section enables us to state the following other proposition (recalling also that, in this case, the density is uniform):

Proposition 6.3.8 *During the entire evolution of centrally symmetric dust, the Hubble's law is valid if and only if it is valid at an initial time.*

6.4 Physical aspects of the reference frames in harmonic coordinates

6.4.1 Relative expressions of the harmonicity conditions

This section is devoted to relatively express the first condition of harmonicity (??).

To this aim, we consider the following decomposition of $\Delta\varphi$

$$\Delta\varphi = \text{Div}(\widetilde{\text{grad}} \varphi) - \text{Div}(\bar{\partial}_4 \varphi \gamma) . \quad (6.4.1)$$

Using the natural projections of the covariant derivative of time-like and space-like tensor fields, (??) can be written as:

$$\Delta\varphi = \tilde{\Delta}^* \varphi + C^i \tilde{\partial}_i \varphi - \left[(\gamma^4)^2 \partial_4^2 \varphi + \left(\partial_4 + \frac{1}{2} \tilde{K}_i^i \right) \gamma^4 \partial_4 \varphi \right] \quad (6.4.2)$$

where the spatial laplacian operator $\tilde{\Delta}^*$ is constructed with the use of the spatial metric tensor $\gamma_{\alpha\beta}$ and of the transverse partial derivative.

To express relatively the second condition of harmonicity (??), we have to take into account the natural projections of Christoffel symbols, explicitly computed in [?]:

$$\Gamma_h = g_{hk} \Gamma^k = -g^{ij} (ij, h) = -\tilde{\Gamma}_h^* + C_h + \frac{1}{2} \gamma_h \left(\tilde{Q}_i^i - 2\partial_4 \gamma^4 - \tilde{K}_i^i \right) ; \quad (6.4.3)$$

where we have set

$$\tilde{\Gamma}_h^* = \widetilde{(ij, h)}^* \gamma^{ij} \quad ; \quad \tilde{Q}_i^i = \gamma_4 \left[\tilde{\partial}_i \left(\frac{\gamma_j}{\gamma_4} \right) + \tilde{\partial}_j \left(\frac{\gamma_i}{\gamma_4} \right) \right] \gamma^{ij} .$$

In other words, the condition of harmonicity is equivalent to:

$$\begin{cases} \tilde{\Gamma}_h^* = C_h , \\ \partial_4 \gamma^4 = \frac{1}{2} (\tilde{Q}_i^i - \tilde{K}_i^i) . \end{cases} \quad (6.4.4)$$

The operators Δ and $\tilde{\Delta}^*$ are coincident in the statical case if and only if $C^i = 0$. The same operators never can coincide in dynamical cases because γ^4 is different from zero. In order to compare solutions of these two operators, let us introduce the following definition.

Definition 6.4.1 *A system of coordinates is called spatially harmonic if it is “harmonic” in accordance either with the following equations*

$$\tilde{\Delta}^* x^\alpha = 0$$

or with the following other

$$\tilde{\Gamma}^{*\alpha} = 0.$$

An harmonic system of reference, as it is evident, is also spatially harmonic if the curvature vanishes. On the contrary, considered a spatially harmonic frame of

reference, the question arises whether this reference system may be harmonic also in the usual sense of Eqs. (??), (??). To this aim, we note that in this case Eq. (??)₁ is satisfied if and only if the curvature vanishes. Hence, being geodesic the congruence of reference, it is always possible to arrange γ^4 such that $\partial_4\gamma^4 = 0$, by using internal Gaussian coordinates. Consequently, Eq. (??)₂ is satisfied if and only if $\tilde{K}_i^i = \tilde{Q}_i^i$. Collecting all these results, we can conclude:

- i. A frame of reference, that is harmonic and geodesic, is also spatially harmonic.
- ii. A frame of reference, that is spatially harmonic, geodesic and such that $\tilde{K}_i^i = \tilde{Q}_i^i$, is also entirely harmonic.

