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Local and Global Properties of Jacobi Related Geometries

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Abstract

In this thesis local and global properties of Jacobi and related geometries are discussed, which means for us that so-called *Dirac-Jacobi* bundles are considered. The whole work is roughly divided in three parts, which are independent of each other up to preliminaries. In the first part local and semi-local properties of Dirac-Jacobi bundles are considered, in particular it is proven that a Dirac-Jacobi bundle is always of a certain form close to suitable transversal manifolds. These semi-local structure theorems are usually refered to as *normal form theorems*. Using the normal form theorems, we prove local splitting theorems of Jacobi brackets, generalized contact bundles and homogeneous Poisson manifolds. The second part is dedicated to the study of *weak dual pairs* in Dirac-Jacobi geometry. It is proven that weak dual pairs give rise to an equivalence relation in the category of Dirac-Jacobi bundles. After that, the similarities of equivalent Dirac-Jacobi bundles are discussed in detail. The goal of the last part is to find global obstructions for existence of generalized contact structures. With the main result of this chapter it is easy to find nontrivial examples of theses structures and two classes are discussed in detail.

Sommario

Nella presente tesi si discutono proprietà locali e globali delle geometrie di Jacobi. In particolare, si considerano i cosiddetti fibrati di Dirac-Jacobi. I fibrati di Dirac-Jacobi sono una immediata generalizzazione delle parentesi di Jacobi, che, in letteratura, sono anche note come strutture di Kirillov. Il presente lavoro è diviso in tre parti, che sono indipententi tra di loro tranne per i preliminari. Nella prima parte, si considerano proprietà locali e semi-locali dei fibrati di Dirac-Jacobi. In particolare, si dimostra che i fibrati di Dirac-Jacobi sono sempre di una determinata forma, simile ad un opportuna varietà trasversale. I teoremi di struttura semi-locale in genere sono teoremi di forma normale. Utilizzando questi ultimi, si dimostrano: teoremi locali di splitting delle parentesi di Jacobi, fibrati generalizzati di contatto, l'analogo in dimensione dispari dei risultati sulle varietà complesse generalizzate e le varietà omogenee di Poisson. La seconda parte della tesi è incentrata sullo studio delle coppie deboli duali nella geometria di Dirac-Jacobi. Si dimostra che le coppie deboli duali danno luogo ad una relazione di equivalenza nella categoria dei fibrati di Dirac-Jacobi. L'obiettivo dell'ultima parte della tesi è di trovare ostruzioni globali all'esistenza di strutture generalizzate di contatto non banali. Il risultato principale è la descrizione di due classi di questo tipo di strutture.

Contents

Introduction

Classical mechanics is considered to be one of the best understood theories in physics, which also depends on the fact that there are very good mathematical techniques available to describe mechanical systems. Even though, classical mechanics is not suitable to describe large-scale physics and and physics on a atomic scale, it is used in many occasions and is suitable to describe a huge amount of physical phenomena in daily life. In fact, it was even enough to consider Newtonian Mechanics to fly to the moon in 1969. From a mathematical point of view, or better said geometric point of view, the conceptual description of classical mechanics started with the works of Hamilton and Lagrange in the 17th and 18th century. In the 20th century there was a Renaissance for classical mechanics in mathematics which started with the works [2], [4] and [42] in the 60's.

Many different branches in geometry developed from this considerations, probably the two most important ones are *Geometric Mechanics* and *Poisson Geometry*. Note that this two subjects do have a more than non-trivial intersection and up to now they profited a lot from each other. This thesis focuses on the latter and its generalizations. Let us give a brief introduction to this subject and let us discuss the relation to mechanics. Let us consider a particle moving in the configuration space $\mathbb{R}^3$ with coordinates $(q^1(t), q^2(t), q^3(t))$. In order to describe its motion, we need to fix a Hamiltonian $H \in \mathscr{C}^\infty(T^*\mathrm{R}^3) = \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Usually, H is of the form

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V(q) \tag{$*$}$$

for the standard coordinates (q, p) of $\mathbb{R}^3 \times \mathbb{R}^3$. In the Hamiltonian formalism of classical mechanics the motion $(q^1(t), q^2(t), q^3(t))$ of the particle is a solution to the ordinary differential equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}(t) = \frac{\partial H}{\partial p_i}(q(t), p(t)) \quad \text{and} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t}(t) = -\frac{\partial H}{\partial q^i}(q(t), p(t)). \tag{$**$}$$

If we define the binary operation $\{-,-\}\colon \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \times \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \to \mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$\{f, g\} = \sum_{i=1}^{3} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i},$$

then we can write the Equations ($**$) in the form

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}(t) = \{q^i, H\}(q(t), p(t)) \quad \text{and} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t}(t) = \{p_i, H\}(q(t), p(t)).$$

Note that $\{-,-\}$ is a Lie bracket which is a derivation in both slots, and this is basically the starting point of Poisson geometry. In fact, to do mechanics, we need to fix three things:

i.) a *phase space*, which has sufficiently nice properties, i.e. is a smooth manifold M,

ii.) a *Poisson bracket*, i.e. a Lie bracket $\{-,-\}\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, which is a derivation in both slots,

iii.) a energy function $H \in \mathscr{C}^\infty(M)$ (the *Hamiltonian*).

From the geometric point, we forget the chosen energy function and call the pair $(M, \{-,-\})$ a Poisson manifold. Even though, Poisson brackets appeared in the late 19th century in a work by Lie, see [32], and their systematic study began with the seminal work of Weinstein [51]. Poisson geometry has a lot of intersections to other fields of mathematics and some can even be seen as a subbranch of Poisson geometry, such as the theory of Lie algebras (see [51]), deformation theory (see [22]), symplectic geometry and *Jacobi geometry*. This thesis is dedicated to the latter, namely Jacobi geometry, which was first introduced by Kirillov in [28] and independently by Lichnerowicz in [31]. Jacobi manifolds can be seen both as generalizations or as specific cases of Poisson manifolds, see [10]. A Jacobi manifold is a manifold M together with a line bundle $L \to M$ and a Lie bracket

$$\{-,-\}\colon \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$$

which is a first order differential operator in both slots. The similarities of Poisson manifolds and Jacobi manifolds are evident and there are even very classical geometries which are special cases of Jacobi manifolds, which do not fit into Poisson geometry: contact and locally conformal symplectic manifolds. Symplectic manifolds can be seen as non-degenerate Poisson structures (in a suitable sense), a contact manifold on the other hand is a non-degenerate Jacobi manifold. So, loosely speaking, Jacobi brackets are in relation to contact structures as Poisson brackets are in relation with symplectic structures. Note that contact manifolds are always odd dimensional and symplectic manifolds are always even dimensional, so we refer to contact manifolds as the odd dimensional analogues of symplectic manifolds.

But there are also purely physical motivations to study Jacobi manifolds, which do not fit into the framework of classical Poisson geometry, for example thermodynamics (for a detailed overview see [9] and its references). But also in classical mechanics of a moving particle contact geometry, and hence Jacobi geometry, plays an important role. In fact, there are situations where they naturally appear:

- If one considers a Hamiltonian like $(*)$, particles move along trajectories of constant energy, so a possibly non-smooth hypersurface such that $H = const$. Under some regularity assumptions on the constant, this is actually a smooth manifold, which admits a contact structure and the motion of a particle on this hypersurface can be completely encoded by the contact structure.

- If the Hamiltonian is explicitly time-dependent, it is sometimes useful to enlarge the phase space by $\mathbb{R}$ and see the Hamiltonian as a function on it. On this new phase space classical mechanics is now described by what is called contact mechanics, see [29] and [9].

Nevertheless Jacobi manifolds are much less studied than Poisson manifolds. Many recent results in Poisson and/or related geometries have mirror statements in Jacobi and/or related geometries. This is exactly the aim of this thesis: filling in some gaps in Jacobi geometry whose analogues have been studied in Poisson geometry. This is part of a series of a long term project aiming at translating from Poisson to Jacobi geometry whenever possibe, see e.g. [43], [46] and references therein.

Before we discuss the content of the single chapters, let us be more precise about the term *related* geometries. In this thesis we are discussing the following geometric structures, which we consider to be related to Jacobi geometry:

Dirac-Jacobi bundles are the Jacobi geometric analogue of *Dirac structures* in Poisson geometry. Dirac structures play an omnipresent role in Poisson geometry, since they generalize Poisson structures, pre-symplectic forms, complex structures, etc. For an introduction to Dirac geometry see [11]. Besides the generalizing aspect, they appear naturally in various situation in Poisson and symplectic geometry, for instance a coisotropic submanifold of a Poisson (resp. symplectic) has no Poisson (resp. symplectic) structure, but it has a canonical Dirac structure which contains all the information needed for *coisotropic reduction*. On the other hand Dirac-Jacobi structures have only been considered recently, see [46], as a generalization of Wade's $\mathcal{E}^1$-Dirac structures [49].

Generalized contact bundles are the odd dimensional counterpart of generalized complex manifolds such as contact manifolds are the odd-dimensional analogue of symplectic manifolds. Generalized complex geometry provides a generalized framework of symplectic and complex geometry and was first systematically studied in [26]. Generalized contact bundles were introduced recently in [47] and very few is known about them and in particular very few (non-trivial) examples are available. In [47] it is proven that every generalized contact bundle induces a Jacobi bracket, which puts generalized contact geometry in the framework of Jacobi geometry and is a common generalization of contact geometry and Atiyah-complex structures, a slight generalization of *normal almost contact structures*, which we disuss in A.3.

Homogeneous Poisson manifolds are Poisson manifolds with a given primitive in the Poisson complex, see [30]. To be precise this is a pair (π, Z) consisting of a

Poisson tensor π and a vector field Z, such that $\mathscr{L}_Z\pi = -\pi$. There are a lot of examples, for example the dual of a Lie algebra with the KKS Poisson structure together with the Euler vector field as well as the canonical symplectic structure on cotangent bundles with the Euler vector field. The question which arises now is why homogeneous Poisson manifolds are more related to Jacobi geometry then arbitrary Poisson manifolds? The answer to this gives the homogenization trick: there is a one-to-one correspondence of Jacobi related geometries and homogeneous Poisson related geometries on $\mathbb{R}^\times$-principal fiber bundles. Precise statements can be found in [10] and references therein as well as Appendix A.2.

We proceed as follows: The first chapter is dedicated to fix notation and recall known facts in the topic, this includes a quick reminder of the D-functor, representation theory of Lie algebroids, as well as the definitions of Jacobi bundles, Dirac-Jacobi bundles and generalized contact structures. The second chapter is based on [41] and [38], where we follow the lines of [38] in order to provide a normal form theorem for Dirac-Jacobi bundles and apply it directly to Jacobi structures and to generalized contact structures in order to re-obtain the results from [41] in a slightly more conceptual way. Moreover, we provide splitting theorems for Jacobi structures, originally obtained in [17], for generalized contact structures from [41] and finally for homogeneous Poisson structures also obtained in [17].

The third chapter is the state of the art of an ongoing collaboration with Alfonso Tortorella [40]. We introduce the notion of (weak) dual pairs in Dirac-Jacobi geometry, which is a triple of Dirac-Jacobi structures, and deduce some of the first properties, such as an alternative proof of the normal form theorem for Dirac-Jacobi, the existence of a so-called *self dual pair* and probably most importantly, we prove that for two Dirac-Jacobi structures sitting in a (weak) dual pair the transverse geometry are isomorphic in a suitable sense. Since dual pairs are deeply connected to Morita equivalence, their study is supposed to be the beginning of a systematic study of Morita equivalence of Jacobi manifolds.

In the last chapter, we just concentrate on one specific Jacobi related geometry: generalized contact bundles. This chapter is based on [39]. After having studied local and semi-local structure of Jacobi related geometries in the previous chapters, we want to end this thesis with some global considerations in generalized contact geometry. The main purpose is to develop an obstruction theory of their existence in a given framework and apply this theory to obtain examples. And in fact, we can prove that all five dimensional nilpotent Lie Groups possess an invariant generalized contact structure and moreover that every contact fiber bundle over a complex base possesses a canonical generalized contact structure.

How to read this Thesis

This thesis consists basically out of three preprints/publications (see [41], [39] and [38]) and a work in progress (see [40]) which are merged together. This means in particular that the chapters 2, 3 and 4 can be read independently. Necessary for all of the three chapters is Chapter 1, in which basic differential geometric are discussed and specific notations are introduced. Due to the fact that everything is very basic and and known, we will not refer, for the sake of readability, to all the notions introduced in Chapter 1 all the time in the following chapters. Moreover, there is an interaction between the chapters, which can be summarized by:

$$\begin{tikzcd}
 & \boxed{\text{Chapter 1}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{\text{Sections 2.1, 2.2}} \arrow[dr] & \boxed{\text{Chapter 3}} \arrow[d] & \boxed{\text{Chapter 4}} \\
 & \boxed{\text{Section 2.3}} \arrow[dl] \arrow[dr] & \\
\boxed{\text{Section 2.4}} & & \boxed{\text{Section 2.5}}
\end{tikzcd}$$

The path $\boxed{\text{Chapter 1}} \to \boxed{\text{Chapter 3}} \to \boxed{\text{Section 2.3}}$ is possible in principle, but not recommended. Chapter 4 is a bit remote, since it contains global properties of generalized contact bundles, whereas the remaining chapters treat the local and semi-local structure of several Jacobi-related geometries.

Chapter 1

Preliminaries

This first chapter is meant to fix the notation and establish the language which allows us to do Dirac geometry in the category of line bundles. Most of the statements are known or at least folklore, see [46] and its references. In this spirit, we first take a closer look to the category of line bundles and afterwards define the analogue in this category of the tangent bundle, the so-called *Atiyah* or *Gauge* algebroid and observe that it is dual, again in the category of line bundles, to the first jet bundle. These are exactly the ingredients in order to study Dirac structures on line bundles, which are Lagrangian subbundles of the so-called *omni* Lie algebroid, see [14]. After that, we make a small excursus to the representation theory of Lie algebroids and fit our framework inside this theory. The next part is dedicated to give motivating examples of this, what we will call them, *Dirac-Jacobi* bundles, which include Jacobi brackets, contact structures and generalized contact structures. As a final section, we add some properties of the category of Dirac-Jacobi bundles.

1.1 Derivations and the Der-Complex

Even tough derivations on vector bundles and their corresponding de Rham complexes are very classical topics in differential geometry, we give the basic definitions and properties, which will be necessary throughout this section. This section is far from being a complete introduction to this topic. A more detailed discussion can be found for example in [33] and [36].

1.1.1 The Category of Line Bundles

The category of line bundles should not be seen as a full subcategory of vector bundles, at least not for our purposes. The reason is that we want to shrink the Hom sets in order to get a category admitting products and a reasonable amount of pull-backs. So let us make this precise:

Definition 1.1.1 *The category $\mathfrak{Line}$ consists of smooth line bundles over manifolds as objects and regular, i.e. fiber-wise invertible, line bundle morphisms as arrows.*

Note that if we allow non-fiberwise invertible line bundle morphisms, we lose one of the most important properties for our purposes: the pull-back of sections, i.e. for a fiberwise invertible line bundle morphism $\Phi\colon L_1 \to L_2$ covering the smooth map $\phi\colon M_1 \to M_2$, we can define

$$\Phi^*\lambda \in \Gamma^\infty(L_1) \quad \text{by} \quad \Phi^*\lambda(m) := \Phi_m^{-1}\lambda(\phi(m)) \tag{1.1.1}$$

for $\lambda \in \Gamma^\infty(L_2)$. As we have announced before, this category has nice properties; in fact we have

Theorem 1.1.2 *The category $\mathfrak{Line}$ admits products. Moreover, if for two line bundles $P_i\colon (L_i \to M_i) \to (L \to M)$, the pull-back (i.e. the fibered product)*

$$\begin{tikzcd} M_1 \times_M M_2 \arrow[r, dashed] \arrow[d, dashed] & M_1 \arrow[d] \\ M_2 \arrow[r] & M \end{tikzcd}$$

in the category of manifolds $\mathfrak{Man}$, the category of smooth manifolds with smooth maps as morphisms, exists, then also the pull-back of $P_i\colon (L_i \to M_i) \to (L \to M)$ in $\mathfrak{Line}$ exists.

Proof: First, we prove that $\mathfrak{Line}$ admits products. Let us therefore consider two line bundles $L_i \to M_i$ for $i = 1, 2$ and the set

$$M^\times = \{\phi_{x,y}\colon L_{1,x} \to L_{2,y} \mid \phi_{x,y} \text{ is a linear isomorphism}\}$$

with the obvious projections $p_i\colon M^\times \to M_i$. Note that $M^\times$ is a smooth manifold, since one can realize it as

$$M^\times = \frac{L_1^*\backslash\{0\} \times L_2^*\backslash\{0\}}{\mathbb{R}^\times}$$

with the diagonal action of $\mathbb{R}^\times$, which is clearly free and proper. The next step is to construct the line bundle $L^\times \to M^\times$ by

$$L^\times = p_1^* L_1$$

together with the regular line bundle morphisms $P_i\colon L^\times \to L_i$ defined by

$$P_1\colon L^\times \ni (\phi_{x,y}, \lambda_x) \mapsto \lambda_x \in L_1$$

and

$$P_2\colon L^\times \ni (\phi_{x,y}, \lambda_x) \mapsto \phi_{x,y}(\lambda_x) \in L_1.$$

Now we want to prove that $L^\times \to M$ has the universal property of a product. Let therefore $L_3 \to M_3$ be a line bundle with regular line bundle morphisms $K_i\colon L_3 \to L_i$ covering $k_i\colon L_3 \to L_i$. Then we can define the map

$$J\colon L_3 \ni l_x \mapsto (K_{2,x} \circ K_{1,x}^{-1}, K_{1,x}(l_x)) \in L^\times.$$

It is easy to see that J is the unique regular line bundle morphism making the diagram

$$\begin{tikzcd}
L_3 \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& L^\times \arrow[r] \arrow[d] & L_1 \\
& L_2 &
\end{tikzcd}$$

commute and hence $(L^\times \to M^\times)$ is the product in $\mathfrak{Line}$.