The importance of these considerations lies in the possibility to control the global harmonic character of a metric only by analyzing the spatially harmonic character of its space-like sections, in the case where the curvature and expansion vanish.

6.4.2 Harmonicity conditions and the initial data problem

In this section we want to formulate the Cauchy problem for a perfect fluid with state equation $p = p(\mu)$ and $p = 0$ (dust case).

Case $p = p(\mu)$

Let's consider a perfect fluid with one parameter state equation $p = p(\mu)$. First of all, let's recall that all the general projection of the Einstein Equations for the evolution

of a perfect fluid can be written as [?, ?]:

$$\begin{cases} s_{\alpha\rho} \equiv \mathcal{P}_{\Sigma\Sigma}(R_{\alpha\rho}) \quad , \quad \left({}^*s_{4h} = 0 \right) ; \\ \bar{S}_\alpha = 0 ; \\ \tilde{R} + \mathcal{I} = -2\mu ; \end{cases} \quad (6.4.5)$$

where $R_{\alpha\rho}$ is the Ricci tensor,

$$\bar{S}_\alpha \equiv \frac{1}{2} \left[\tilde{\nabla}_\alpha \tilde{K}_\nu{}^\nu - \tilde{\nabla}_\beta \left(\tilde{K}_\alpha{}^\beta + \tilde{\Omega}_\alpha{}^\beta \right) \right] + C^\beta \tilde{\Omega}_{\beta\alpha} ; \quad (6.4.6)$$

and the scalar invariants \tilde{R} and \mathcal{I} have respectively the expressions

$$\begin{aligned} \tilde{R} &\equiv \gamma^{\alpha\beta} \tilde{P}_{\alpha\beta} = \\ &= \gamma^{\alpha\beta} \left[-\tilde{\partial}_\beta \left\{ \widetilde{\alpha^\rho} \right\} + \tilde{\partial}_\rho \left\{ \widetilde{\beta^\alpha} \right\} - \left\{ \widetilde{\alpha^\rho} \right\} \left\{ \widetilde{\beta^\sigma} \right\} + \left\{ \widetilde{\beta^\alpha} \right\} \left\{ \widetilde{\rho^\sigma} \right\} \right]. \end{aligned} \quad (6.4.7)$$

$$\mathcal{I} = \frac{1}{4} \left[\left(\tilde{K}_\alpha{}^\alpha \right)^2 - \tilde{K}^{\alpha\rho} \tilde{K}_{\alpha\rho} + 3\tilde{\Omega}^{\alpha\rho} \tilde{\Omega}_{\alpha\rho} \right]. \quad (6.4.8)$$

In a recent paper [?] it has been proved that the four equations deduced from

$$\begin{cases} \mathcal{P}_{\Sigma\Theta}(G_{hr}) = -\mathcal{P}_{\Sigma\Theta}(T_{hr}) \\ \mathcal{P}_{\Theta\Sigma}(G_{hr}) = -\mathcal{P}_{\Theta\Sigma}(T_{hr}) \\ \mathcal{P}_{\Theta\Theta}(G_{hr}) = -\mathcal{P}_{\Theta\Theta}(T_{hr}) \end{cases} \quad (6.4.9)$$

and calculated on \mathcal{C} , that is the equations

$$\left\{ \begin{array}{l} (\bar{S}_\alpha)_c \equiv \left\{ \frac{1}{2} \left[\tilde{\nabla}_\alpha \tilde{K}_\nu^\nu - \tilde{\nabla}_\beta \left(\tilde{K}_\alpha^\beta + \tilde{\Omega}_\alpha^\beta \right) \right] + C'^\beta \tilde{\Omega}_{\beta\alpha} \right\}_c = 0 \quad (\alpha, \beta = 1, 2, 3) \\ (\tilde{R} + \mathcal{I})_c = -2\mu_c \end{array} \right. \quad (6.4.10)$$

solve *initial conditions problem*. More precisely they assign the Cauchy data on the reference configuration \mathcal{C} .

So, once assigned on a given hypersurface Σ an unitary controvariant vector field γ and three symmetric tensor fields γ_{ij} , \tilde{K}_{ij} , satisfying the equations of the initial data problem plus the equations of the relative conditions of harmonicity, we can write:

$$\left\{ \begin{array}{l} \frac{1}{2} \left[\tilde{\nabla}_\alpha^* \left(\frac{1}{2} \partial_4 \gamma^4 \right) - \tilde{\nabla}_\beta^* \left(\tilde{K}_\alpha^\beta + \tilde{\Omega}_\alpha^\beta \right) \right] + \tilde{\Gamma}^{*\beta} \tilde{\Omega}_{\beta\alpha} = 0 \\ \gamma^{\alpha\beta} \tilde{P}_{\alpha\beta}^* + \frac{1}{4} \left[\left(\frac{1}{2} \partial_4 \gamma^4 \right)^2 - \tilde{K}^{\alpha\beta} \tilde{K}_{\alpha\beta} + 3\tilde{\Omega}^{\alpha\beta} \tilde{\Omega}_{\alpha\beta} \right] = -2\mu \\ \tilde{\Gamma}_h^* = C_h \\ \partial_4 \gamma^4 - \frac{1}{2} (\tilde{Q}_i^i - \tilde{K}_i^i) = 0 \end{array} \right. \quad (6.4.11)$$

A well known theorem of Bruhat (see [?, ?]) affirms that the conditions of harmonicity, if initially satisfied on Σ , then are satisfied also in all the neighbourhood of Σ where exists and is unique (see [?]) the solution of the restricted evolution problem. Consequently, under the assumption of comoving reference system, Eqs. (??) can be regarded as the explicit constraints on the geometrical objects characterizing the

first order differential properties of stream lines, which, if initially satisfied, ensure the harmonicity of the motions during the entire evolution.

Dust $p = 0$

Using gaussian coordinates we easily see that the reference congruence (i.e. the congruence of the stream lines) is geodesic [?]. The initial data are hence to be chosen satisfying the following equations:

$$\left\{ \begin{array}{l} \tilde{\nabla}_{\beta}^* (\tilde{K}_{\alpha}^{\beta} + \tilde{\Omega}_{\alpha}^{\beta}) = 0 \\ \gamma^{\alpha\beta} \tilde{P}_{\alpha\beta}^* + \frac{1}{4} [-\tilde{K}^{\alpha\beta} \tilde{K}_{\alpha\beta} + 3\tilde{\Omega}^{\alpha\beta} \tilde{\Omega}_{\alpha\beta}] = -2\mu \\ C_h = \tilde{Q}_i^i - \tilde{K}_i^i = 0 \end{array} \right. \quad (6.4.12)$$

6.4.3 Harmonic frame of reference for spherical symmetry

In a spherically symmetric background, a set of polar coordinates $(r, \vartheta, \varphi, t)$ may be considered as a natural set of coordinates. Hence, the spherically symmetric line-element can be written as

$$ds^2 = Y^2(r, t) d\Omega^2 + e^{2\lambda(r, t)} dr^2 - e^{2v(r, t)} dt^2, \quad (6.4.13)$$

where, as usual, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$.