Let us now show that $\mathfrak{Line}$ admits a reasonable amount of pull-backs. We consider the pull-back of principal fiber bundle

$$\begin{tikzcd}
M^\times_M \arrow[r] \arrow[d] & M^\times \arrow[d] \\
M_1 \times_M M_2 \arrow[r] & M_1 \times M_2
\end{tikzcd}$$

via the map $M_1 \times_M M_2 \to M_1 \times M_2$, where $M_1 \times_M M_2$ is a smooth manifold by assumption. It is easy to see that for a line bundle $L \to M$ with two line bundles $L_i \to L$ for $i = 1, 2$, that

$$\begin{tikzcd}
(L^\times_M \to M^\times_M) \arrow[r, dashed] \arrow[d, dashed] & (L_1 \to M_1) \arrow[d] \\
(L_2 \to M_2) \arrow[r] & (L \to M)
\end{tikzcd}$$

commutes and is moreover a a pull-back in $\mathfrak{Line}$. XΞΣ

1.1.2 The D-Functor

In this subsection we want to treat the D-functor, which is a functor from the category of vector bundles with fiber-wise invertible vector bundle morphisms $\mathfrak{Vect}$ into the category of Lie algebroids $\mathfrak{LieAlg}$. Even though we are mainly interested in line bundles, we will treat the D-functor in full generality. Given a vector bundle $E \to M$ and a point $p \in M$, we consider the set

$$D_pE := \{\delta_p \colon \Gamma^\infty(E) \to E_p \mid \exists! v_p \in T_pM : \delta_p(fs) = v_p(f)s + f\delta_p(s)\},$$

for $f \in \mathscr{C}^\infty(M)$, $s \in \Gamma^\infty(E)$. Note that all $\delta_p \in D_pE$ are local operators and D_pE is a vector space. Moreover, we have the short exact sequence

$$0 \to \mathsf{End}(E_p) \to D_pE \to T_pM \to 0$$

where the last non-trivial arrow is the assignment $\delta_p \mapsto v_p$, called the symbol and is denoted by σ. The disjoint union

$$DE := \coprod_{p \in M} D_pE$$

can be given a unique smooth vector bundle structure, such that

$$0 \to \mathsf{End}(E) \to DE \xrightarrow{\sigma} TM \to 0$$

is a short exact sequence of vector bundles, the so-called *Spencer Sequence*. Note that this implies that

$$\Delta(\lambda) := (p \to \Delta_p(\lambda)) \in \Gamma(E)$$

is a smooth section for all $\Delta \in \Gamma^\infty(DE)$ and $\lambda \in \Gamma^\infty(E)$. Additionally, the sections of $DE \to M$ posses a Lie bracket given by the commutator, which turns DE, together with $\sigma\colon DE \to TM$, into a Lie algebroid. The sections of $DE \to M$ are refered to as *derivations* and $DE \to M$ is called *Atiyah* or *gauge* algebroid. So this gives us an assignment $E \to DE$, which is exactly the D-functor on objects in $\mathfrak{Vect}$. The next step is to clarify what it does on morphisms. So let $\Phi\colon E \to E'$ be a fiber-wise invertible vector bundle morphism covering $\phi\colon M \to M'$ and let $\delta_p \in D_pE$, then we define the map

$$D\Phi\colon DE \ni \delta_p \mapsto (\lambda \mapsto \Phi_p(\delta_p \Phi^* \lambda)) \in DE',$$

which is a vector bundle map covering $\phi\colon M \to M'$. Moreover, it is easy to see that $D\Phi\colon DE \to DE'$ is a Lie algebroid morphism, i.e. $D\Phi^*\colon \Gamma^\infty(\Lambda^\bullet(DE')^*) \to \Gamma^\infty(\Lambda^\bullet(DE)^*)$ intertwines the Lie algebroid differentials and $\sigma' \circ D\Phi = T\phi \circ \sigma$ for the symbols $\sigma\colon DE \to TM$ and $\sigma'\colon DE' \to TM'$. The functoriality follows by a simple computation. Since a section $\Delta \in \Gamma^\infty(DE)$ can be applied to a section of $E \to M$, we have $\Delta \in \mathsf{Hom}_{\mathbb{R}}(\Gamma^\infty(E), \Gamma^\infty(E))$, in fact we have $\Gamma^\infty(DE) \subseteq \mathrm{DiffOp}^1(E, E)$, the first differential operators of the sections of the vector bundle E.

Let us go back to line bundles, since in this case there are simplifications and more features of the D-functor.

Lemma 1.1.3 *Let $L \to M$ be a line bundle. Then* $\mathrm{DiffOp}^1(L, L) = \Gamma^\infty(DL)$.

Proof: This is an easy consequence of the fact that, for a line bundle $L \to M$, we have $\mathsf{End}(L) \cong \mathscr{C}^\infty(M)$. ΧΞΣ

Note that the first-order differential operators $\mathrm{DiffOp}^1(L, L)$ can be understood as sections of the vector bundle

$$(J^1L)^* \otimes L \to M,$$

where we denote by J^1L the first jet bundle of L. If we denote by $j^1\colon \Gamma^\infty(L) \to \Gamma^\infty(J^1L)$ the first jet prolongation, the $1:1$ correspondence of $\Gamma^\infty((J^1L)^* \otimes L)$ and $\mathrm{DiffOp}^1(L, L)$ is realized by

$$\Gamma^\infty((J^1L)^* \otimes L) \ni \alpha \mapsto (\lambda \mapsto \alpha(j^1\lambda)) \in \mathrm{DiffOp}^1(L, L).$$

Moreover, the dual of the Spencer sequence, after tensorizing by L, can be written as

$$0 \to T^*M \otimes L \to J^1L \to L \to 0.$$

Now we want to discuss the local structure of $DL \to M$ and $J^1L \to M$ for a line bundle $L \to M$. Let us choose a local trivialization

$$L_U \cong U \times \mathbb{R}$$

so we may identify $\Gamma^\infty(L_U) \cong \mathscr{C}^\infty(U)$. Let us moreover, assume that U is a chart domain with coordinates $x = (x^1, \ldots, x^n)$. The claim is now to show that the maps

$$\delta_i \colon \mathscr{C}^\infty(U) \ni f \to \frac{\partial f}{\partial x^i} \in \mathscr{C}^\infty(U)$$

together with the identity $\mathbb{1} \colon \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$ form a local basis of DL_U. Let $\Delta \in \Gamma^\infty(DL_U)$. Let us denote its symbol by $\sigma(\Delta) = X^i \frac{\partial}{\partial x^i}$. It is easy to see that, first $X^i\delta_i \in \Gamma^\infty(DL_U)$ and moreover

$$(\Delta - X^i\delta_i)(f) = f \cdot (\Delta - X^i\delta_i)(1) = f \cdot \Delta(1).$$

Therefore $\Delta = X^i\delta_i + g \cdot \mathbb{1}$ with $g = \Delta(1)$. This allows us to identify

$$DL_U \cong TU \oplus \mathbb{R}_U$$

locally, where we denote by $\mathbb{R}_U$ the trivial line bundle $\mathbb{R} \times U \to U$. Moreover, this identification holds true for trivial line bundles $\mathbb{R}_M \to M$ globally. Identifying $J^1L = (DL)^* \otimes L$, we see that locally

$$J^1L_U \cong T^*U \oplus \mathbb{R}_U.$$

There are many similarities between DL and the tangent bundle. The last similarity we want to mention is the existence of flows of derivations. The flow of $\Delta \in \Gamma^\infty(DL)$ is defined as the unique one-parameter family of line bundle automorphism $\Phi_t^\Delta \in \mathsf{Aut}(L)$, fulfilling

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\Phi_t^\Delta)^*(\lambda) = \Delta(\lambda),$$

which is explained in more detail in [46]. Let us now discuss the Atiyah algebroid of some special line bundles. We start with products in the category of line bundles.

Lemma 1.1.4 *Let $L_i \to M_i$ be line bundles for $i = 1, 2$ and denote by $P_i \colon L^\times \to L_i$ their product covering $p_i \colon M^\times \to M_i$, then*

$$DL^\times = \ker DP_1 \oplus \ker DP_2.$$

PROOF: Note that the map

$$p\colon M^\times \to M_1 \times M_2$$

is a surjective submersion and hence the kernel of its tangent map has dimension one. But we have $\ker Tp = \ker Tp_1 \cap \ker Tp_2$, which is given by the fundamental vector field of the principal action

$$\phi\colon \mathbb{R}^\times \times M^\times \ni (\alpha, \psi_{x,y}) \mapsto \alpha^{-1}\psi_{x,y} \in M^\times$$

which is covered by a one-parameter group of regular line bundle morphism

$$\Phi\colon \mathbb{R}^\times \times L^\times \ni (\alpha, (\psi_{x,y}, \lambda_x)) \mapsto (\alpha^{-1}\psi_{x,y}, \lambda_x) \in L^\times.$$

By the defintion of $L^\times$, we have that

$$P_1 \circ \Phi_\alpha = P_1 \text{ and } P_1 \circ \Phi_\alpha = \alpha^{-1} P_1 \text{ for all } \alpha \in \mathbb{R}^\times. \tag{1.1.2}$$

Let us introduce the derivation $\Delta \in \Gamma^\infty(DL^\times)$ by

$$\Delta(\lambda) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi^*_{\exp(t)}\lambda.$$

Using Equations (1.1.2), we have

$$DP_1(\Delta_{\psi_{x,y}}) = 0 \text{ and } DP_2(\Delta_{\psi_{x,y}}) = \mathbb{1}_y.$$

Let now $\square \in \ker DP_1 \cap \ker DP_2$, then we have that $\sigma(\square) \in \ker Tp_1 \cap \ker Tp_2$ and hence there exist $k, l \in \mathbb{R}$, such that $\square = k\mathbb{1} + l\Delta$, but then we have that $DP_1(\square) = k\mathbb{1} = 0$ and hence $k = 0$. Furthermore, we have that $DP_2(\square) = l\mathbb{1} = 0$ and hence $l = 0$. This means that $\ker DP_1 \cap \ker DP_2 = \{0\}$ and counting dimensions the claim follows. ΧΞΣ

Lemma 1.1.5 *Let $L_i \to M_i$ be line bundles for $i = 1, 2$ and denote by $P_i\colon L^\times \to L_i$ their product and by $p_i\colon M^\times \to M_i$ the maps covered by P_i. Then*

$$DL^\times \cong p^*(DL_1 \oplus DL_2)$$

where $p\colon M^\times \ni m \mapsto (p_1(m), p_2(m)) \in M_1 \times M_2$.

PROOF: Let us simply write down the map

$$I\colon DL^\times \ni \Delta_{\phi_{x,y}} \mapsto (\phi_{x,y}, (DP_1(\Delta_{\phi_{x,y}}), DP_2(\Delta_{\phi_{x,y}}))) \in p^*(DL_1 \oplus DL_2),$$

which is injective by Lemma 1.1.4. Comparing the ranks of $p^*(DL_1 \oplus DL_2)$ and $DL^\times$, the claim follows. ΧΞΣ

As a final Corollary, we discuss the canonical splitting of pull-backs

Corollary 1.1.6 *Let $\Phi_i\colon L_i \to L$ be regular line bundle morphisms covering $\phi_i\colon M_i \to M$, such that the pull-back in $\mathfrak{Line}$*

$$\begin{array}{ccc} L^\times_M & \xrightarrow{P_2} & L_2 \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle \Phi_2} \\ L_1 & \xrightarrow{\Phi_1} & L \end{array}$$

exists, then

$$DL^\times_M \cong p^*(DL_1 \times_{DL} DL_2),$$

where $p\colon M^\times_M \ni m \mapsto (p_1(m), p_2(m)) \in M_1 \times_M M_2$.

1.1.3 Representations of Lie Algebroids

In the literature, see for example [15], a Lie algebroid representation of a Lie algebroid $(A \to M, \rho_A, [-,-]_A)$ is a vector bundle $E \to M$ together with a so-called A-connection, which is flat. An A-connection is a map

$$\nabla\colon \Gamma^\infty(A) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$$

which is $\mathscr{C}^\infty(M)$-linear in the first argument and

$$\nabla_a fe = \rho_A(a)(f) \cdot e + f \cdot \nabla_a e.$$

Flatness means in this setting

$$[\nabla_a, \nabla_b] = \nabla_{[a,b]_A}.$$

By the very definition of DE, this notion is equivalent to have a Lie algebroid morphism

$$\nabla\colon A \to DE$$

covering the identity, i.e. its adjoint

$$\nabla^*\colon \Gamma^\infty(\Lambda^\bullet A^*) \to \Gamma^\infty(\Lambda^\bullet DE^*)$$

is a chain map with respect to the de Rham differentials of A and DE, respectively. Note that this implies in particular that $\sigma \circ \nabla = \rho_A$. We will not examine in depth the theory of representations of Lie algebroids; a much more detailed discussion can be found in [15]. Let us anyway give two (trivial) examples:

Example 1.1.7 Let $L \to M$ be a line bundle, then $\mathsf{id}\colon DL \to DL$ is a Lie algeboid representation of DL on L. In the following, we will refer to this as the *tautological* representation.

Example 1.1.8 Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. We can interpret them as a Lie algebroid over a point and a vector bundle over a point. Note that for the Gauge algebroid, we have the exact sequence

$$0 \to \mathsf{End}(V) \to DV \to T\{*\} = 0$$

and hence $End(V) = DV$. This means that a representation of $\mathfrak{g}$ on V is a Lie algebra map $\nabla\colon \mathfrak{g} \to \mathsf{End}(V)$, which coincides with the usual definition of a Lie algebra representation.

Even though most of the following constructions works for arbitrary vector bundles, we limit ourselves to the case of line bundles. The next step is to associate a complex with a representation, Which generalizes the Chevalley-Eilenberg complex of a Lie algebra representation.

Definition 1.1.9 *Let $L \to M$ be a line bundle and let $A \to M$ be a Lie algebroid with a representation $\nabla\colon A \to DL$. Then $\Omega^{\bullet}_{(A,L)}(M) := \Gamma^\infty(\Lambda^\bullet A^* \otimes L)$ together with the differential*

$$\mathrm{d}_{(A,L)}\colon \Omega^{\bullet}_{(A,L)}(M) \ni \alpha \otimes \lambda \mapsto \mathrm{d}_A\alpha \otimes \lambda + e^i \wedge \alpha \otimes \nabla_{e_i}(\lambda) \in \Omega^{\bullet+1}_{(A,L)}(M),$$

where d_A is the usual Lie algebroid differential of $A \to M$ and $\{e_i\}_{i\in I}$ is a local basis with dual $\{e^i\}_{i\in I}$, is said to be the the de Rham complex of A with coefficients in L.

Note that it is easy to see that, first, $\mathrm{d}_{(A,L)}$ is independent of the choice of the local basis and second that it is in fact a differential. Moreover, by definition we can see that the de Rham complex with coefficients is a graded module for the usual de Rham complex, i.e. we can multiply

$$\Gamma^\infty(\Lambda^k A^*) \times \Omega^{\ell}_{(A,L)}(M) \ni (\alpha, \beta \otimes \lambda) \mapsto \alpha \cdot (\beta \otimes \lambda) := \alpha \wedge \beta \otimes \lambda \in \Omega^{k+\ell}_{(A,L)}(M)$$

and, additionally, we have

$$\mathrm{d}_{(A,L)}(\alpha \cdot B) = \mathrm{d}_A(\alpha) \cdot B + (-1)^{|\alpha|} \alpha \cdot \mathrm{d}_{(A,L)} B$$

for $\alpha \in \Gamma^\infty(\Lambda^\bullet A)$ and $B \in \Omega^{\bullet}_{(A,L)}(M)$. Before we consider examples, we briefly discuss morphisms. So let $L_i \to M_i$ be two line bundles and let $\nabla_i\colon A_i \to DL_i$ be two flat connections. A morphism between the triples $(L_i \to M_i, A_i, \nabla_i)$ is a pair (P, Φ) consisting of a Lie algebroid morphism $P\colon A_1 \to A_2$ and a regular line bundle morphism $\Phi\colon L_1 \to L_2$ covering the same map $\phi\colon M_1 \to M_2$, such that

$$\begin{tikzcd} A_1 \arrow[r, "P"] \arrow[d, "\nabla_i"] & A_2 \arrow[d, "\nabla_2"] \\ DL_1 \arrow[r, "D\Phi"] & DL_2 \end{tikzcd} \tag{1.1.3}$$

commutes. We can define the (pull-back) map

$$(K,\Phi)^*\colon \Omega^\bullet_{(A_2,L_2)}(M_2) \to \Omega^\bullet_{(A_1,L_1)}(M_1)$$

by

$$((K,\Phi)^*\alpha)_p(a_1,\dots,a_k) = \Phi_p^{-1}\alpha_{\phi(p)}(Ka_1,\dots,Ka_k)$$

for $\alpha \in \Omega^k_{(A_2,L_2)}(M_2)$ and $a_1,\dots,a_k \in A_{1,p}$. One can show that for $\alpha \in \Gamma^\infty(\Lambda^k A_2)$ and $\lambda \in \Gamma^\infty(L_2)$, we have that

$$(K,\Phi)^*(\alpha\otimes\lambda) = K^*\alpha \otimes \Phi^*\lambda, \tag{1.1.4}$$

where $K^*\colon \Gamma^\infty(A_2) \to \Gamma^\infty(A_1)$ is the usual pull-back of sections of vector bundles and Φ^* is the pull-back of sections of a line bundle along a regular line bundle morphism, see Equation (1.1.1).

Lemma 1.1.10 *Let $L_i \to M_i$ be two line bundles and let $\nabla_i\colon A_i \to DL_i$ be two flat connections. Then*

$$\mathrm{d}_{(A_1,L_1)} \circ (K,\Phi)^* = (K,\Phi)^* \circ \mathrm{d}_{(A_2,L_2)}$$

PROOF: This is a consequence of Equation (1.1.4). ΧΞΣ

The next two examples and their interaction are crucial throughout the whole thesis, so we will discuss them in quite some detail.