In this scenario, the conditions of harmonicity, where directly imposed on polar coordinates, cause anisotropic relations [?, ?]. Hence, in the centrally symmetric case, we

have to differently deal with the conditions of harmonicity. To this aim, according to the polar transformation of coordinates:

$$x^1 = r \sin\theta \cos\varphi; \quad x^2 = r \sin\theta \sin\varphi; \quad x^3 = r \cos\theta; \quad x^4 = t \quad (6.4.14)$$

we can give the following definition:

Definition 6.4.2 *A set of polar coordinates is called polar harmonic if they are the polar transformation of harmonic coordinates.*

Hence, we can verify if a physical frame of reference \mathcal{R} admits or not harmonic coordinates by checking if \mathcal{R} admits or not polar harmonic coordinates¹⁰. In order to write the conditions of polar harmonicity one might directly use Eqs. (??). It is convenient, however, to follow a different method in order to obtain several useful relations. In particular, we will find the most general change of coordinates enabling one to pass from polar coordinates to polar harmonic coordinates. Let us insert into Eqs. (??) the most general change of coordinates adapted to spherical symmetry:

$$r' = f_1(r, t); \quad t' = f_2(r, t);$$

next let us insert the (??), thus transformed, into (??). A straightforward calculation gives:

$$\begin{cases} \partial_r (e^{(v-\lambda)} Y^2 f_1') - 2e^{(\lambda+v)} f_1 - \partial_t (e^{(\lambda-v)} Y^2 \dot{f}_1) = 0, \\ \partial_r (e^{(v-\lambda)} Y^2 f_2') - \partial_t (e^{(\lambda-v)} Y^2 \dot{f}_2) = 0. \end{cases} \quad (6.4.15)$$

¹⁰We note that the polar transformation is internal to a given system of reference.

In particular, if f_1 and f_2 are not dependent the former of t and the latter of r (i.e. changes internal to a given frame of reference) we obtain:

$$\begin{cases} \partial_r (e^{(v-\lambda)} Y^2 f_1') - 2e^{(\lambda+v)} f_1 = 0 \\ \partial_t (e^{(\lambda-v)} Y^2 \dot{f}_2) = 0 \end{cases} \quad (6.4.16)$$

Lastly, for $r = r'$ and $t = t'$ (i.e. considering directly polar harmonic coordinates) we obtain:

$$\begin{cases} \partial_r (e^{(v-\lambda)} Y^2) - 2r e^{(\lambda+v)} = 0 \\ \partial_t (e^{(\lambda-v)} Y^2) = 0 \end{cases} \quad (6.4.17)$$

Eqs. (??), (??), (??) are respectively: the condition which a transformation of polar coordinates has to satisfy to determine a set of polar harmonic coordinates; the condition which a transformation of polar coordinates, internal to a spherical frame of reference, has to satisfy to determine a set of polar harmonic coordinates; the direct condition which the coefficients of an harmonic line–element have to verify.

The previous observations lead to prove the following:

Proposition 6.4.3 *More than a single system of reference, but not the totality of systems, admits polar harmonic coordinates.*

PROOF – The first part of the assertion is obtained by observing that the linear differential Eqs. (??) always admit solutions. Instead, since the coefficients in Eqs. (??) in general depend both on r and t , do not always admit solutions for which f_1 depends only on r and f_2 depends only on t .

6.4.4 Exact solutions in harmonic coordinates: some integration examples

The methods that we have above discussed are not only useful in order to clarify the geometrical and physical meaning of the harmonic frames of reference, but also to more easily look for possible exact solutions in harmonic coordinates. We will analyze hereafter some simple examples.

First of all, let us consider static universes generated by an insular mass or by a mass distributed on a sphere of radius \bar{r} . The unknown quantities λ , Y , v , must satisfy the conditions of polar harmonicity and Einstein field equations. In particular, if we accept the Fock's heuristical assumption $\lambda = -v$ we obtain the following system:

¹¹

$$\left\{ \begin{array}{l} e^{2v}Y^2 = r^2 + c_1 \\ Y(1 - Y'^2 e^{2v}) = c_2 \\ Y'' = 0 \end{array} \right. \quad (6.4.18)$$

where c_1 and c_2 are integration constants. The constant c_2 does not depend on the choice of the radial coordinate, so it can be computed by using intrinsic radius and assuming the usual asymptotic condition $e^v = 1 - \frac{\alpha}{r}$. Hence, we obtain $c_2 = 2\alpha$, with $\alpha = \frac{M}{4\pi}$ (where M is the central mass).