Example 1.1.11 (Tautological Representation) Let $L \to M$ be a line bundle. Let us denote the de Rham complex of DL with coefficients in L by $(\Omega_L(M), \mathrm{d}_L)$. Throughout this thesis we refer to elements of $\Omega_L(M)$ as *Atiyah forms*. In this particular case, we are able to compute its cohomology: if we denote by $\mathbb{1}\colon \Gamma^\infty(L) \to \Gamma^\infty(L)$ the identity operator, it is easy to show that

$$\mathrm{d}_L\iota_{\mathbb{1}} + \iota_{\mathbb{1}}\,\mathrm{d}_L = \mathsf{id} \tag{1.1.5}$$

and hence the cohomology of d_L is tivial. Moreover, for a regular line bundle morphism $\Phi\colon L_1 \to L_2$ between two line bundles $L_i \to M_i$, $(D\Phi,\Phi)$ is a morphism between the tautological representations. Moreover, every morphism of triples $(L_i \to M_i, DL_i, \mathsf{id})$ is of the form $(D\Phi,\Phi)$ for a regular line bundle morphism $\Phi\colon L_1 \to L_2$. In the rest of the thesis, we will denote

$$(D\Phi,\Phi)^* := \Phi^*.$$

Having a (local) trivialization $L_U \cong U \times \mathbb{R}$, we have seen in Subsection 1.1.2, that

$$(J^1L_U)^* \otimes L_U \cong DL_U \cong TU \oplus \mathbb{R}_U$$

and moreover that

$$J^1 L_U \cong (DL_U)^* \otimes L_U \cong T^*U \oplus \mathbb{R}_U.$$

Hence, we have that $\Lambda^\bullet (DL_U)^* \otimes L_U \cong \Lambda^\bullet(T^*U \oplus \mathbb{R}_U) \cong \Lambda^\bullet T^*U \oplus \Lambda^{\bullet-1}T^*U$. Throughout this thesis, we will write this splitting as

$$\alpha = \alpha_1 + \mathbb{1}^* \wedge \alpha_2,$$

for $\alpha \in \Lambda^k(DL_U)^* \otimes L_U$ and $(\alpha_1, \alpha_2) \in \Lambda^k T^*U \oplus \Lambda^{k-1}T^*U$, where $\mathbb{1}^* \in \Gamma^\infty(T^*U \oplus \mathbb{R}_U)$ is the canonical generator of the second factor. In this trivialization the differential d_L can be simply written as

$$\mathrm{d}_L(\alpha_1 + \mathbb{1}^* \wedge \alpha_2) = \mathrm{d}\alpha_1 + \mathbb{1}^* \wedge (\alpha_1 - \mathrm{d}\alpha_2),$$

for $\alpha_1 + \mathbb{1}^* \wedge \alpha_2 \in \Omega_{L_U}(U)$ and for the usual de Rham differential $\mathrm{d}\colon \Gamma^\infty(\Lambda^\bullet T^*M) \to \Gamma^\infty(\Lambda^{\bullet+1}T^*M)$.

The next example, which will occur in this thesis are flat connections on line bundles.

Example 1.1.12 (Flat TM-Connection) Let $L_i \to M_i$ be two line bundles and let $\nabla_i\colon TM_i \to DL_i$ be flat connections. First, we note that in this case we have that $\sigma_i \circ \nabla_i = \mathsf{id}_{TM_i}$ for the symbol maps $\sigma_i\colon DL_i \to TM_i$. Now let us consider a morphism (P, Φ) between $(L_1 \to M_1, TM_1, \nabla_1)$ and $(L_2 \to M_2, TM_2, \nabla_2)$ covering $\phi\colon M_1 \to M_2$. We conclude

$$\begin{aligned} P &= \sigma_2 \circ \nabla_2 \circ P = \sigma_2 \circ D\Phi \circ \nabla_1 \\ &= T\phi \circ \sigma_1 \circ \nabla_1 \\ &= T\phi, \end{aligned}$$

so a morphism of two flat connections on line bundles is completely determined by a regular line bundle morphism. In the case of a flat connection $\nabla\colon TM \to L$, we denote the de Rham complex with values in L by

$$(\Omega_\nabla(M), \mathrm{d}^\nabla).$$

Moreover, there is an interplay between Example 1.1.11 and the connection differential. Let us denote by $\mathbb{1}^* \in \Gamma^\infty((DL)^*)$ the section which is defined by

$$\mathbb{1}^*(\mathbb{1}) = 1 \quad \text{and} \quad \mathbb{1}^*(\nabla_X) = 0 \ \forall X \in TM,$$

which is well defined since $DL = \operatorname{im}(\nabla) \oplus \langle \mathbb{1} \rangle$. Note that this coincides with the $\mathbb{1}^*$ defined above for the trivial line bundle. Let us moreover use the symbol $\sigma\colon DL \to TM$, in order to define

$$\sigma^*\colon \Omega^k_\nabla(M) \ni \psi \mapsto \sigma^*\psi = \psi \circ (\sigma \otimes \ldots \otimes \sigma) \in \Omega^k_L(M).$$

Note that this map is not a chain map, since $\nabla \circ \sigma \neq \mathsf{id}$ and thus the diagram (1.1.3) does not commute. But we have that for all $\alpha \in \Omega^k_L(M)$, there exist unique $\alpha_1 \in \Omega^k_\nabla(M)$ and $\alpha_2 \in \Omega^{k-1}_\nabla(M)$, such that

$$\alpha = \sigma^*\alpha_1 + \mathbb{1}^* \wedge \sigma^*\alpha_2,$$

which are defined by

$$\alpha_1(X_1, \ldots, X_k) := \alpha(\nabla_{X_1}, \ldots, \nabla_{X_k})$$

and

$$\alpha_2(X_1, \ldots, X_{k-1}) = \alpha(\mathbb{1}, \nabla_{X_1}, \ldots, \nabla_{X_{k-1}})$$

for $X_i \in TM$, $i = 1, \ldots, k$. Moreover, the differential d_L can be computed by

$$\mathrm{d}_L\alpha = \mathrm{d}_L(\sigma^*\alpha_1 + \mathbb{1}^* \wedge \sigma^*\alpha_2) = \sigma^*(\mathrm{d}^\nabla\alpha_1) + \mathbb{1}^* \wedge \sigma^*(\alpha_1 - \mathrm{d}^\nabla\alpha_2).$$

Now the local structure of Example 1.1.11 shines in a new light. By identifying $\mathscr{C}^\infty(U)$ with $\Gamma^\infty(L\big|_U)$ we already have chosen the canonical connection which makes the trivializing section flat and the connection differential is just the de Rham differential.

Note that a lot of different geometric structures can be understood via the de Rham complex with coefficients in a representation. This includes locally conformal symplectic structures and contact bundles, which we discuss at a later stage of this chapter. The de Rham complex of a Lie algbroid with coefficients in a line bundle is a modification of the usual de Rham complex of a Lie algebroid. In classical differential geometry, the de Rham complex structure on $\Gamma^\infty(\Lambda^\bullet A^*)$ is equivalent to have a Gerstenhaber-like structure on $\Gamma^\infty(\Lambda^\bullet A)$, i.e. a bracket

$$[-,-]\colon \Gamma^\infty(\Lambda^\bullet A) \times \Gamma^\infty(\Lambda^\bullet A) \to \Gamma^\infty(\Lambda^{\bullet+\bullet-1} A)$$

which fulfills for homogeneous elements a_i with degree $|a_i|$:

i.) $[a_1, a_2] = -(-1)^{(|a_1|-1)(|a_2|-1)}[a_2, a_1]$

ii.) $[a_1, a_2 \wedge a_3] = [a_1, a_2] \wedge a_3 + (-1)^{(|a_1|-1)|a_2|} a_2 \wedge [a_1, a_3]$

iii.) $[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + (-1)^{(|a_1|-1)(|a_2|-1)}[a_2, [a_1, a_3]]$

We want to follow now [43] and mimic this construction for the de Rham complex with coefficients in a line bundle. Let us now introduce the notion of a *Gerstenhaber-Jacobi* algebra and see that our case fits into this framework.

Definition 1.1.13 ([43, Def. 1.9]) *A Gerstenhaber-Jacobi algebra is given by a graded commutative unital algebra $\mathscr{A}$ and a graded $\mathscr{A}$-module $\mathscr{L}$ together with a graded Lie bracket $[-,-]\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ and an action by derivations, $\lambda \mapsto X_\lambda$ of $\mathscr{L}$ on $\mathscr{A}$, such that*

$$[\lambda, a\mu] = X_\lambda(a)\mu + (-1)^{|\lambda||a|}a[\lambda, \mu]$$

for homogeneous $a \in \mathscr{A}$ and $\lambda, \mu \in \mathscr{L}$.

Our case is a special case of a Gerstenhaber-Jacobi algebra. Let us from now on denote $A_L := A \otimes L^*$ for a Lie alebroid A and a line bundle L.

Lemma 1.1.14 ([23, Thm. 5]) *Let $L \to M$ be a line bundle and let $A \to M$ be a Lie algebroid with a representation $\nabla\colon A \to DL$. Then there is a graded Lie bracket $[-,-]_{(A,L)}$ on $\Gamma^\infty(\Lambda^\bullet A_L \otimes L)$ uniquely determined by*

i.) $[\lambda, \mu]_{(A,L)} = 0$ *for* $\lambda, \mu \in \Gamma^\infty(L)$

ii.) $[\Box, \lambda]_{(A,L)} = \nabla_\Box \lambda$ *for* $\lambda \in \Gamma^\infty(L)$ *and* $\Box \in \Gamma^\infty(A) = \Gamma^\infty(A \otimes L^* \otimes L)$

iii.) $[\Delta, \Box]_{(A,L)} = [\Delta, \Box]$ *for* $\Delta, \Box \in \Gamma^\infty(A)$

Moreover, the pair $\Gamma^\infty(\Lambda^\bullet A_L)$ and $\Gamma^\infty(\Lambda^\bullet A_L \otimes L)$ forms a Gerstenhaber-Jacobi algebra.

For our purposes, we shall mention the case of the tautological representation from Example 1.1.7. In this case, using $DL_L = DL \otimes L^* = (J^1L)^*$, we get a Gerstenhaber-Jacobi structure on

$$\Gamma^\infty(\Lambda^\bullet (J^1L)^* \otimes L),$$

where we will denote the Gerstenhaber-Jacobi bracket simply by $[\![-,-]\!]_L$. This is the Gerstenhaber-Jacobi algebra of first order multidifferential operators and $[\![-,-]\!]_L$ is given by the usual Gerstenhaber formula.

1.2 Jacobi Related Geometries

The framework for generalized geometry in odd dimensions is the so-called *omni-Lie algebroid* of a line bundle $L \to M$ and a specific type of subbundles of them, so-called *Dirac-Jacobi bundles.*

They were introduced in [46] by Vitagliano and are a slight generalization of Wade's $\mathcal{E}^1(M)$-Dirac structures (see [49]). Moreover, these bundles are a straightforward Dirac theoretic generalization of Jacobi bundles, as usual Dirac structures are for Poisson manifolds. The aim of this section is to, first, define the omni-Lie algebroid and afterwards discuss Dirac-Jacobi bundles and their characteristic foliation.

1.2.1 The Omni-Lie Algebroid of a Line Bundle and its Automorphisms

The omni-Lie algebroid was first introduced in [14] in order to connect Dirac-like subbundles to Lie algebroids. Our aim is a little bit different, since we are interested in the Dirac analogue in Jacobi geometry, which are not Lie algebroids. Nevertheless, we adapt the notion of Chen and Liu from [14] and combine them with notions coming from Dirac geometry. This sections follows exactly the same lines as [38].

Definition 1.2.1 *Let $L \to M$ be a line bundle and let $H \in \Omega^3_L(M)$ be closed. The vector bundle $\mathbb{D}L := DL \oplus J^1L$ together with* *i.) the (Dorfman-like, H-twisted) bracket*

$$[\![(\Delta_1, \psi_1), (\Delta_2, \psi_2)]\!]_H = ([\Delta_1, \Delta_2], \mathscr{L}_{\Delta_1}\psi_2 - \iota_{\Delta_2} \mathrm{d}_L\psi_1 + \iota_{\Delta_1}\iota_{\Delta_2}H)$$

ii.) the non-degenerate L-valued pairing

$$\langle\!\langle(\Delta_1, \psi_1), (\Delta_2, \psi_2)\rangle\!\rangle := \psi_1(\Delta_2) + \psi_2(\Delta_1)$$

iii.) the canonical projection $\mathrm{pr}_D\colon \mathbb{D}L \to DL$

is called the H-twisted omni-Lie algebroid of $L \to M$*.*

We collect now some of the main properties of the defining structures of the Courant-Jacobi algebroid.

Lemma 1.2.2 *Let* $L \to M$ *be a line bundle and let* $H \in \Omega^3_L(M)$ *be closed. Then*

i.) $\langle\!\langle -, - \rangle\!\rangle$ *is an L-valued bilinear form of split signature* $(\dim(M)+1, \dim(M)+1)$

ii.) $\mathrm{pr}_D[\![(\Delta_1, \psi_1), (\Delta_2, \psi_2)]\!]_H = [\Delta_1, \Delta_2]$ *for* $(\Delta_i, \psi_i) \in \Gamma^\infty(\mathbb{D}L)$

Proof: These are straightforward computations using the very definitions of $\langle\!\langle -, - \rangle\!\rangle$ and $[\![-, -]\!]$. ΧΞΣ

Remark 1.2.3 If $H = 0$, we will refer to $(\mathbb{D}L, [\![-, -]\!], \langle\!\langle -, - \rangle\!\rangle)$ as the omni-Lie algebroid.

Remark 1.2.4 Identifying $J^1L = (DL)^* \otimes L$, we can see the omni-Lie algebroid as a special case of

$$A \oplus (A^* \otimes L)$$

for a Lie algebroid A and a representation $\nabla\colon A \to DL$ and the obvious adaptions of its Cartan calculus. In a later stage, we will discuss also the case $A = TM$, nevertheless we prefer not to discuss these objects in general, since we are not going to use it in its full generality.

Let us now discuss (auto-)morphisms of the omni-Lie algebroid.

Definition 1.2.5 *Let* $L \to M$ *be a line bundle and let* $H \in \Omega^3_L(M)$ *be closed. A pair* $(F, \Phi) \in \mathsf{Aut}(\mathbb{D}L) \times \mathsf{Aut}(L)$ *is called a (H-twisted) Courant-Jacobi automorphism, if*

i.) $D\Phi\colon \mathrm{pr}_D = \mathrm{pr}_D \circ F$

ii.) $\Phi^*\langle\!\langle -, - \rangle\!\rangle = \langle\!\langle F-, F- \rangle\!\rangle$

iii.) $F^*[\![-, -]\!]_H = [\![F^*-, F^*-]\!]_H$

The group of H-twisted Courant-Jacobi automorphism is denoted by $\mathsf{Aut}^H_{CJ}(L)$*.*

It is easy to see that an automorphism $\Phi \in \mathsf{Aut}(L)$ defines an automorphism of the omni-Lie algebroid by

$$\mathbb{D}\Phi\colon \mathbb{D}L \ni (\Delta, \alpha) \mapsto (D\Phi(\Delta), (D\Phi^{-1})^*\alpha) \in \mathbb{D}L.$$

This might fail for the H-twisted omni-Lie algebroid for an arbitrary H. In fact the it is an H-twisted Courant-Jacobi automorphism if and only if $\Phi^*H = H$.

The 2-form $B \in \Omega^2_L(M)$ defines the map

$$\exp(B)\colon \mathbb{D}L \ni (\Delta, \alpha) \mapsto (\Delta, \alpha + \iota_\Delta B) \in \mathbb{D}L,$$

which also fulfills conditions *i.)* and *ii.)* in Definition 1.2.5 and fulfills condition *iii.)* if and only if $\mathrm{d}_L B = 0$. In this case, we refer to B as a B-field.

The semi-direct product of these special kind of automorphisms span all the Courant-Jacobi automorphism group:

Lemma 1.2.6 *Let $L \to M$ be a line bundle and let $H \in \Omega^3_L(M)$ be closed. Then*

$$\mathcal{I}_H\colon Z^2_L(M) \rtimes \mathsf{Aut}(L) \ni (B, \Phi) \mapsto (\exp(B + \iota_{\mathbb{1}}(H - \Phi_* H)) \circ \mathbb{D}\Phi, \Phi) \in \mathsf{Aut}^H_{CJ}(L)$$

is an ismorphism of groups.

Proof: The proofs are a straightforward verification using the deifintion of Courant-Jacobi automorphism. Moreover, it can be found in [41]. ΧΞΣ

Let us now focus on infinitesimal automorphisms of the omni-Lie algebroid.

Definition 1.2.7 *Let $L \to M$ be line bundle and let $H \in \Omega^3_L(M)$ be closed. A pair $(D, \Delta) \in \Gamma^\infty(D\mathbb{D}L) \times \Gamma^\infty(DL)$ is called infinitesimal (H-twisted) Courant-Jacobi automorphism, if*

i.) $[\Delta, \mathrm{pr}_D(\varepsilon)] = \mathrm{pr}_D(D(\varepsilon))$

ii.) $\Delta\langle\!\langle \varepsilon, \chi \rangle\!\rangle = \langle\!\langle D(\varepsilon), \xi \rangle\!\rangle + \langle\!\langle \epsilon, D(\chi) \rangle\!\rangle$

iii.) $D([\![\varepsilon, \chi]\!]_H) = [\![D(\epsilon), \chi]\!]_H + [\![\epsilon, D(\chi)]\!]_H$

for all $\varepsilon, \chi \in \Gamma^\infty(\mathbb{D}L)$. The Lie algebra of infinitesimal (H-twisted) Courant-Jacobi automorphisms is denoted by $\mathfrak{aut}^H_{CJ}(L)$.

It is not very surprising that we can also find here an easy description of them.

Lemma 1.2.8 *Let $L \to M$ be line bundle and let $H \in \Omega^3_L(M)$ be closed. Then the map*

$$\mathfrak{i}_H\colon Z^2_L(M) \rtimes \Gamma^\infty(DL) \to \mathfrak{aut}^H_{CJ}(L)$$

with

$$(B, \Delta) \to ((\Box, \beta) \mapsto ([\Delta, \Box], \mathscr{L}_\Delta \beta + \iota_\Box(B - \mathscr{L}_\Delta \iota_{\mathbb{1}} H)))$$

is an isomorphism of Lie algebras.