Moreover, from Eq. (??)₃ it follows $Y = hr + k$. Therefore, the further unknown v ,

¹¹System (??) is obtained by considering (??), the suitable projection of Einstein's field equations in spherical symmetry (see [?]) and the harmonicity conditions.

have to satisfy the following system:

$$\begin{cases} e^{2v} (hr + k)^2 = r^2 + c_1 \\ (1 - h^2 e^{2v}) (hr + k) = 2\alpha \end{cases} \quad (6.4.19)$$

Such equations are algebraic in e^v . Their condition of consistence causes the constants c_1 , h , k , and the functions e^v , to assume the following values:

$$c_1 = -\alpha^2, \quad h = 1, \quad k = \alpha,$$

and

$$e^v = \frac{r - \alpha}{r + \alpha}.$$

So we have easily obtained the classical external Schwarzschild solution. This solution was already written by Fock and several authors (e.g. [?, ?]) with different methods consisting essentially in a change of the radius, and by Graif with the hamiltonian formalism [?].

Let us consider the De Sitter's universe. Its line-element in a comoving frame of reference takes the form:

$$ds^2 = \frac{dr^2}{1 - \frac{1}{3}Rr^2} + r^2 d\Omega^2 - \left(1 - \frac{1}{3}Rr^2\right) dt^2 \quad (6.4.20)$$

where R is a constant. The associated frame of reference can be considered static and harmonic since in a static system it is always possible to choose harmonic coordinates (see Eqs. (??)). Hence through a suitable change of coordinates, the De Sitter line-

element can be written as (see e.g. [?]):

$$ds^2 = \left(\frac{t}{a}\right)^4 (dr^2 + r^2 d\Omega^2 - dt^2) \quad (6.4.21)$$

where a is a constant. This new line-element can be interpreted as the De Sitter's universe referred to a dynamical, but no longer comoving frame of reference. It can be also interpreted, remaining in the class of comoving frames of reference, as a dynamical evolution corresponding to a certain "dynamical" choice of the initial data.

Now, we want to prove that such motions are harmonic (in the sense that the comoving systems of reference are harmonic). In fact, the transformation from the coordinates of the metric (??) to harmonic coordinates has to satisfy the following equations (see (??)):

$$\begin{cases} \partial_r (r^2 f_1') = 2f_1 \\ \partial_t \left(\frac{t^4}{a^4} \dot{f}_2\right) = 0 \end{cases} \quad (6.4.22)$$

The first equation is satisfied by: $f_1 = r$. The second equation admits the following solution: $f_2 = \frac{h_1}{t^3} + k_1$ with h_1 and k_1 constants. The metric of De Sitter can be thus given the following form:

$$ds^2 = \frac{h_1^{4/3}}{a^4} (t' - k_1)^{-4/3} \left[dr^2 + r^2 d\Omega^2 - \frac{1}{9} h_1^{2/3} (t' - k_1)^{-8/3} dt'^2 \right] \quad (6.4.23)$$

The above is the unique line-element generated by the following harmonic choice of

the initial data:¹²

$$e^{\bar{\lambda}} = e^{\lambda(r,0)} = \frac{h_1^{2/3}}{a^2} (k_1)^{-2/3}; \quad \bar{Y} = Y(r,0) = \frac{h_1^{2/3}}{a^2} r (k_1)^{-2/3};$$

$$e^{\bar{v}} = e^{v(r,0)} = \frac{1}{3} \frac{h_1}{a^2} (k_1)^{-2}; \quad \bar{\psi}(r) = \dot{\lambda}(r,0) = \frac{2}{3} (k_1)^{-1}; \quad (6.4.24)$$

$$\bar{\chi}(r) = \dot{Y}(r,0) = \frac{2}{3} \frac{h_1^{2/3}}{a^2} r (k_1)^{-5/3}$$