Proof: Similarly to Lemma 1.2.6, the proof can be found in [41]. XΞΣ

If $(\Delta, \alpha) \in \Gamma^\infty(\mathbb{D}L)$ then we see that the map $[\![(\Delta, \alpha), -]\!]_H$ is an infinitesimal (H-twisted) Courant-Jacobi automorphism. Applying the inverse of the isomorphism from Lemma 1.2.9 to this element, we find

$$\mathrm{i}_H(\mathrm{d}_L(\iota_\Delta \iota_\mathbb{1} H - \alpha), \Delta) = [\![(\Delta, \alpha), -]\!]_H$$

The term "infinitesimal automorphisms" suggests that they integrate to automorphisms. In fact we can compute the flow of an infinitesimal automorphism fairly explicit which is discussed in the following

Lemma 1.2.9 *Let $L \to M$ be line bundle and let $H \in \Omega^3_L(M)$ be closed. Let additionally $(\alpha, \Delta) \in Z^2_L(M) \rtimes \Gamma^\infty(DL)$. The flow of $\mathrm{i}_H(B, \Delta)$ is given by*

$$\begin{aligned}\mathcal{I}_H(\gamma_t, \Phi^\Delta_t) &= \mathcal{I}_H\bigg(- \int_0^t (\Phi^\Delta_{-\tau})^* B \, \mathrm{d}\tau, \Phi^\Delta_t \bigg) \\ &= \exp\bigg(- \int_0^t (\Phi^\Delta_{-\tau})^*(B) \, \mathrm{d}\tau + \iota_\mathbb{1}(H - (\Phi^\Delta_t)_* H) \bigg) \circ \mathbb{D}\Phi^\Delta_t.\end{aligned}$$

Proof: The proof of this Lemma is an obvious adaption of the corresponding statement for Courant automorphisms in [12] or equivalently an easy computation just by deriving both sides of the equation by t. XΞΣ

Corollary 1.2.10 *Let $L \to M$ be a line bundle and let $H \in \Omega^3_L(M)$ be closed. For every $(\Delta, \alpha) \in \Gamma^\infty(\mathbb{D}L)$ the flow of $[\![(\Delta, \alpha), -]\!]_H$ is given by*

$$\exp\bigg(\int_0^t (\Phi^\Delta_{-\tau})^*(\mathrm{d}_L \alpha + \iota_\Delta H) \, \mathrm{d}\tau \bigg) \circ \mathbb{D}\Phi^\Delta_t.$$

1.2.2 Dirac-Jacobi bundles and their characteristic Foliation

Let us now discuss the subbundles of the omni-Lie algebroid we are interested in: the so-called *Dirac-Jacobi bundles*. They are, roughly speaking, Dirac-like subbundles of the omni-Lie algebroid and they were also introduced in [14]. The first time Dirac-like structures appeared in order to model the Dirac analogue in Jacobi geometry was in [49], which were called $\mathcal{E}^1(M)$-Dirac structures. Note that these bundles are special cases of the Dirac-Jacobi bundles we will define, if the line bundle of the omni-Lie algebroid is trivial.

Dirac-Jacobi Bundles

Definition 1.2.11 *Let $L \to M$ be a line bundle and let $H \in \Omega^3_L(M)$ be closed. A subbundle $\mathcal{L} \subseteq \mathbb{D}L$ is called a (H-twisted) Dirac-Jacobi structure, if*

i.) $\mathcal{L}$ is involutive with respect to $[\![-, -]\!]_H$, *ii.)* $\mathcal{L}$ *is maximally isotropic with respect to* $\langle\!\langle -, - \rangle\!\rangle$.

Moreover, if $H = 0$*, we will call* $\mathcal{L}$ *simply Dirac-Jacobi structure.*

Remark 1.2.12 We call a line bundle equipped with a Dirac-Jacobi structure simply *Dirac-Jacobi bundle.*

The isotropy condition in the definition of Dirac-Jacobi structures makes sense, because the pairing $\langle\!\langle -, - \rangle\!\rangle$ has split signature. Moreover, the involutivity of a maximally isotropic subbundle $\mathcal{L}$ is equivalent to the vanishing of the tensor field

$$N_{\mathcal{L}} \colon \Lambda^3 \mathcal{L} \to L,$$

which is defined by

$$N_{\mathcal{L}}(X, Y, Z) = \langle X, [\![Y, Z]\!]_H \rangle. \tag{1.2.1}$$

Example 1.2.13 Let $L \to M$ be a line bundle and $\omega \in \Omega^2_L(M)$ be an Atiyah 2-form. Then $\mathcal{L}_\omega := \{(\Delta, \iota_\Delta \omega) \in \mathbb{D}L \mid \Delta \in DL\}$ is a maximally isotropic subbundle, which is involutive if and only if $\mathrm{d}_L \omega = 0$. Moreover, a Dirac-Jacobi structure $\mathcal{L}$ is the graph of a 2-form if and only if $J^1 L \cap \mathcal{L} = \{0\}$. We omit the proof of this statement here and refer to the proof of the upcoming Lemma 1.2.35, which is in the same spirit.

Example 1.2.14 Let $L \to M$ be a line bundle and let $K \subseteq DL$ be an involutive subbundle, i.e. for all $\Delta, \Box \in \Gamma^\infty(K)$ the commutator $[\Delta, \Box]$ is a section of K. The subbundle

$$K \oplus \mathrm{Ann}(K) \subseteq \mathbb{D}L$$

for $\mathrm{Ann}(K) := \{\alpha \in J^1 L \mid \alpha(K) = 0\}$ is a Dirac-Jacobi structure.

We shall now focus on morphisms between Dirac-Jacobi bundles. As in the Dirac case there are two different kinds

Definition 1.2.15 *Let* $L_i \to M_i$ *be line bundles for* $i = 1, 2$ *and let* $\mathcal{L}_i \in \mathbb{D}L_i$ *be* H_i*-twisted Dirac-Jacobi bundles. A regular line bundle morphism* $\Phi \colon L_1 \to L_2$ *is called*

i.) *a* forward Dirac-Jacobi *map, if*

$$\mathfrak{F}_\Phi(\mathcal{L}_1) := \{(D\Phi(\Delta), \psi) \in \mathbb{D}L_2 \mid (\Delta, D\Phi^*\psi) \in \mathcal{L}_1\} = \mathcal{L}_2\big|_{\phi(M_1)}$$

and $H_1 = \Phi^* H_2$.

ii.) *a* backward Dirac-Jacobi *map, if*

$$\mathfrak{B}_\Phi(\mathcal{L}_2) := \{(\Delta, D\Phi^*\psi) \in \mathbb{D}L_1 \mid (D\Phi(\Delta), \psi) \in \mathcal{L}_2\} = \mathcal{L}_1$$

and $H_1 = \Phi^* H_2$.

It is worth mentioning that with both morphisms Dirac-Jacobi bundels become a category. The morphisms which are more interesting for us are the backward maps. So let us discuss them in some detail. For a regular line bundle morphism $\Phi\colon L_1 \to L_2$ ($L_i \to M_i$) and a H_2-twisted Dirac-Jacobi structure $\mathcal{L}_2 \subseteq \mathbb{D}L_2$ the family of vector spaces, one may define

$$\mathcal{L}_1 := \mathfrak{B}_\Phi(\mathcal{L}_2).$$

If $\mathcal{L}_1$ is a Φ_1^H-twisted Dirac-Jacobi structure, then Φ is clearly a backwards map. But $\mathcal{L}_1$ is not necessarily a Driac-Jacobi structure:

Lemma 1.2.16 *Let $L_i \to M_i$ be line bundles for $i = 1, 2$ and let $\mathcal{L} \in \mathbb{D}L_2$ be an H-twisted Dirac-Jacobi structure. Then the family of vector spaces $\mathfrak{B}_\Phi(\mathcal{L})$ is maximally isotropic and if it is a subbundle, then it is a Φ^*H-twisted Dirac-Jacobi structure.*

PROOF: The isotropy is a pointwise condition and is fulfilled, which can be seen by an elementary computation. The proof of the involutivity follows the same lines as the proof in the Dirac case (see e.g. [11, Section 5.2.1]). ΧΞΣ

From now on we will refer to $\mathfrak{B}_\Phi(\mathcal{L})$ as the backward transform of the Dirac-Jacobi structure $\mathcal{L}$. Let us examine under which circumstances the backward transform of a Dirac-Jacobi bundle is a subbundle. A very useful tool is the following

Theorem 1.2.17 *[46, Prop. 8.4] Let $\Phi\colon L_1 \to L_2$ be a regular line bundle morphism over $\phi : M_1 \to M_2$ and let $\mathcal{L} \in \mathbb{D}L_2$ be a Dirac-Jacobi bundle. If $\ker D\Phi^* \cap \phi^*\mathcal{L}$ has constant rank, then $\mathfrak{B}_\Phi(\mathcal{L})$ is a Dirac-Jacobi bundle.*

Let us deduce a very useful corollary from this theorem

Corollary 1.2.18 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi bundle and let $\Phi\colon L_N \to L$ be a regular line bundle morphism covering a smooth map $\phi\colon N \to M$, such that*

$$D\Phi(DL_N) + \mathrm{pr}_D\mathcal{L} = DL\big|_{\phi(N)}.$$

Then $\mathfrak{B}_\Phi(\mathcal{L}) \subseteq \mathbb{D}L_N$ is a Dirac-Jacobi structure.

PROOF: Let $(n, (0, \alpha)) \in \ker D\Phi^* \cap \phi^*\mathcal{L}$, thus $(0, \alpha) \in \mathcal{L}_{\phi(n)}$ and $\alpha(\operatorname{im} D\Phi) = 0$. Using $\mathcal{L} \cap J^1L = \operatorname{Ann}(\mathrm{pr}_D\mathcal{L})$, which is a consequence of $\mathcal{L}$ being maximally isotropic, we see that $\alpha(\operatorname{im}(D\Phi) + \mathrm{pr}_D\mathcal{L}) = 0$. By the condition

$$D\Phi(DL_N) + \mathrm{pr}_D\mathcal{L} = DL\big|_{\phi(N)}$$

we conclude $\alpha = 0$ and hence $\ker D\Phi^* \cap \phi^*\mathcal{L} = \{0\}$. By Thereom 1.2.17 the claim follows. ΧΞΣ

Remark 1.2.19 For a Dirac-Jacobi bundle $(L \to M, \mathcal{L})$ and a regular line bundle morphism $\Phi\colon L_N \to L$ covering a smooth map $\phi\colon M \to N$, such that

$$D\Phi(DL_N) + \mathrm{pr}_D\mathcal{L} = DL\big|_{\phi(N)},$$

we will say Φ is transversal to $\mathcal{L}$. If $N \hookrightarrow M$ is a submanifold, we say that N is a transversal.

Corollary 1.2.18 implies for example that the backwards transform for a submersion is always a Dirac-Jacobi structure. Note that we have in special cases that backward and forward transforms are inverse to each other, which will be useful throughout this thesis.

Corollary 1.2.20 *Let $L_i \to M_i$ be line bundles for $i = 1, 2$, let $\mathcal{L}_2 \subseteq \mathbb{D}L_2$ be a Dirac-Jacobi bundle and let $\Phi\colon L_1 \to L_2$ be a regular line bundle morphism. If Φ covers a surjective submersion, then*

$$\mathfrak{F}_\Phi(\mathfrak{B}_\Phi(\mathcal{L}_2)) = \mathcal{L}_2$$

Let us now discuss the relation of a Courant-Jacobi automorphims of the form

$$\mathbb{D}\Phi\colon \mathbb{D}L \to \mathbb{D}L$$

to backward transforms. For a Dirac-Jacobi structure $\mathcal{L} \subseteq \mathbb{D}L$, we obtain that

$$\mathbb{D}\Phi(\mathcal{L}) = \mathfrak{B}_{\Phi^{-1}}(\mathcal{L})$$

which is a Dirac-Jacobi structure again. Moreover, given a closed 2-form $B \in \Omega^2_L(M)$, we get that

$$\mathcal{L}^B := \exp(B)\mathcal{L}$$

is again a Dirac-Jacobi structure, which can be shown by an easy computation.

The characteristic Foliation of a Dirac-Jacobi Bundle

For a Dirac-Jacobi structure $\mathcal{L} \subseteq \mathbb{D}L$ on $L \to M$, we define

$$K_\mathcal{L} = \sigma(\mathrm{pr}_D(\mathcal{L})) \subseteq TM.$$

Since both maps, $\sigma\colon DL \to TM$ and $\mathrm{pr}_D\colon \mathcal{L} \to DL$, are Lie algebroid maps, we see that $K_\mathcal{L}$ is a singular involutive distribution.

Lemma 1.2.21 ([46, Chapt. 5]) *Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq \mathbb{D}L$ be a Dirac-Jacobi structure. Then the singular involutive distribution $D_\mathcal{L}$ is integrable in the sense of Stefan-Sussmann.*

PROOF: It is easy to show that $\sigma \circ \mathrm{pr}_D$ turns $\mathcal{L}$ into a Lie algebroid. But the characteristic distribution of a Lie algebroid is always integrable, see e.g. [19]. ΧΞΣ

The leaves of $K_{\mathcal{L}}$ assemble what will be referred to as the characteristic foliation of $\mathcal{L}$.

For a give leaf $S \hookrightarrow M$ of the foliation of a Dirac-Jacobi structure $\mathcal{L}$, we define the pull-back line bundle $L_S \to S$ via the diagram

$$\begin{tikzcd} L_S \arrow[r, "I"] \arrow[d] & L \arrow[d] \\ S \arrow[r] & M \end{tikzcd}.$$

The map $I\colon L_S \to L$ is clearly a regular line bundle morphism, which allows us to consider the backward transform of the Dirac-Jacobi structure $\mathcal{L}$.

Lemma 1.2.22 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}L$ be a Dirac-Jacobi structure and let $\iota\colon S \hookrightarrow M$ be one of its leaves. Then the backwards transform $\mathfrak{B}_I(\mathcal{L})$ is a subbundle and hence a Dirac-Jacobi structure.*

PROOF: We want to make use of Theorem 1.2.17. We have that the $\ker(DI^*) \subseteq \mathrm{Ann}(\mathrm{im}(DI)) \subseteq J^1L$ and moreover, using the maximal isotropy of $\mathcal{L}$, we get $J^1L \cap \mathcal{L} = \mathrm{Ann}(\mathrm{pr}_D(\mathcal{L})$ and hence

$$\ker(DI^*) \cap \mathcal{L}\big|_S = \mathrm{Ann}(\mathrm{im}(DI)) \cap \mathrm{Ann}(\mathrm{pr}_D(\mathcal{L}_S)) = \mathrm{Ann}(\mathrm{im}(DI) + \mathrm{pr}_D(\mathcal{L}\big|_S)).$$

But since S is a leaf, we have that $\sigma(\mathrm{pr}_D(\mathcal{L}\big|_N)) = T\iota(TS)$ and thus $\mathrm{pr}_D(\mathcal{L}\big|_N) \subseteq \mathrm{im}(DI)$. Therefore, we have

$$\ker(DI^*) \cap \mathcal{L}\big|_S = \mathrm{Ann}(\mathrm{im}(DI))$$

and the intersection has constant rank. ΧΞΣ

Since the symbol map has a one dimensional kernel, we can distinguish two kinds of leaves:

Definition 1.2.23 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}L$ be a Dirac-Jacobi structure and $\iota : S \hookrightarrow M$ be a leaf. Then S is said to be*

i.) pre-contact, if $\mathrm{rank}(\mathrm{pr}_D(\mathfrak{B}_I(\mathcal{L}))) = \dim(S) + 1$.

ii.) locally conformal pre-symplectic, if $\mathrm{rank}(\mathrm{pr}_D(\mathfrak{B}_I(\mathcal{L}))) = \dim(S)$.

Note that this distinction is the first and probably the most significant conceptual difference between Dirac-Jacobi structures and classical Dirac structures. Before we explain the names of the different leaves, we want to ensure that this definition makes sense at all, i.e. if $\mathrm{pr}_D(\mathfrak{B}_I(\mathcal{L}))$ is constant along the leaves. Let us choose a point

$s_0 \in S$ and an arbitrary vector field $X \in \Gamma^\infty(TS)$. Since S is a leaf, we have that $\sigma(\mathrm{pr}_D(\mathcal{L})\big|_S) = TS$ and hence there is a section $(\Delta, \alpha) \in \Gamma^\infty(\mathcal{L})$ such that $\sigma(\Delta)\big|_S = X$. Moreover, since (Δ, α) is a section of $\mathcal{L}$, its flow $\exp(\gamma_t) \circ \mathbb{D}\Phi_t^\Delta$ preserves $\mathcal{L}$, but this means

$$\mathrm{pr}_D(\mathcal{L}_{\phi_t^X(s_0)}) = \mathrm{pr}_D(\exp(\gamma_t)(\mathbb{D}\Phi_t^\Delta(\mathcal{L}\big|_{s_0}))) = D\Phi_t^\Delta(\mathrm{pr}_D\mathcal{L}\big|_{s_0}),$$

where we denote by ϕ_t^X the flow of X. Since Φ_t^Δ is an automorphism also $D\Phi_t^\Delta$ is. Thus the rank of $\mathrm{pr}_D\mathcal{L}\big|_{s_0}$ gets preserved along the flowlines of X, but X was arbitrary and the claim follows by the connectedness of S.

Remark 1.2.24 For a Dirac-Jacobi bundle $(L \to M, \mathcal{L})$, we can distinguish two kinds of points:

i.) $p \in M$ is called pre-contact point, if $\mathbb{1}_p \in \mathrm{pr}_D(\mathcal{L})$.

ii.) $p \in M$ is called locally conformal pre-symplectic point, if $\mathbb{1}_p \notin \mathrm{pr}_D(\mathcal{L})$.

Moreover, every point in M is either pre-contact or locally conformal pre-symplectic.

In Dirac geometry the leaves have an induced pre- symplectic form, which is induced via the backward transform of the Dirac structure via the inclusion. In the case of Dirac-Jacobi bundles it is a bit different, since they admit two different kind of leaves which have different induced structures. Let us start with pre-contact leaves, which are very similar to pre-symplectic leaves in Dirac geometry.

Lemma 1.2.25 *Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq \mathbb{D}L$ be an H-twisted Dirac structure, such that $\mathrm{pr}_D(\mathcal{L}) = DL$. Then there exists a unique 2-form $\omega \in \Omega_L^2(M)$, such that*

$$\mathcal{L} = \{(\Delta, \iota_\Delta \omega) \in \mathbb{D}L \mid \Delta \in DL\}$$

and $\mathrm{d}_L\omega = H$.

PROOF: We only prove the existence of such an ω, since $\mathrm{d}_L\omega = H$ follows by involutivity. Since we have $\mathrm{rank}(\mathcal{L}) = \mathrm{rank}(DL) = \dim(M) + 1$, we see that $\mathrm{pr}_D\big|_{\mathcal{L}} \colon \mathcal{L} \to DL$ is an isomorphism, hence there exists a unique inverse $\tau \colon DL \to \mathcal{L}$. We define the map $J \colon DL \to J^1L$ by

$$J = \mathrm{pr}_{J^1L} \circ \tau.$$

Note that this means $\tau(\Delta) = (\Delta, J(\Delta))$, since for $\Delta_1, \Delta_2 \in DL$, we have, using the isotropy,

$$0 = \langle(\Delta_1, J(\Delta_1)), (\Delta_2, J(\Delta_2))\rangle = J(\Delta_1)(\Delta_2) + J(\Delta_2)(\Delta_1)$$

and hence

$$\omega(\Delta_1, \Delta_2) := J(\Delta_1)(\Delta_2)$$

is a well-defined Atiyah 2-form and the claim follows. ΧΞΣ

We can apply Lemma 1.2.25 directly to the case of pre-contact leaves, since (by their very definition) they are equipped with Dirac-Jacobi structures of this kind. We will call the corresponding 2-form a *pre-contact* form and we will see later why. Let us now turn to locally conformal pre-symplectic leaves. The following lemma can be found in [46] for $H = 0$.

Lemma 1.2.26 *Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq \mathbb{D}L$ be an H-twisted Dirac structure, such that $\sigma\colon \mathrm{pr}_D(\mathcal{L}) \to TM$ is an isomorphism. Then there exists a canonical flat connection $\nabla\colon TM \to DL$ and a L-valued 2-form $\omega \in \Gamma^\infty(\Lambda^2 TM \otimes L)$, such that*

$$\mathcal{L} = \{\nabla_X, (\iota_X\omega)\circ\sigma + \alpha \in \mathbb{D}L \mid X \in TM \text{ and } \alpha \in \mathrm{Ann}(\mathrm{im}(\nabla))\}$$

and $\mathrm{d}^\nabla\omega(X_1, X_2, X_3) = H(\nabla_{X_1}, \nabla_{X_2}, \nabla_{X_3})$ for all $X_i \in TM$.

Proof: As in Lemma 1.2.25, we just prove the existence of ω and the additional properties follow immediately from the maximal isotropy and the involutivity. Since $\sigma\big|_{\mathrm{pr}_D(\mathcal{L})}\colon \mathrm{pr}_D(\mathcal{L}) \to TM$ is an isomorphism, it has an inverse $\nabla\colon TM \to \mathrm{pr}_D(\mathcal{L}) \subseteq DL$. This is in fact a connection by definition, we just have to show that it is flat. Let $X, Y \in \Gamma^\infty(TM)$, then we consider $\nabla_X, \nabla_Y, \nabla_{[X,Y]} \in \Gamma^\infty(\mathrm{pr}_D(\mathcal{L}))$. We have, since σ inverts∇,

$$\sigma(\nabla_{[X,Y]}) = [X, Y] = [\sigma(\nabla_X), \sigma(\nabla_Y)] = \sigma([\nabla_X, \nabla_Y]).$$

Using that $\Gamma^\infty(\mathrm{pr}_D(\mathcal{L}))$ is closed under the commutator, we can conclude that ∇ is flat. We claim now that for each $X \in TM$ there is a unique $J(X) \in J^1L$ such that $(\nabla_X, J(X)) \in \mathcal{L}$ and $J(X)(\mathbb{1}) = 0$. First we recall that a connection always induces a splitting $DL = \mathrm{im}(\nabla) \oplus \langle \mathbb{1}\rangle$. Let us define the element $\mathbb{1}^* \in \Gamma^\infty((DL)^*)$ by

$$\mathbb{1}^*(\mathbb{1}) = 1 \text{ and } \mathbb{1}^*(\mathrm{im}(\nabla)) = 0.$$

We consider $X \in TM$ and choose a $\psi \in J^1L$ such that $(\nabla_X, \psi) \in \mathcal{L}$. It is easy to see, using the maximal isotropy of $\mathcal{L}$, that $(0, \mathbb{1}^* \otimes \psi(\mathbb{1})) \in \mathcal{L}$, which means that

$$(\nabla_X, \psi - \mathbb{1}^* \otimes \psi(\mathbb{1})) \in \mathcal{L}.$$

It is easy to see that $J(X) := \psi - \mathbb{1}^* \otimes \psi(\mathbb{1})$ is independent of the choice of ψ and hence unique, moreover it vanishes clearly on $\mathbb{1}$. We can prove, using isotropy again, that $J(X)(\nabla_Y) = -J(Y)(\nabla_X)$. Defining $\omega \in \Gamma^\infty(\Lambda^2 T^*M \otimes L)$ by

$$\omega(X, Y) = J(X)(\nabla_Y),$$

the claim follows. ΧΞΣ

A locally conformal pre-symplectic leaf is clearly equipped with one of these Dirac-Jacbobi structures and hence in the form of Lemma 1.2.26. Let us summarize the previous results and discussions in the following

Corollary 1.2.27 *Let $L \to M$ be line bundle, let $\mathcal{L} \subseteq DL$ be a H-twisted Dirac-Jacobi bundle and let $\iota\colon S \hookrightarrow M$ be a leaf of its characteristic foliation. If S is a*

i.) *pre-contact leaf, then there exists a unique $\omega \in \Omega^2_{L_S}(S)$, such that*

$$\mathfrak{B}_I(\mathcal{L}) = \{(\Delta, \iota_\Delta \omega) \in \mathbb{D}L_S \mid \Delta \in DL_S\}$$

and $\mathrm{d}_{L_S}\omega = \Phi^* H$.

ii.) *locally conformal pre-symplectic leaf, then there exists a flat connection $\nabla\colon TS \to DL_S$ and a unique L_S-valued 2-form $\omega \in \Gamma^\infty(\Lambda^2 TS \otimes L_S)$, such that*

$$\mathfrak{B}_I(\mathcal{L}) = \{\nabla_X, \sigma^*(\iota_X \omega) + \alpha \in \mathbb{D}L_S \mid X \in TS \textit{ and } \alpha \in \operatorname{Ann}(\operatorname{im}(\nabla))\}$$

and $\mathrm{d}^\nabla \omega(X_1, X_2, X_3) = I^* H(\nabla_{X_1}, \nabla_{X_2}, \nabla_{X_3})$.

Remark 1.2.28 For a Dirac-Jacobi bundle $(L \to M, \mathcal{L})$ the manifold M is a disjoint union of two sets: the set of pre-contact points, i.e. points which are contained in a pre-contact leaf, and the set of locally conformal presymplectic points, i.e. points which are contained in a locally conformal presymplectic leaf.

The locally conformal pre-symplectic leaves differ from the pre-contact leaves a lot. Let us introduce a notion, or generalization, of Dirac structures where they fit in and behave similarly to pre-contact leaves. Consider a line bundle $L \to M$, a flat connection $\nabla\colon TM \to DL$ and a d^∇-closed 3-form H. With this we can consider $\mathbb{T}^L M := TM \oplus (T^*M \otimes L)$ and equip it, similarly as the omni-Lie algebroid, with:

i.) the non-degenerate L-valued pairing $\langle\!\langle (X,\alpha), (Y,\beta) \rangle\!\rangle = \alpha(X) + \beta(Y)$

ii.) the map $\mathrm{pr}_T\colon \mathbb{T}^L M \to TM$

iii.) the bracket $[\![(X,\alpha), (Y,\beta)]\!]_H = ([X,Y], \mathscr{L}^\nabla_X \beta - \iota_Y \mathrm{d}^\nabla \alpha) + \iota_X \iota_Y H)$

Similarly to Dirac-Jacobi bundles, we can define

Definition 1.2.29 *Let $L \to M$ be a line bundle and let $\nabla\colon TM \to DL$ be a flat connection. A subbundle $\mathcal{D} \subseteq \mathbb{T}^L M$ is said to be a H-twisted locally conformal Dirac structure, if $\mathcal{D}$*

i.) *is involutive with respect to $[\![-,-]\!]_H$*

ii.) *is maximally isotropic with respect to $\langle\!\langle -,- \rangle\!\rangle$*

Example 1.2.30 Let $L \to M$ be a line bundle, $\nabla : TM \to DL$ be a flat connection, $H \in \Gamma^\infty(\Lambda^3 TM \otimes L)$ be a d^∇-closed and let ω be L-valued 2-form such that $\mathrm{d}^\nabla \omega = H$. Then

$$\mathcal{D}_\omega = \{(X, \iota_X \omega) \in \mathbb{T}^L M \mid X \in TM\}$$

is an H-twisted locally conformal Dirac structure.

Out of a H-twisted locally conformal Dirac structure $\mathcal{D}$ on $(\mathbb{T}^L_{\mathbb{C}}M, \nabla)$ we can cook up a K-twisted Dirac-Jacobi structure if H is exact with primitive α and $K := \mathrm{d}_L\sigma^*\alpha$ by

$$\mathcal{L}_{\mathcal{D}} = \{(\nabla_X, \sigma^*\alpha + \beta) \in \mathbb{D}L \mid (X, \alpha) \in \mathcal{D},\ \beta \in \operatorname{Ann}(\operatorname{im}(\nabla))\}. \tag{1.2.2}$$

In fact, it is easy to check that $\mathcal{L}_{\mathcal{D}}$ is a K-twisted Dirac-Jacobi structure. We get immediatly

Lemma 1.2.31 *Let $L \to M$ be a line bundle and let $\nabla\colon TM \to DL$ be a flat connection. If $\mathcal{L} \subseteq \mathbb{D}L$ is a Dirac-Jacobi structure such that $\mathrm{pr}_D\mathcal{L} \subseteq \operatorname{im}(\nabla)$, then*

$$\mathcal{L} = \mathcal{L}_{\mathcal{D}}$$

for a unique locally conformal Dirac-Jacobi structure $\mathcal{D} \subseteq \mathbb{T}^L M$.

1.2.3 Jacobi Brackets and Contact Structures

In this section we want to discuss a certain kind of Dirac-Jacobi bundles, the so-called *Jacobi* bundles. They play the same role as Poisson manifolds in Dirac geometry and have been introduced first by Kirillov as a special kind of *local Lie algebra*, see [28].They are also called *Kirillov manifolds* in the literature, see for instance [10]. Afterwards we put Jacobi structures in the framework of Dirac-Jacobi geometry and discuss their characteristic foliation. As a last part, we will discuss contact geometry in the Jacobi framework. This is very similar to the study of symplectic structures in the Poisson setting. At this moment it is worth mentioning that, historically speaking, Jacobi structures where introduced earlier than Dirac-Jacobi structures and hence our proofs in this section, which usually use Dirac-Jacobi techniques, are non-standard. Nevertheless, most of the results which just concern Jacobi structures can already be found in [28]. For the results which concerns the interaction of Dirac-Jacobi structures and Jacobi bracket we refer to [46].

Definition 1.2.32 *Let $L \to M$ be a line bundle. A Jacobi bracket on L is a $\mathbb{R}$-bilinear operation*
$\{-,-\}\colon \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$*, such that*

i.) $\{-,-\}\colon \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$ *is a Lie bracket*

ii.) $\{\lambda, -\}\colon \Gamma^\infty(L) \to \Gamma^\infty(L)$ *is a section of DL for all $\lambda \in \Gamma^\infty(L)$*

For two Jacobi brackets $\{-,-\}_i$ on $L_i \to M_i$ for $i = 1, 2$, we call a regular line bundle morphism $\Phi\colon L_1 \to L_2$ a Jacobi map, if

$$\Phi^*\{\lambda, \mu\}_2 = \{\Phi^*\lambda, \Phi^*\mu\}_1.$$

Remark 1.2.33 We use the words *Jacobi bracket* and *Jacobi structure* synonymously. Moreover, a *Jacobi bundle* is a pair consisting of a line bundle and a Jacobi bracket on its sections.

For a Jacobi bracket $\{-,-\}$, we can define the bundle map

$$J^\sharp \colon J^1L \ni j^1_p\lambda \mapsto \{\lambda, -\}_p \in DL.$$

Of course, we have to show that this map is well defined, i.e. if there are $\lambda_1, \lambda_2 \in \Gamma^\infty(L)$, such that $j^1_p\lambda_1 = j^1_p\lambda_2$, then $J^\sharp(j^1_p\lambda_1) = J^\sharp(j^1_p\lambda_2)$. But this follows immediatly, since $\{-,-\}$ is a derivation in both slots. We refer to the tensor J as the *Jacobi tensor*. Note that not every section of $\Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ induces a Jacobi bracket by

$$\{\lambda, \mu\} := J(j^1\lambda, j^1\mu),$$

in fact axiom *(ii)* of Definition 1.2.32 is always fulfilled, while the defined bracket needs not to be a Lie bracket, since the Jacobi identity might fail to hold in general. Using the definition of the Gerstenhaber-Jacobi bracket $[\![-,-]\!]_L \colon \Gamma^\infty(\Lambda^i(J^1L)^* \otimes L) \times \Gamma^\infty(\Lambda^j(J^1L)^* \otimes L) \to \Gamma^\infty(\Lambda^{i+j-1}(J^1L)^* \otimes L)$ from Subsection 1.1.3, the Jacobi identity is equivalent to

$$[\![J, J]\!]_L = 0.$$

Nevertheless, we want to go in another direction: having a tensor field $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$, we can define

$$\mathcal{L}_J := \{(J^\sharp(\psi), \psi) \in \mathbb{D}L \mid \psi \in J^1L\}, \tag{1.2.3}$$

which is always a maximally isotropic subbundle.

Lemma 1.2.34 *Let $L \to M$ be a line bundle and let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$. Then $\mathcal{L}_J$ is involutive if and only if the bracket $\{-,-\} \colon \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$ defined by*

$$\{\lambda, \mu\} = J(j^1\lambda, j^1\mu)$$

is a Jacobi bracket.

PROOF: Note that from the above discussion it is clear that the only thing to show for the bracket to be Jacobi is the Jacobi identity. We know that involutivity is equivalent to the vanishing of the tensor field defined in (1.2.1). Let us compute

$$\begin{aligned}
N_{\mathcal{L}_J}&\Big((J^\sharp(j^1\lambda), j^1\lambda), (J^\sharp(j^1\mu), j^1\mu), (J^\sharp(j^1\nu), j^1\nu)\Big) = \\
&= \langle (J^\sharp(j^1\lambda), j^1\lambda), [\![(J^\sharp(j^1\mu), j^1\mu), (J^\sharp(j^1\nu), j^1\nu)]\!]\rangle \\
&= \langle (J^\sharp(j^1\lambda), j^1\lambda), ([J^\sharp(j^1\mu), J^\sharp(j^1\nu)], \mathscr{L}_{J^\sharp(j^1\mu)} j^1\nu - \iota_{J^\sharp(j^1\nu)} \mathrm{d}_L j^1\mu)\rangle \\
&= \langle (J^\sharp(j^1\lambda), j^1\lambda), ([J^\sharp(j^1\mu), J^\sharp(j^1\nu)], j^1\iota_{J^\sharp(j^1\mu)} j^1\nu\rangle \\
&= \langle (J^\sharp(j^1\lambda), j^1\lambda), ([J^\sharp(j^1\mu), J^\sharp(j^1\nu)], j^1\{\mu, \nu\})\rangle \\
&= j^1\lambda([J^\sharp(j^1\mu), J^\sharp(j^1\nu)]) + j^1\{\mu, \nu\}(J^\sharp(j^1\lambda))
\end{aligned}$$

$$= [J^\sharp(j^1\mu), J^\sharp(j^1\nu)](\lambda) + \{\lambda, \{\mu, \nu\}\}$$
$$= \{\mu, \{\nu, \lambda\}\} - \{\nu, \{\mu, \lambda\}\} + \{\lambda, \{\mu, \nu\}\}$$

$\lambda, \mu, \nu \in \Gamma^\infty(L)$. Since $N_{\mathcal{L}_J}$ is a tensor field, it uniquely determined by values on generators of $\Gamma^\infty(\mathcal{L}_J)$ such as $(J^\sharp(j^1\lambda), j^1\lambda)$. ΧΞΣ

We can give an easy characterization of Dirac-Jacobi structures, that are induced by a Jacobi tensor in the sense of 1.2.3.