Finally we consider the Einstein-De Sitter universe in order to deal with a direct dynamical example, too. By means of a comoving system of reference, this metric takes the form (see e.g. [?]):

$$ds^2 = \left(\frac{b}{t}\right)^2 (dr^2 + r^2 d\Omega^2 - dt^2) \quad (6.4.25)$$

where b is a constant. Eq. (??)₁ also in this case is satisfied by $f_1 = r$. Eq. (??)₂ is satisfied by: $f_2 = h_2 t^3 + k_2$ where h_2 and k_2 are constants. Hence, we obtain the following harmonic expression of the Einstein-De Sitter universe:

$$ds^2 = \frac{b^2 h_2^{2/3}}{(t' - k_2)^{2/3}} \left[dr^2 + r^2 d\Omega^2 - \frac{1}{9 h_2^{2/3}} (t' - k_2)^{-4/3} dt'^2 \right] \quad (6.4.26)$$

which represents in a comoving frame of reference, the unique solution correspondent

¹²See e.g. [?, ?] for a complete treatment about the initial data problem and hence the functions introduced in (??).

to the following harmonic initial data:

$$e^{\bar{\lambda}} = -bh_2^{1/3}(k_2)^{-1/3}; \quad \bar{Y} = -r b h_2^{1/3} (k_2)^{-1/3}; \quad e^{\bar{v}} = -\frac{b}{3} (k_2)^{-1} \tag{6.4.27}$$

$$\bar{\psi} = \frac{1}{3} (k_2)^{-1}; \quad \bar{\chi} = -\frac{1}{3} b h_2^{1/3} r (k_2)^{-4/3}$$

Bibliography

- [1] E. Mach, La Meccanica nel suo sviluppo storico-critico, *Universale Scientifica Boringhieri*, Torino, 1977.
- [2] H. Bondi D. Sciama, Cosmologie a confronto, *Universale Scientifica Boringhieri*, Torino, 1976, pp 136–139.
- [3] G. Caricato, Fondamenti di Meccanica Newtoniana, *CISU*, Roma, 1984.
- [4] C. Kittel et al., La fisica di Berkeley, *Zanichelli*, Bologna, **1**, 1970.
- [5] L.P. Eisenhart, Riemannian Geometry, *Princeton University Press*, 1960.
- [6] A. Einstein, Il significato della Relatività, *Giulio Einaudi Editore*, Torino, 1950, pp 89-91.
- [7] S. Lang, Algebra lineare, *Bollati Boringhieri*, Chap. XIII, 1984.
- [8] L. P. Eisenhart, Continuous groups of transformations, *Princeton University Press*, 1933.
- [9] G. Caricato, Un criterio di stazionarietà per le varietà d'universo a simmetria spaziale sferica, *Riv. Mat. Univ. Parma*, **4**, 1963, pp 107–113.

- [10] D. Kramer H. Stefany M. MacCallum E. Herlt, Exact solutions of Einstein's field equations, *Cambridge University Press*, 1980.
- [11] A. Longobardi E. Mazziotti, On geometrical and physical consequences of using certain special classes of comoving physical frames of reference, *Journal of Interdisciplinary Mathematics*, **12**, n.6, 2009, pp. 883–900.
- [12] G. Caricato, Sul significato fisico del parametro affine per le geodetiche di lunghezza nulla V_4 einsteiniana, *Riv. Mat. Univ. Parma*, Parma, **4**, 1963, 271-274.
- [13] I. Cattaneo Gasparini, Projections naturelles des tenseurs de courbure d'une variété V_{n+1} á métrique hyperbolique normale, *Comptes rendus Academie des Sciences*, Edizioni Cremonese, **252**, 1961, pp 3722-3724.
- [14] C. Cattaneo, Proiezioni naturali e derivazione trasversa in una varietà riemanniana a metrica iperbolica normale, *Annali di Matematica pura ed applicata*, Springer, **XLVIII**, 1959.
- [15] C. Cattaneo, Conservation Laws in General Relativity, *Il Nuovo Cimento*, **X**, **13**, 1959, pp 237-240.
- [16] C. Cattaneo, Formulation relative des lois physiques en relativité générale, *Cours professé au Collégé de France*, 1962.
- [17] P. Benvenuti, Formulazione relativa delle equazioni dell'elettromagnetismo in relatività generale, *Annali Scuola Normale Superiore di Pisa*, **XIV**, 1960, pp 171-193.