Lemma 1.2.35 *Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq \mathbb{D}L$ be a Dirac-Jacobi structure. Then $\mathcal{L}$ is induced by a Jacobi tensor in the sense of 1.2.3, if and only if*

$$\mathcal{L} \cap DL = \{0\}.$$

Proof: We just have to prove that a Dirac-Jacobi structure $\mathcal{L}$ fulfilling

$$\mathcal{L} \cap DL = \{0\}.$$

is induced by a tensor field $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ since then J automatically a Jacobi tensor by the previous lemma. We claim first that the map $\mathrm{pr}_{J^1L}\big|_{\mathcal{L}} \colon \mathbb{D}L \to J^1L$ is bijective. Let $(\lambda, \psi) \in \ker \mathrm{pr}_{J^1L} \cap \mathcal{L}$, then $\psi = 0$, but by the hypothesis of the lemma then also $\Delta = 0$ and hence $\mathrm{pr}_{J^1L}\big|_{\mathcal{L}} \colon \mathcal{L} \to J^1L$ is injective. By dimensional reasons it is also bijectve and hence an ismorphism of vector bundles covering the identity. Let us denote by $\tau \colon J^1L \to \mathcal{L}$ its (smooth) inverse. As in the proof of Lemma 1.2.25, we define

$$J(\psi, \chi) = \chi(\mathrm{pr}_D \tau(\psi))$$

for $\psi, \chi \in J^1L$ and the claim follows. ΧΞΣ

Now we want to encode also the morphisms of Jacobi structures in the language of Dirac-Jacobi structures

Lemma 1.2.36 *Let $(L_i \to M_i, \{-,-\}_i)$ be Jacobi bundles for $i = 1, 2$. A regular line bundle map $\Phi \colon L_1 \to L_2$ is a Jacobi map if and only if it is a forward Dirac-Jacobi map of the corresponding Dirac-Jacobi structures.*

Proof: Let us denote by J_i the Jacobi tensors corresponding to the brackets $\{-,-\}_i$ and let us choose an arbitrary section $\lambda \in \Gamma^\infty(L_2)$, then $(J_1^\sharp(j^1\Phi^*\lambda), j^1\Phi^*\lambda) \in \Gamma^\infty(\mathcal{L}_{J_1})$. Assuming that Φ is a forward Dirac-Jacobi map, we have that

$$(D\Phi(J_1^\sharp(j_p^1\Phi^*\lambda)), j^1_{\phi(p)}\lambda) \in \mathcal{L}_{J_2},$$

which means that $J_2^\sharp(j_p^1\lambda) = D\Phi(J_1^\sharp(j_p^1\Phi^*\lambda))$. A short computation shows that this is equivalent to Φ being a Jacobi map. ΧΞΣ

Let us now focus on the characteristic foliation of a Jacobi structure. Since it is a special kind of Dirac-Jacobi structure the definition is straight-forward, but in this case the leaves have more structures. But before talking about the leaves give a

Definition 1.2.37 *Let $(L \to M, \{-,-\})$ be a line bundle equipped with a Jacobi bracket. The characteristic foliation of the corresponding Dirac-Jacobi structure is called characteristic foliation of the Jacobi structure.*

Lemma 1.2.38 *Let $(L \to M, \{-,-\})$ be a Jacobi bundle and let $\iota\colon S \hookrightarrow M$ be a leaf of its characteristic foliation. Then for the Dirac-Jacobi structure $\mathcal{L}_J$ corresponding to the Jacobi bracket*

$$\mathfrak{B}_I(\mathcal{L}_J) \cap DL_S = \{0\}$$

holds.

Proof: Let $(\Delta, 0) \in \mathfrak{B}_I(\mathcal{L}_J)$, then there exists $\psi \in J^1L$, such that $DI(\Delta) = J^\sharp(\psi)$ and $DI^*\psi = 0$. But since S is a leaf, we have that $\operatorname{im}(J^\sharp) \subseteq \operatorname{im}(DI)$, but this implies for $\chi \in J^1L$ that

$$\chi(J^\sharp(\psi)) = -\psi(J^\sharp(\chi))$$

which vanishes since ψ vanishes on the image of DI and thus $J^\sharp(\psi) = 0$ and finally, since DI is injective, $\Delta = 0$. ΧΞΣ

So on the leaves there are induced Jacobi structures, which is the mirror statement to the one in Poisson geometry, but there the leaves have more structure than just Poisson, they are *symplectic*. In our case we have two different kind of leaves, which carry two different kinds of structures.

Lemma 1.2.39 *Let $(L \to M, \{-,-\})$ be a line bundle equipped with a Jacobi bracket, let $\iota\colon S \hookrightarrow M$ be a leaf of its characteristic foliation and let $\mathcal{L}_J$ be the corresponding Dirac-Jacobi structure. If S is*

i.) a pre-contact leaf, then there exists a unique $\omega \in \Omega^2_{L_S}(S)$, such that

$$\mathfrak{B}_I(\mathcal{L}_J) = \{(\Delta, \iota_\Delta\omega) \in \mathbb{D}L_S \mid \Delta \in DL_S\}$$

with $\mathrm{d}_L\omega = 0$ and ω is non-degenerate, i.e. $\omega^\flat\colon DL \to J^1L$ is invertible.

ii.) a locally conformal pre-symplectic leaf, then there exists a flat connection $\nabla\colon TS \to DL_S$ and a unique L_S-valued 2-form $\Omega \in \Gamma^\infty(\Lambda^2T^*S \otimes L_S)$, such that

$$\mathfrak{B}_I(\mathcal{L}_J) = \{(\nabla_X, \sigma^*(\iota_X\Omega) + \alpha) \in \mathbb{D}L_S \mid X \in TS \text{ and } \alpha \in \operatorname{Ann}(\operatorname{im}(\nabla))\}$$

with $\mathrm{d}^\nabla\omega = 0$ and ω is non-degenerate, i.e. $\omega^\flat\colon TM \to T^*M \otimes L$ is invertible.

PROOF: First we note that, besides the non-degeneracy of the forms, everything is proven already in Corollary 1.2.27. The proof of the non-degeneracy is in both cases the same, so we just do it for pre-contact leaves. Assume that there is $\Delta \in DL_S$, such that $\iota_\Delta \omega = 0$. Then we have

$$(\Delta, 0) = (\Delta, \iota_\Delta \omega) \in \mathfrak{B}_I(\mathcal{L}_J)$$

using Lemma 1.2.38, we get that $\Delta = 0$ and the claim follows. ΧΞΣ

Corollary 1.2.40 *Let $L \to M$ be a line bundle and let $J \colon \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi tensor. Then its*

i.) pre-contact leaves are odd dimensional.

ii.) locally conformal pre-symplectic leaves are even dimensional.

PROOF: This follows immediately from the non-degeneracy of the induced 2-forms on the leaves. ΧΞΣ

In fact the leaves carry very classical geometries, which we will recall now: contact structures and locally conformal symplectic structures. This geometries are interesting in themselves and are not just studied in the context of Jacobi manifolds. See for example [16] for the contact case and for the locally conformal symplectic case we refer to [45]. Let us recall both of these geometries. We start with contact structures:

Remark 1.2.41 (Contact Geometry) A contact structure on a manifold M is a codimension one subbundle $H \subseteq TM$, such that for the projection $\Theta \colon TM \to L := TM/H$, the tensor field

$$C \colon H \times H \to L,$$

which is defined on vector fields $X, Y \in \Gamma^\infty(H)$ by $C(X,Y) = \Theta([X,Y])$, is non-degenerate. (It is easy to check that C is indeed a tensor field). The condition that C is non-degenerate is called *maximal non-integrable*, which makes sense since the vanishing of C is equivalent to the integrability of H. Now we consider

$$\omega = \mathrm{d}_L \sigma^* \Theta$$

where σ^* is the pull-back with the symbol and we claim that ω is non-degenerate. Indeed, let $\Delta \in DL$, such that $\iota_\Delta \omega = 0$. By equation (1.1.5), we have that $\iota_{\mathbb{1}} \omega = \sigma^* \Theta$ and hence $0 = \omega(\mathbb{1}, \sigma(\Delta)) = \sigma^*\Theta(\Delta)$ and thus $\sigma(\Delta) \in H$. For two elements $\Delta, \Box \in DL$ with $\sigma(\Delta), \sigma(\Box) \in H$, we have $\omega(\Delta, \Box) = C(\sigma(\Delta), \sigma(\Box))$, therefore, using the non-degeneracy of C, $\sigma(\Delta) = 0$. Thus $\Delta = k\mathbb{1}$ for $k \in \mathbb{R}$, but then $0 = \iota_\Delta \omega = k\sigma^*\Theta$ and thus $k = 0$ and the claim follows. In fact, one can cook up, for a non-degenerate closed two form $\omega \in \Omega^2_L(M)$, a contact structure $H \subseteq TM$ by putting

$$H := \sigma(\ker(\iota_{\mathbb{1}} \omega)).$$

These two constructions are inverse to each other, so contact structures identify with non-degenerate closed Atiyah 2-forms. From now on, we will use this identification and if we are considering contact structures on M, we always refer to a line bundle $L \to M$ together with a non-degenerate closed 2-form $\omega \in \Omega^2_L(M)$.

We have seen that one kind of leaf of a Jacobi manifold has the structure of a contact manifold, but according to Lemma 1.2.38 it carries a structure of a Jacobi manifold. This is not a coincidence:

Lemma 1.2.42 *Let $(L \to M, \omega)$ be a contact manifold. Then $J \in \Gamma^\infty(\Lambda^2 (J^1 L)^* \otimes L)$, which is defined via its sharp map by*

$$J^\sharp = (\omega^\flat)^{-1},$$

is a Jacobi tensor.

PROOF: It is easy to see that the Dirac-Jacobi structure of J and ω coincide. Hence, since the Dirac-Jacobi structure induced by ω is involutive, the one induced by J also is. Using Lemma 1.2.34, we get the claim. ΧΞΣ

So it follows that Jacobi brackets generalize contact structure, the same way Poisson brackets generalize symplectic structures. Contact structures are themselves an active field of research, since their application to (mathematical) physics are hard to overestimate. For a later use we want to discuss at least one specific example of a contact structure: the first jet of a line bundle.

Example 1.2.43 (Canonical Contact Structure) First, we would like to mention that this is not the standard approach, i.e. using the Cartan distribution, to obtain the closed Atiyah 2-form on the first jet bundle of a line bundle, instead we use techniques from symplectic geometry. Let us consider a line bundle $L \to M$ and its pullback

$$\begin{tikzcd} \pi^* L \arrow[r, "\Pi"] \arrow[d] & L \arrow[d] \\ J^1 L \arrow[r, "\pi"] & M \end{tikzcd}$$

together with the canonical $\lambda_{\mathrm{can}} \in \Gamma^\infty(J^1 \pi^* L)$ defined by

$$\lambda_{\mathrm{can}} \colon D\pi^* L \ni \Delta_{\alpha_p} \mapsto (\alpha_p, \alpha_p(D\Pi(\Delta_{\alpha_p}))) \in \pi^* L.$$

Note that this section has a similar universal property as the Liouville 1-form on the cotangent bundle: given a $\psi \in \Gamma^\infty(J^1 L)$, we can define a canonical regular line bundle morphism

$$\Phi_\psi \colon L \ni \lambda_p \mapsto (\psi(p), \lambda_p) \in \pi^* L,$$

which fulfills $\Pi \circ \Phi_\psi = \mathsf{id}$. Moreover, λ_{can} has the universal property that for all $\psi \in \Gamma^\infty(J^1L)$, we have

$$\Phi_\psi^* \lambda_{\mathrm{can}} = \psi.$$

Moreover, interpreting λ_{can} as a contact version of the Liouville 1-form it is not surprising that

$$\omega_{\mathrm{can}} := -\,\mathrm{d}_{p^*L}\lambda_{\mathrm{can}} \in \Omega^2_{\pi^*L}(J^1L)$$

is a contact structure. Furthermore, an easy computation shows that it agrees with the well-known contact structure on the first jet of a line bundle.

Let us now turn towards the locally conformal pre-symplectic leaves of a Jacobi manifold. The structure on its leaves is given by a flat connection and a non-degenerate line bundle valued 2-form which is closed with respect to the connection differential. This structures are known as locally conformal symplectic structures, see e.g. [45]. Note however, that this is not the standard approach to locally conformal contact structures. They always induce a Jacobi structure:

Lemma 1.2.44 *Let $L \to M$ be a line bundle and let (∇, ω) be a locally conformal symplectic structure on L. Then*

$$\mathcal{L} := \{(\nabla_X, \sigma^*(\iota_X\omega) + \alpha) \in \mathbb{D}L \mid X \in TM \text{ and } \alpha \in \mathrm{Ann}(\mathrm{im}(\nabla_X))\}$$

is a Dirac-Jacobi structure which is induced by a Jacobi tensor.

PROOF: The subbundle is obviously isotropic and $\mathrm{rank}(\mathcal{L}) = \dim(M) + 1$ and hence also maximal. The ivolutivity follows from the closedness of ω. Moreover, the non-degeneracy of ω implies that

$$DL \cap \mathcal{L} = \{0\}.$$ ΧΞΣ

At this point, we shall introduce the notion of locally conformal Poisson structures, since they will come across this thesis several times. Roughly speaking they are the Poisson version of locally conformal symplectic structures.

Definition 1.2.45 *Let $L \to M$ be a line bundle with a flat connection $\nabla\colon TM \to DL$. An element $\pi \in \Gamma^\infty(\Lambda^2(TM \otimes L^*) \otimes L)$ is said to be a locally conformal Poisson structure, if*

$$[\![\pi, \pi]\!]_{(TM,L)} = 0.$$

Note that, by definition, a locally conformal symplectic structures $\omega \in \Gamma^\infty(\Lambda^2 T^*M \otimes L)$ is non- degenerate and hence

$$\omega^\flat\colon TM \to T^*M \otimes L$$

is invertible and one can show that its inverse is sharp map of a locally conformal Poisson tensor. As one may expect, a locally conformal Poisson structure also induces a Jacobi bracket. Let us make this precise.

Corollary 1.2.46 *Let* $(L \to M, \nabla)$ *be a line bundle with a flat connection and let* $\pi \in \Gamma^\infty(\Lambda^2(TM \otimes L^*) \otimes L)$ *be a locally conformal Poisson tensor. Then*

$$\{\lambda, \mu\} := \pi(\mathrm{d}^\nabla \lambda, \mathrm{d}^\nabla \mu)$$

for $\lambda, \mu \in \Gamma^\infty(L)$ *defines a Jacobi bracket.*

Proof: This is an easy computation, which is paralleling the computation in Poisson geometry, using the properties of the bracket $[\![-,-]\!]_{(TM,L)}$ and the fact that π commutes with itself. ΧΞΣ

Now we want to translate locally conformal Poisson in (locally conformal) Dirac language, as we did for Jacobi bundles. It is easy to show that

$$\mathcal{D}_\pi = \{(\pi^\sharp(\alpha), \alpha) \in \mathbb{T}^L M \mid \alpha \in T^*M \otimes L\}$$

is a maximally isotropic subbundle for an element $\pi \in \Gamma^\infty(\Lambda^2(TM \otimes L^*) \otimes L)$. Moreover, it is involutive if and only if π is a locally conformal Poisson structure. So $\mathcal{D}_\pi$ is a locally conformal Dirac structure in the sense of Definition 1.2.29. Furthermore, one can show that the induced Dirac-Jacobi structure from Equation 1.2.2

$$\mathcal{L}_\mathcal{D} = \{(\nabla_{\pi^\sharp(\alpha)}, \sigma^*\alpha + \beta) \mid \alpha \in T^*M \otimes L,\ \beta \in \mathrm{Ann}(\mathrm{im}(\nabla))\}$$

comes from a Jacobi bracket, which is exactly the one constructed in Corollary 1.2.46.

Remark 1.2.47 From now on we refer to the pre-contact leaves (resp. locally conformal pre-symplectic leaves) as the contact leaves (respectively locally conformal symplectic leaves) of a Jacobi bundle. Note that this justifies also the name we gave the leaves in the Dirac-Jacobi setting, in fact the names are inspired by the *pre-symplectic* foliation in Dirac geometry, i.e. a pre-contact structure on a line bundle is a closed Atiyah 2-form and a locally conformal pre-symplectic structure is a flat connection and a closed (with respect to the connection) line bundle valued two form.

If L is the trivial line bundle, then the notion of Jacobi bracket boils down to that of *Jacobi pair*, which was first introduced in [31].

Remark 1.2.48 (Trivial Line bundle) Let $\mathbb{R}_M \to M$ be the trivial line bundle and let J be a Jacobi tensor on it. Note that from the discussion in Subsection 1.1.2, we have $DL \cong TM \oplus \mathbb{R}_M$ and

$$J^1\mathbb{R}_M = (D\mathbb{R}_M)^* \otimes \mathbb{R}_M = T^*M \oplus \mathbb{R}_M.$$

With this splitting, we see that

$$J = \Lambda + \mathbb{1} \wedge E$$

for some $(\Lambda, E) \in \Gamma^\infty(\Lambda^2 TM \oplus TM)$ and the canonical section $\mathbb{1} = (0,1) \in \Gamma^\infty(TM \oplus \mathbb{R}_M)$. The Jacobi identity is equivalent to $[\Lambda, \Lambda] + E \wedge \Lambda = 0$ and $\mathscr{L}_E \Lambda = 0$. The pair (Λ, E) is often referred to as *Jacobi pair*, see [31]. Moreover, if we denote by $\mathbb{1}^* \in \Gamma^\infty(J^1\mathbb{R}_M)$ the canonical section then we can write any $\psi \in J^1\mathbb{R}_M$ as $\psi = \alpha + r\mathbb{1}^* \in \Gamma^\infty(J^1\mathbb{R}_M)$, for some $\alpha \in T^*M$ and $r \in \mathbb{R}$. We obtain

$$J^\sharp(\alpha + r\mathbb{1}^*) = \Lambda^\sharp(\alpha) + rE - \alpha(E)\mathbb{1}.$$

A change of the basis of the line bundle by a non-vanishing function $f \in \mathscr{C}^\infty(M)$, seen as a line bundle automorphism by multiplication, induces a different Jacobi pair (Λ^f, E^f) which is connected to the first one by $(\Lambda^f, E^f) = (f\Lambda, E - \Lambda^\sharp(\mathrm{d}f))$. In the literature (see [17]) this is referred to as *conformally* equivalent Jacobi pairs. A more detailed discussion about Jacobi structures on trivial line bundles can be found in [43, Chapter 2].

1.2.4 Generalized Contact Bundles

Generalized contact bundles were introduced recently in [47], as a slight generalization of *generalized almost contact structures* from Wade and Iglesias in [27], and so far very little is known about them. They basically mimic the notion of generalized complex structures, see [26], in the framework of Dirac-Jacobi bundles. They are moreover deeply connected to generalized complex manifolds via the homogenezation, see [10] or Appendix A.2. Additionally, they are modeled to be the analogue to generalized complex structures in odd dimensions. Moreover, their characteristic foliation provides submanifolds of generalized contact bundles, which are (local) generalized complex manifolds.

Definition 1.2.49 *Let $L \to M$ be a line bundle. A subbundle $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L := \mathbb{D}L \otimes \mathbb{C}$ is called generalized contact structure on L, if*

i.) *$\mathcal{L}$ is a (complex) Dirac-Jacobi structure and*

ii.) $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$.

A line bundle equipped with a generalized contact structure is called generalized contact bundle.

Remark 1.2.50 In Definition 1.2.49 all the structures of the omni-Lie algebroid are extended $\mathbb{C}$-linearly, in particular this holds for the non-degenerate pairing $\langle\!\langle -, - \rangle\!\rangle$, so it is *not* $\mathbb{C}$-antilinear in one of its arguments.

Equivalently, generalized contact structures can be described by an endomorphism of $\mathbb{D}L$ squaring to minus the identity, paralleling the generalized complex case.

Lemma 1.2.51 *Let $(L \to M, \mathcal{L})$ be a generalized contact bundle, then it is the $+\mathrm{i}$-eigenbundle of a unique endmorphism $\mathbb{K}\colon \mathbb{D}L \to \mathbb{D}L$, fulfilling*

i.) $\mathbb{K}^2 = -\mathsf{id}$

ii.) $\langle\!\langle \mathbb{K}-, \mathbb{K}- \rangle\!\rangle = \langle\!\langle -, - \rangle\!\rangle$

iii.) $0 = N_{\mathbb{K}}(A, B) := [\![\mathbb{K}A, \mathbb{K}B]\!] - \mathbb{K}[\![\mathbb{K}A, B]\!] - \mathbb{K}[\![A, \mathbb{K}B]\!] - [\![A, B]\!]$ *for all* $A, B \in \Gamma^\infty(\mathbb{D}L)$

Conversely, the $+\mathrm{i}$*-Eigenbundle of an endomorphism fulfilling i.)-iii.) is a generalized contact bundle.*

Proof: Given a generalized contact structure, we define

$$\mathbb{K}\big|_{\mathcal{L}} = \mathrm{i} \cdot \mathsf{id} \text{ and } \mathbb{K}\big|_{\overline{\mathcal{L}}} = -\mathrm{i} \cdot \mathsf{id}\,.$$

Note that this definition implies immediately that $\overline{\mathbb{K}} = \mathbb{K}$ and hence that $\mathbb{K}$ is a complexification of a real endomorphism. The claim follows immediately by the isotropy and involutivity. ΧΞΣ

Remark 1.2.52 If we use a local trivialization of the line bundle $L \to M$ around a point p, we can identify $V := D_pL \cong T_pM \oplus \mathbb{R}$, $V^* := J^1L \cong T_p^*M \oplus \mathbb{R}$ and $\mathbb{K}_p$ as an endomorphism $\mathsf{End}(V \oplus V^*)$. We see that $\mathbb{K}$ fulfills all the axioms of being a linear generalized complex structure in the sense of [26, Section 1]. This means in particular, that V is even dimensional and hence M is odd dimensional.

From now on, we will refer frequently to the endomorphism as the generalized contact structure, if we need to. The endomorphism $\mathbb{K}$ splits canonically according to the splitting of $\mathbb{D}L = DL \oplus J^1L$ and its components have interesting properties.

Lemma 1.2.53 *Let* $L \to M$ *be a line bundle and let* $\mathbb{K} \in \mathsf{End}(\mathbb{D}L)$ *be a generalized contact structure. Then*

$$\mathbb{K} = \begin{pmatrix} \phi & J^\sharp \\ \alpha^\flat & -\phi^* \end{pmatrix}$$

for an endomorphism $\phi \in \mathsf{End}(DL)$*, a tensor field* $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ *and an Atiyah 2-form* $\alpha \in \Omega_L(M)$*, where* $\phi^* \in \mathsf{End}(J^1L)$ *is the adjoint of* ϕ *with respect to the* L*-valued pairing of* J^1L *and* DL*. Moreover, the tensor field* J *is a Jacobi tensor.*

Proof: Note that this proof originally appeared in [47] and is identical to the analogous statement in generalized complex geometry. The only things to prove for the first statement are that the off-diagonal maps are skew-symmetric and that the diagonal maps are adjoint to each other, but both facts follow from condition *ii.)* from Lemma 1.2.51. Let us now prove that J is a Jacobi tensor. To do so we exploit the equation $0 = \mathcal{N}_{\mathbb{K}}((0, j^1\lambda), (0, j^1\mu))$ for $\lambda, \mu \in \Gamma^\infty(L)$, which reads

$$\begin{aligned} 0 &= \mathcal{N}_{\mathcal{I}}((0, j^1\lambda), (0, j^1\mu)) \\ &= \big([J^\sharp(j^1\lambda), J^\sharp(j^1\mu)] - J^\sharp(\mathscr{L}_{J^\sharp(j^1\lambda)} j^1\mu), \psi\big) \end{aligned}$$

$$= \big([J^\sharp(j^1\lambda), J^\sharp(j^1\mu)] - J^\sharp(j^1 J(j^1\lambda, j^1\mu)), \psi\big)$$

where ψ is the element such that the above equality holds (we are just interested in the DL-component of the above element). Let us apply the identity $[J^\sharp(j^1\lambda), J^\sharp(j^1\mu)] = J^\sharp(j^1 J(j^1\lambda, j^1\mu))$ to a third $\nu \in \Gamma^\infty(L)$ as in the following:

$$\begin{aligned}
\{\lambda\{\mu,\nu\}\} - \{\mu,\{\lambda,\nu\}\} &= [J^\sharp(j^1\lambda), J^\sharp(j^1\mu)](\nu) \\
&= J^\sharp(j^1 J(j^1\lambda, j^1\mu))(\nu) \\
&= J^\sharp(j^1\{\lambda,\mu\})(\nu) \\
&= \{\{\lambda,\mu\},\nu\}.
\end{aligned} \qquad \text{X}\Xi\Sigma$$

This means, roughly speaking, that a generalized contact structure always induces a Jacobi bracket. The question of when a Jacobi structure induces a generalized contact structure is in turn very hard to answer, but partial results can be found in Chapter 4 Before we go on, let us comment on the previous Lemma in order to avoid confusion.

Remark 1.2.54 The components of the endomorphism $\mathbb{K}$ fulfill many more algebraic and differential compatibilities, but for us at this moment this is not particularly interesting. The complete list of relations of ϕ, J, α can be found in [47, Section 3].

The next thing to do is to look for examples. Fortunately, we already came across one of them in this chapter.

Example 1.2.55 Let $L \to M$ be a line bundle and let $\omega \in \Omega^2_L(M)$ be a contact 2-form. Then

$$\mathcal{L} = \{(\Delta, \mathrm{i}\iota_\Delta\omega) \in \mathbb{D}_{\mathbb{C}}L \mid \Delta \in D_{\mathbb{C}}L\}$$

is a generalized contact structure. The corresponding endomorphism $\mathbb{K} \in \Gamma^\infty(\mathsf{End}\,\mathbb{D}L)$ is given by

$$\mathbb{K} = \begin{pmatrix} 0 & J^\sharp \\ -\omega^\flat & 0 \end{pmatrix},$$

where J is the Jacobi tensor of ω.

Example 1.2.56 Let $L \to M$ be a line bundle and let $\phi \in \Gamma^\infty(\mathsf{End}\,DL)$ be a complex structure, i.e. an almost complex structure whose Nijenhuis torsion with respect to the Lie algebroid bracket vanishes. If we denote by $DL^{(1,0)}$ its $+$i-eigenbundle of $\phi\colon D_{\mathbb{C}}L \to D_{\mathbb{C}}L$, then

$$\mathcal{L} = DL^{(1,0)} \oplus \mathrm{Ann}(DL^{(1,0)})$$

is a generalized contact structure with corresponding endomorphism $\mathbb{K} \in \Gamma^\infty(\mathsf{End}\,\mathbb{D}L)$ given by

$$\mathbb{K} = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix}.$$

In the following, we will refer to ϕ as an *Atiyah complex* structure on L.

In the rest of this thesis we call the last two examples, i.e. contact structures and Atiyah complex structures, the extreme cases of generalized contact structures. Note that in generalized complex geometry the extreme cases are complex structures and symplectic structures. Before we turn towards the characteristic foliation, we want to drop a word on morphisms of generalized contact bundles. This is a non-trivial issue, but for most of our purposes isomorphisms of line bundles will be enough.

Corollary 1.2.57 *Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ be a generalized contact structure. Then for a closed real $B \in \Omega_L(M)$ the (complex) Dirac-Jacobi structure $\mathcal{L}^B$ is a generalized contact structure, which has the same induced Jacobi structure as $\mathcal{L}$.*

PROOF: This is an easy verification using the endomorphism $\mathbb{K}$ corresponding to $\mathcal{L}$ and show that the endomorphism $\mathbb{K}'$ corresponding to $\mathcal{L}^\omega$ is given by

$$\mathbb{K}' = \exp(\omega)\mathbb{K}\exp(-\omega).$$

ΧΞΣ

Let us now turn to the characteristic foliation of generalized contact bundles. Since they are a particular kind of complex Dirac-Jacobi bundles, they of course induce an integrable distribution in the complexified tangent bundle. But this is not what we want: we want an integrable distribution of the real tangent bundle of the base manifold. For a generalized contact structure $\mathcal{L} \subseteq \mathbb{D}L$ we have now two canonical choices: The symbols of

i.) $\mathsf{Re}(\mathrm{pr}_D\mathcal{L} \cap \mathrm{pr}_D\overline{\mathcal{L}})$ and

ii.) $\mathrm{pr}_D(\mathcal{L}_J)$, where J is the Jacobi tensor induced by the generalized contact structure

The following Lemma shows that they coincide.

Lemma 1.2.58 *Let $(L \to M, \mathcal{L})$ be a generalized contact bundle. Then*

$$\mathsf{Re}(\mathrm{pr}_D\mathcal{L} \cap \mathrm{pr}_D\overline{\mathcal{L}}) = \mathrm{pr}_D(\mathcal{L}_J),$$

where J is the Jacobi tensor induced by the generalized contact structure.

PROOF: This is an easy consequence of the fact that $\mathcal{L}$ is the +i-Eigenbundle of the unique endomorphism

$$\mathbb{K} = \begin{pmatrix} \phi & J^\sharp \\ \alpha^\flat & -\phi^* \end{pmatrix},$$

which is ensured by Lemma 1.2.53. ΧΞΣ

From now on, we will refer to the foliation integrating $\sigma(\mathrm{pr}_D\mathcal{L} \cap \mathrm{pr}_D\overline{\mathcal{L}})$ as the characteristic foliation of the generalized contact structure $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$. If we examine the leaves, we immediately realize, that some of them may be odd and some may be even dimensional according to Corollary 1.2.40, since it is the foliation induced by a Jacobi structure. But from Remark 1.2.52, we know that the base manifold of a generalized contact structure is always odd dimensional. This means that a locally conformal symplectic leaf, the kind of leaf which is even-dimensional, of the generalized contact structure cannot carry the structure of a generalized contact structure. But let us start with the contact leaves, which indeed carry the structure of a generalized contact structure.

Lemma 1.2.59 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ be a generalized contact structure and let $\iota\colon S \hookrightarrow M$ be a contact leaf of its characteristic foliation. Then*

$$\mathfrak{B}_I(\mathcal{L}) \subseteq \mathbb{D}_{\mathbb{C}}L_S$$

is a generalized contact structure. Moreover, there exists a unique closed $B \in \Omega^2_{L_S}(S)$, such that $\mathfrak{B}_I(\mathcal{L}) = \mathcal{L}_{\mathrm{i}\omega+B}$, where $\omega \in \Omega^2_{L_S}(S)$ is the contact structure on the leaf induced by the Jacobi structure induced by $\mathcal{L}$.

Proof: Note that we have to prove also the smoothness of $\mathfrak{B}_I(\mathcal{L})$, since it is not covered by Lemma 1.2.22 as we are not considering the distribution $\sigma(\mathrm{pr}_D(\mathcal{L}))$, nevertheless we make use of Theorem 1.2.17. For a leaf $\iota\colon S \hookrightarrow M$, we have that $T_{\mathbb{C}}S = \sigma(\mathrm{pr}_D(\mathcal{L}) \cap \mathrm{pr}_D(\overline{\mathcal{L}}))\big|_S$ and since S is a contact leaf, we even have $\mathrm{pr}_D(\mathcal{L}) \cap \mathrm{pr}_D(\overline{\mathcal{L}}) = \mathrm{im}(DI)_{\mathbb{C}}$. The complexified version of Theorem 1.2.17 reads: If $(\ker D\Phi^*)_{\mathbb{C}} \cap \phi^*\mathcal{L}$ has constant rank, then $\mathfrak{B}_\Phi(\mathcal{L})$ is a Dirac-Jacobi bundle. As in the proof of Lemma 1.2.22, this happens if and only if

$$\mathrm{Ann}(\mathrm{im}(DI)_{\mathbb{C}}) \cap \mathrm{Ann}(\mathrm{pr}_D(\mathcal{L})\big|_S)$$

has constant rank. But we have

$$\begin{aligned}\mathrm{Ann}(\mathrm{im}(DI)_{\mathbb{C}}) \cap \mathrm{Ann}(\mathrm{pr}_D(\mathcal{L}\big|_S)) &= \mathrm{Ann}(\mathrm{im}(DI)_{\mathbb{C}} + \mathrm{pr}_D(\mathcal{L}\big|_S)) \\ &= \mathrm{Ann}(\mathrm{pr}_D(\mathcal{L}))\big|_S).\end{aligned}$$

So we have to show that $\mathrm{pr}_D(\mathcal{L}\big|_S)$ has constant rank. But this follows from $\mathrm{pr}_D(\mathcal{L}\big|_S) \cap \mathrm{pr}_D(\overline{\mathcal{L}}\big|_S) = \mathrm{im}(DI)_{\mathbb{C}}$ and $\mathrm{pr}_D(\mathcal{L}\big|_S) + \mathrm{pr}_D(\overline{\mathcal{L}}\big|_S) = D_{\mathbb{C}}L$, this is canonically fulfilled. Let us now make use of Example 1.2.13: let $(\Delta, \alpha) \in \mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}}))$ be real, then there exists a $\psi \in J^1_{\mathbb{C}}L$, such that $(DI(\Delta), \psi) \in \mathcal{L}$ and $DI^*\psi = \alpha$. This allows us to define $\chi \in J^1_{\mathbb{C}}L$ by

$$\chi\big|_{\mathrm{pr}_D(\mathcal{L})} = \psi \text{ and } \chi\big|_{\mathrm{pr}_D(\overline{\mathcal{L}})} = \overline{\psi}.$$

Note that this element is well-defined, since $\psi\big|_{\mathcal{L}\cap\overline{\mathcal{L}}} = \overline{\psi}\big|_{\mathcal{L}\cap\overline{\mathcal{L}}}$, which follows from $DI^*\psi = \alpha$ and by α being real. We want to show that $(DI(\Delta), \chi) \in \mathcal{L} \cap \overline{\mathcal{L}}$. Let $(\Box, \beta) \in \mathcal{L}$, then

$$\langle\!\langle (DI(\Delta), \chi), (\Box, \beta) \rangle\!\rangle = \beta(DI\Delta) + \chi(\Box) = \beta(DI\Delta) + \psi(\Box) = 0,$$

where we used $\chi\big|_{\mathrm{pr}_D(\mathcal{L})} = \psi$ and that $(DI(\Delta), \psi) \in \mathcal{L}$. This means in particular that $\langle\!\langle (DI(\Delta), \chi), \mathcal{L} \rangle\!\rangle = 0$ and hence, by the maximal isotropy of $\mathcal{L}$, that $(DI(\Delta), \chi) \in \mathcal{L}$. With the same argument, we can show that $(DI(\Delta), \chi) \in \overline{\mathcal{L}}$. We conclude that $(DI(\Delta), \chi) \in \mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$ and in particular $(\Delta, \alpha) = 0$ and thus $\mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}}) = \{0\}$.

In a similar way we can show that $\mathfrak{B}_I(\mathcal{L}) \cap J^1_{\mathbb{C}} L_S = \{0\}$. As a consequence, we have that $\mathfrak{B}_I(\mathcal{L}) = \mathcal{L}_\Omega$ for $\Omega \in \Omega^2_{L_{\mathbb{C}}}(M)$ and the last thing to show is that $\mathsf{Im}(\Omega) = \omega$, where ω is the contact structure induced by J. Let us therefore, consider the unique endomorphism

$$\mathbb{K} = \begin{pmatrix} \phi & J^\sharp \\ \alpha^\flat & -\phi^* \end{pmatrix}$$

defining $\mathcal{L}$ and let us choose an arbitrary real $\psi \in J^1L\big|_S$. Note that $(J^\sharp(\psi), i\psi - \phi^*\psi) \in \mathcal{L}$ and since $\mathrm{im}(J^\sharp)_{\mathbb{C}}\Big|_S = \mathrm{pr}_D(\mathcal{L}\big|_S) \cap \mathrm{pr}_D(\overline{\mathcal{L}}\big|_S) = \mathrm{im}(DI)_{\mathbb{C}}$, there is a unique real $\Delta \in DL_S$, such that $DI(\Delta) = J^\sharp(\psi)$. Hence, we conclude that

$$(\Delta, DI^*(\mathrm{i}\psi - \phi^*\psi)) \in \mathfrak{B}_I(\mathcal{L}) = \mathcal{L}_\Omega,$$

which implies $\iota_\Delta \mathsf{Im}(\Omega) = DI^*(\psi)$. The exact same computation can be done for $\mathcal{L}_J$ and we get $\iota_\Delta \omega = DI^*\psi$ and the claim follows by defining $\mathsf{Re}(\Omega) = B$. ΧΞΣ

Let us discuss locally conformal symplectic leaves of the Jacobi structure of a generalized contact structure. But before, in order to capture the full information on the leaf, we shall discuss the generalized complex analogue in the conformal setting, similarly as we discussed the locally conformal Dirac setting at the end of Subsection 1.2.2 and locally conformal Poisson at the end of Subsection 1.2.3. These objects have been considered for the trivial line bundle case in [44].