-
- [18] C. Cattaneo, Saggio di una teoria relativistica dell'elasticità, *Atti del I Congresso Nazionale dell'A.I.M.E.T.A.*, 1971.
- [19] C. Cattaneo, Teoria macroscopica dei continui relativistici, *Pitagora Editrice*, Bologna, 1980.
- [20] A. Signorini, Lezioni di Fisica Matematica, *Veschi*, Roma, a.y. 1952-53.
- [21] G. Caricato, Lezioni di Calcolo differenziale assoluto, *Ist. Mat. G. Castelnuovo*, Roma, a.y. 1981-82.
- [22] J. Schauder, Cauchy'sches Problem fur partielle Differentialgleichungen erster Ordnung. Anwendung einiger sich auf die Absolutbeträge der Lösungen beziehenden Abschätzungen, *Commentarii Mathematici Helvetici*, **9**, 1936, pp 263-283.
- [23] Y. Bruhat, Theoreme d'existence pour certains systemes d'equations aux derivees partielles non lineaires, *Acta Mathematica*, **88**, 1952, pp.141-225.
- [24] Y. Bruhat, Sur l'integrations des equations de la relativite generale, *Journal of rational mechanics and analysis*, **5**, 1956, pp 951-966.
- [25] G. Caricato, Sulla definizione intrinseca di una varietà d'universo a simmetria sferica, *Rend. di Mat.*, **XXII**, 1963 pp. 253-265.
- [26] G. Caricato, Sur le problème de Chauchy pour les equations de la relativité generale dans les schèmes matériels fluid parfait et matiere pure, *Ann. Inst. Henry Poincaré*, **IX**, n. 3, 1968, pp 283-301.

- [27] A. Longobardi E. Mazziotti, On geometrical and physical consequences of using certain special classes of comoving physical frames of reference II, *Journal of Interdisciplinary Mathematics*, **14**, n.2, 2011, pp 189-204.
- [28] I. Bochicchio E. Laserra A.Longobardi E. Mazziotti, Reference frame in harmonic coordinates: Geometric and Physical aspects, *in print on Ricerche Mat.*, doi: 10.1007/s11587-011-0125-x
- [29] E. Laserra, Sul problema di Cauchy relativistico in un universo a simmetria spaziale sferica, *Rendiconti di Mat.*, **2**, 1982, pp 267-271.
- [30] G. Caricato, Il problema ristretto di evoluzione per un fluido perfetto carico di elettricità, *Ricerche di Matematica*, **XXV**, 1975, pp 51-80.
- [31] C.W. Misner and D.H.Sharp, Relativistic equations for adiabatic spherically symmetric gravitational collapse, *Phys. Rev.*, **36**, n.2B, 1964.
- [32] P.S. Wesson, An exact solution of Einstein's equations with a stiff equation of state, *J. Math. Phys.*, **19**, n. 11, 1978.
- [33] I.I. Gutman, R.M. Bespalko, Some exact spherical symmetric solutions of Einstein equations, *Sbornik Sovrem. Probl. grav. Tblisi*, 1967.
- [34] E.N. Glass, Shear free gravitational collapse, *J.Math.Phys.*, **20**, n.7, 1979, pp 1508-1513.
- [35] H. Stephany, A new interior solution of Einstein's field equations for a spherically symmetric perfect fluid in shear free motion, *J. Phys. A:Math. Gen.*, **16**, 1983, pp 3529-3532.

- [36] M. Wyman, Equations of state for radially symmetric distributions of matter, *Phys. Rev.*, **70**, n.5-6, 1946, pp 396-400.
- [37] R. Mansouri, On the non existence of time dependent fluid spheres in general relativity obeying an equation of state, *Ann. Inst. Henri Poincaré*, **XXVII**, n.2, 1977, pp 175-182.
- [38] D.C. Srivastava and S.S. Prasad, Spherically symmetric perfect fluid distributions in general relativity, *Proceedings of the X GRG Meeting edited by B. Bertotti*, Padova, 1983, pp 360-362.
- [39] P. Kustaanheimo B. Qvist, A note on some general solutions of the Einstein's field equations in a spherically symmetric world, *Com. Phys. Math. Helsingf.*, **XIII**, n.16, 1948, pp 1-12.
- [40] G. Caricato, Principio dell'azione stazionaria e principio di Fermat in relatività generale, *Rendiconti di Matematica*, Edizioni Cremonese, Roma, **3-4**, Vol.22, 1963, pp 416-426.
- [41] I. Cattaneo Gasparini, Proiezioni naturali des tenseurs de courbure d'une variété V_{n+1} à métrique hyperbolique normale, *Comptes rendus Academie des Sciences*, Edizioni Cremonese, **252**, 1961, pp 3722-3724.
- [42] C. Cattaneo, Proiezioni naturali e derivazione trasversa in una varietà riemanniana a metrica iperbolica normale, *Annali di Matematica pura ed applicata*, Springer, **XLVIII**, 1959.

-
- [43] C. Cattaneo, Conservation Laws in General Relativity , *Il Nuovo Cimento*, *X*, **13**, 1959, pp 237-240.
- [44] C. Cattaneo, Formulation relative des lois physiques en relativité générale , *Cours professé au Collège de France*, 1962.
- [45] P. Benvenuti, Formulazione relativa delle equazioni dell'elettromagnetismo in relatività generale , *Annali Scuola Normale Superiore di Pisa*, **XIV**, 1960, pp.171-193.
- [46] C. Cattaneo, Saggio di una teoria relativistica dell'elasticità, *Atti del I Congresso Nazionale dell'A.I.M.E.T.A.*, 1971.
- [47] C. Cattaneo, Teoria macroscopica dei continui relativistici, *Pitagora Editrice*, Bologna, 1980.
- [48] A. Signorini, Lezioni di Fisica Matematica, *Veschi*, Roma, a.y. 1952-53.
- [49] G. Caricato, Lezioni di Calcolo differenziale assoluto, *Ist. Mat. G. Castelnuovo*, Roma, a.y. 1981-82.
- [50] G. Caricato, Translation of absolute evolutions laws of a continuous system in one comoving physical frame of reference. Some exact solutions, *International Journal of Pure and Applied Mathematics*, **59**, n. 1, 2010, pp 19-27.
- [51] I. Bochicchio, On the restricted evolution problem of Einstein relativistic mechanics, *Ricerche Mat.*, **59**, n. 2, 2010, pp 327–341.

-
- [52] I. Bochicchio, Existence and physical uniqueness of the restricted evolution problem of the Einstein relativistic mechanics, *Ricerche Mat.*, **60**, n. 2, 2011, pp 179–192.
- [53] V. Fock, Einstenian static in conformal space. Recent developments in General Relativity, *Pergamon Press*, Oxford, 1962, pp 209-223.
- [54] H. Arzelies, Le champ statique à symmetrie spherique, *Gauthier-Villars*, Paris, 1963, pp 16-20.
- [55] F. Graif, Sull'uso di coordinate armoniche in relatività generale, *Lincci Rend. Sc. fis. e nat.*, **34**, 1963, pp 312–320.
- [56] J.L. Synge, Relativity: The general theory, *North-Holland Publishing Company*, Amsterdam, 1960.