Definition 1.2.60 *Let $L \to M$ be a line bundle and let $\nabla \colon TM \to DL$ be a flat connection. A subbundle $\mathcal{D} \subseteq \mathbb{T}^L_{\mathbb{C}} M$ is said to be locally conformal generalized complex, if $\mathcal{D}$*

i.) is a locally conformal Dirac-structure

ii.) is complex, i.e. $\mathcal{D} \oplus \overline{\mathcal{D}} = \mathbb{T}^L_{\mathbb{C}} M$

We will be very sloppy with the definitions and proofs in this setting, since the results, at least the ones we need, are very similar to the ones in generalized contact and/or generalized complex geometry. Anyway, let us discuss the two most obvious examples first.

Example 1.2.61 Let $L \to M$ be a line bundle and let (∇, ω) be a locally conformal symplectic structure. Then

$$\mathcal{D}_\omega = \{(X, \mathrm{i}\iota_X\omega) \in \mathbb{T}^L_{\mathbb{C}} M \mid X \in T_{\mathbb{C}} M\}$$

is a locally conformal generalized complex structure.

Example 1.2.62 Let $L \to M$ be a line bundle, let $\nabla\colon TM \to DL$ be a flat connection and let $\phi \in \mathsf{End}(TM)$ be a complex structure with holomorphic (resp. anti-holomorphic) tangent bundle $T^{(1,0)}M$ (resp. $T^{(0,1)}M$), then

$$\mathcal{D} := T^{(1,0)}M \oplus (T^{(0,1)}M)^* \otimes L$$

is a locally conformal generalized complex manifold.

As for generalized contact structures a locally conformal generalized complex structures can be defined via an appropriate endomorphism

Lemma 1.2.63 *Let $L \to M$ be a line bundle with a flat connection $\nabla\colon TM \to DL$ and let $\mathcal{D} \subseteq \mathbb{T}^L_{\mathbb{C}}M$ be a locally conformal generalized complex structure. Then $\mathcal{D}$ it is the $+\mathrm{i}$-Eigenbundle of a unique endmorphism $\mathbb{I}\colon \mathbb{T}^L_{\mathbb{C}}M \to \mathbb{T}^L_{\mathbb{C}}M$, fulfilling*

i.) $\mathbb{I}^2 = -\mathsf{id}$

ii.) $\langle\!\langle \mathbb{I}-, \mathbb{I}-\rangle\!\rangle = \langle\!\langle -,-\rangle\!\rangle$

iii.) $0 = \mathcal{N}_{\mathbb{I}}(A,B) := [\![\mathbb{I}A, \mathbb{I}B]\!] - \mathbb{I}[\![\mathbb{I}A, B]\!] - \mathbb{I}[\![A, \mathbb{I}B]\!] - [\![A,B]\!]$ *for all* $A, B \in \Gamma^\infty(\mathbb{T}^L M)$

Moreover, the $+\mathrm{i}$-Eigenbundle of an endmorphism fulfilling i.)-iii.) is a locally conformal generalized complex structure.

Using the splitting $\mathbb{T}^L_{\mathbb{C}}M = TM \oplus (T^*M \otimes L)$, we can write

$$\mathbb{I} = \begin{pmatrix} \phi & \pi^\sharp \\ \sigma^\flat & -\phi^* \end{pmatrix}$$

for $\phi \in \mathsf{End}(TM)$, $\pi \in \Gamma^\infty(\Lambda^2(TM \otimes L^*) \otimes L)$ and $\sigma \in \Gamma^\infty(\Lambda^2 T^*M \otimes L)$. Moreover, we can show that π is a locally conformal Poisson structure.

The Dirac-Jacobi structure induced by a locally conformal Dirac structure as in Equation 1.2.2, if the input is a locally conformal generalized complex structure, can never be generalized contact, simply because of dimensional reasons. Nevertheless, we have

Lemma 1.2.64 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ be a generalized contact structure and let $\iota\colon S \hookrightarrow M$ be a locally conformal symplectic leaf. Then*

$$\mathfrak{B}_I(\mathcal{L}) = \mathcal{L}_{\mathcal{D}}$$

for $\mathcal{D} = \{(X, \iota_X(\mathrm{i}\omega + B)) \in \mathbb{T}^L_{\mathbb{C}}S \mid X \in T_{\mathbb{C}}S\}$, where ω is the locally conformal structure corresponding to $\mathcal{L}$ and $B \in \Gamma^\infty(\Lambda^2 T^*S \otimes L_S)$ is d^∇-closed.

Proof: The proof of the smoothness of $\mathfrak{B}_I(\mathcal{L})$ follows the same lines as the proof of Lemma 1.2.59. Let us denote by (ω, ∇) the locally conformal symplectic structure induced by the Jacobi tensor which corresponds to the generalized contact structure.

It is easy to see that $\mathrm{pr}_D(\mathfrak{B}_I(\mathcal{L})) = \mathrm{im}(\nabla)_{\mathbb{C}}$. By the very same argument as in the proof of Lemma 1.2.26, we see that

$$\mathfrak{B}_I(\mathcal{L}) = \{(\nabla_X, \sigma^*(\iota_X\Omega) + \alpha) \in \mathbb{D}_{\mathbb{C}}L_S \mid X \in T_{\mathbb{C}}S \text{ and } \alpha \in \mathrm{Ann}(\mathrm{im}(\nabla)_{\mathbb{C}})\}$$

for a for a L_S-valued 2-form Ω with $\mathrm{d}^\nabla\Omega = 0$. Applying the same argument as in Lemma 1.2.59, we get the claim. ΧΞΣ

1.3 Products of Dirac-Jacobi Bundles

Unlike in Dirac Geometry, the products of Dirac-Jacobi bundles are rather involved, they involve the product construction of Subsection 1.1.1.

Let $L \to M$ be a line bundle and let $\mathcal{L}_1, \mathcal{L}_2 \subset \mathbb{D}L$ be Dirac-Jacobi structures on L. Put

$$\mathcal{L}_1 \star \mathcal{L}_2 := \{(\Delta, \psi_1 + \psi_2) : (\Delta, \psi_i) \in \mathcal{L}_i, i = 1, 2\} \subset \mathbb{D}L$$

Lemma 1.3.1 *If $\mathcal{L}_1 \star \mathcal{L}_2 \subset \mathbb{D}L$ is smooth, then it is a Dirac-Jacobi structure (called the* sum *of $\mathcal{L}_1$* and *$\mathcal{L}_2$).*

Proof: See [34]. The isotropy of $\mathcal{L}_1 \star \mathcal{L}_2 \subset \mathbb{D}L$ is clear by definition. The product fits in the following (pointwise) exact sequence

$$0 \longrightarrow \mathrm{Ann}(\mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) \cap \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2)) \longrightarrow \mathcal{L}_1 \star \mathcal{L}_2 \longrightarrow \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) \cap \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2) \longrightarrow 0,$$

where we used that $\mathrm{pr}_D(\mathcal{L}_1 \star \mathcal{L}_2) = \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) \cap \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2)$. We conclude that $rk(\mathcal{L}_1 \star \mathcal{L}_2) = n + 1$ and hence it is maximal isotropic. For the involutivity we first choose a point $p \in M$. Let $(\Delta_p, \psi_p^1 + \psi_p^2) \in \mathcal{L}_1 \star \mathcal{L}_2|_p$ be arbitrary. We see that $\Delta_p \in \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) \cap \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2) = \mathrm{pr}_D(\mathcal{L}_1 \star \mathcal{L}_2)$ which is smooth. Therefore there exists a (local) section $\Delta \in \Gamma^\infty(\mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) \cap \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2))$, such that $\Delta(p) = \Delta_p$. Since in particular $\Delta\Gamma^\infty(\mathrm{pr}_D\mathcal{L}_i\big|_U)$ for an open subset U containing p, we can find $\psi^i \in \Gamma^\infty(\in J^1L_i|_U)$, such that $(\Delta, \lambda^i) \in \Gamma^\infty(\mathcal{L}_i|_U)$ and $\psi^i(p) = \psi^i_p$.

So we found a section $(\Delta, \psi^1 + \psi^2) \in \Gamma^\infty(\mathcal{L}_1 \star \mathcal{L}_2)$ which is evaluated at p given by $(\Delta_p, \psi_p^1 + \psi_p^2)$. The proof that the tensor $\mathcal{N}_{\mathcal{L}}$ vanishes at p is now an easy computation using this kind of sections. ΧΞΣ

Corollary 1.3.2 *Let $\mathcal{L}_i \subseteq \mathbb{D}L$, for $i = 1, 2$, be two Dirac-Jacobi structures. If $rk\big(\mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) + \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2)\big)$ is constant, then $\mathcal{L}_1 \star \mathcal{L}_2$ is a Dirac-Jacobi structure.*

Proof: We introduce the map

$$K \colon \mathcal{L}_1 \oplus \mathcal{L}_2 \ni \big((\Delta, \alpha), (\Box, \beta)\big) \mapsto \Delta - \Box \in \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) + \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2).$$

Since $rk(\mathrm{pr}_{\mathrm{D}}(\mathcal{L}_1) + \mathrm{pr}_{\mathrm{D}}(\mathcal{L}_2))$ is constant and K is surjective, D is a regular vector bundle morphism and its kernel is therefore a smooth subbundle. We consider the map

$$S \colon \ker(K) \ni \big((\Delta, \alpha), (\Delta, \beta)\big) \mapsto (\Delta, \alpha + \beta) \in \mathcal{L}_1 \star \mathcal{L}_2$$

Since S is a surjective bundle morphism, $\mathcal{L}_1 \star \mathcal{L}_2$ is smooth and by Lemma 1.3.1 a Dirac-Jacobi structure. XΞΣ

Remark 1.3.3 In fact the previous lemma is just a special case of the fact that $\mathcal{L}_1 \star \mathcal{L}_2$ is smooth if and only if $\mathrm{pr}_D \mathcal{L}_1 \cap \mathrm{pr}_D \mathcal{L}_2$ is, which can be seen by elementary techniques. Namely, let us assume that $\mathrm{pr}_D \mathcal{L}_1 \cap \mathrm{pr}_D \mathcal{L}_2$ is smooth and let $(\delta, \psi^1 + \psi^2) \in \mathcal{L}_1 \star \mathcal{L}_2$. Then we find a smooth section $\Delta \in \Gamma^\infty(\mathrm{pr}_D \mathcal{L}_1 \cap \mathrm{pr}_D \mathcal{L}_2)$, such that $\Delta_p = \delta$. Thus we can find two sections $\Psi^i \in \Gamma^\infty(J^1 L)$ for $i = 1, 2$, such that $(\Delta, \Psi^i) \in \Gamma^\infty(\mathcal{L}_i)$ and $\Psi^i_p = \psi^i$. Hence, $(\Delta, \Psi^1 + \Psi^2)$ is a smooth section and coincides with $(\delta, \psi^1 + \psi^2)$.

Lemma 1.3.4 *Let $(M_i, L_i, \mathcal{L}_i)$ $i = 1, 2$ be two Dirac-Jacobi bundles and let $L^\times \to M^\times$ be the product of L_1 and L_2 in $\mathfrak{Line}$. The subbundle $\mathcal{L}_1 \times^! \mathcal{L}_2 := \mathfrak{B}_{P_1}(\mathcal{L}_1) \star \mathfrak{B}_{P_1}(\mathcal{L}_2) \subseteq \mathbb{D}L^\times$ is smooth and therefore a Dirac-Jacobi structure, moreover the the morphisms $P_i \colon L^\times \to L_i$ are forward Dirac-Jacobi maps.*

Proof: By the definition of the backwards transforms, we have that $\ker DP_i \subseteq \mathfrak{B}_{P_i}(\mathcal{L}_i)$ and hence we have $\mathrm{pr}_D \mathfrak{B}_{P_1}(\mathcal{L}_1) + \mathrm{pr}_D(\mathfrak{B}_{P_2}(\mathcal{L}_2)) \supseteq \ker DP_1 + \ker DP_2$, but from Lemma 1.1.4 we know that $\ker DP_1 \oplus \ker DP_2 = DL^\times$. Applying now Corollary 1.3.2, we see that $\mathfrak{B}_{P_1}(\mathcal{L}_1) \star \mathfrak{B}_{P_2}(\mathcal{L}_2)$ is smooth. Now we want to prove

$$\mathfrak{F}_{P_i}(\mathfrak{B}_{P_1}(\mathcal{L}_1) \star \mathfrak{B}_{P_2}(\mathcal{L}_2)) = \mathcal{L}_i.$$

Let $(\Delta_{p_1(x)}, \alpha_{p_1(x)}) \in \mathcal{L}_{1,p_1(x)}$ be arbitrary. Using again Lemma 1.1.4, we find $\tilde{\Delta}_x \in \ker DP_2\big|_x$, such that $DP_1(\tilde{\Delta}_x) = \Delta_{p_1(x)}$ and therefore $(\tilde{\Delta}_x, Dp_1^* \alpha_{p_1(x)}) \in \mathfrak{B}_{p_1}(\mathcal{L}_1)$. Since $\tilde{\Delta}_x \in \ker Dp_2$, we also have $(\tilde{\Delta}_x, DP_1^* \alpha_{p_1(x)}) \in \mathfrak{B}_{p_1}(\mathcal{L}_1) \star \mathfrak{B}_{p_1}(\mathcal{L}_1)$

We see therefore, that $(\Delta_{p_1(x)}, \alpha_{p_1(x)}) = (DP_1(\tilde{\Delta}_x), \alpha_{p_1(x)}) \in \mathfrak{F}_{P_i}(\mathfrak{B}_{P_1}(\mathcal{L}_1) \star \mathfrak{B}_{P_2}(\mathcal{L}_2))$ and hence

$$\mathfrak{F}_{P_i}(\mathfrak{B}_{P_1}(\mathcal{L}_1) \star \mathfrak{B}_{P_2}(\mathcal{L}_2)) \supseteq \mathcal{L}_i.$$

For dimensional reasons we conclude equality. XΞΣ

Remark 1.3.5 For two Dirac-Jacobi bundles $(L_i \to M_i, \mathcal{L}_i))$ the Dirac-Jacobi bundle $\mathcal{L}_1 \times^! \mathcal{L}_2$ is actually a product in the category of Drac-Jacobi bundles with forward Dirac-Jacobi maps as morphisms. This is easy to verify using the fat that the underlying line bundle is a product in the category of line bundles, see 1.1.1. We refer to $(L^\times \to M^\times, \mathcal{L}^\times)$ as the product of $(L_1 \to M_1, \mathcal{L}_1))$ and $(L_2 \to M_2, \mathcal{L}_2)$.

We now want to explore how this construction behaves, when the Dirac-Jacobi bundles are actually Jacobi structures.

Lemma 1.3.6 *Let $(L_i \to M_i, \mathcal{L}_i)$ be two Dirac-Jacobi bundles coming from two Jacobi tensors J_i. Then $\mathcal{L}_1 \times^! \mathcal{L}_2$ is a Dirac-Jacobi structure coming from a Jacobi structure. Moreover, if the two Jacobi structures are contact also their product is.*

PROOF: We are going to use Lemma 1.2.35, so we just have to show that $(\mathcal{L}_1 \times^! \mathcal{L}_2) \cap DL^\times = \{0\}$, since the product is a Dirac- Jacobi structure be the previous considerations. Let $(\Delta, 0) \in \mathcal{L}_1 \times^! \mathcal{L}_2$. So, using that the P_i's are forward Dirac-Jacobi, we have that

$$(DP_i(\Delta), 0) \in \mathcal{L}_i.$$

But since by assumption the $\mathcal{L}_i$'s come from Jacobi tensors, we can use Lemma 1.2.35 to obtain $DP_i(\Delta) = 0$. Thus $\Delta = 0$, which holds, because $DL^\times = \ker DP_1 \oplus \ker DP_2$. The proof of the last claim follows the same lines. ΧΞΣ

Remark 1.3.7 By the previous Lemma it is clear that the product of Dirac-Jacobi bundles is also a product of Jacobi bundles. Nevertheless, it is not a product in the category of contact bundles, since a Jacobi map between contact bundles is not necessarily a contact map. A similar phenomenon occurs in the Poisson setting with symplectic structures.

Now we want to introduce pull-back diagrams of Dirac-Jacobi bundles, note that pull-backs do not always exist in the category of line bundles.

Corollary 1.3.8 *Let $\Phi_i \colon L_i \to L$ be regular line bundle morphisms covering $\phi_i \colon M_i \to M$ and let $\mathcal{L}_i \subseteq \mathbb{D}L_i$ and $\mathcal{L} \subseteq \mathbb{D}L$ Dirac-Jacobi structures, such that the Φ_i's are forward Dirac-Jacobi maps. If the pull-back*

$$\begin{tikzcd} L^\times_M \arrow[r, "P_2"] \arrow[d, "P_1"] & L_2 \arrow[d, "\Phi_2"] \\ L_1 \arrow[r, "\Phi_1"] & L \end{tikzcd}$$

exists, then $\mathfrak{B}_{P_1}(\mathcal{L}_1)$, $\mathfrak{B}_{P_2}(\mathcal{L}_2)$ and $\mathfrak{B}_{P_1}(\mathcal{L}_1) \star \mathfrak{B}_{P_2}(\mathcal{L}_2)$ are Dirac-Jacobi structures and moreover the P_i's are forward Dirac-Jacobi maps.

PROOF: The proof follows from Lemma 1.3.4 and Corollary 1.1.6. ΧΞΣ

Let us now discuss some properties of the product with respect to backwards and forward Dirac-Jacobi maps. There are many compatibilities wetween them, but we will just discuss the ones which we are going to use throughout this thesis.

Lemma 1.3.9 *Let $(L_i \to M_i, \mathcal{L}_i)$ be two Dirac-Jacobi bundles for $i = 1, 2$ and let $\Phi \colon L_{N_i} \to L_i$ be two regular line bundle morphisms covering $\phi_i \colon N_i \to M_i$, such that $\mathfrak{B}_{\Phi_i}(\mathcal{L})$ are Dirac-Jacobi bundles. Then*

$$\mathfrak{B}_{\Phi_1 \times^! \Phi_2}(\mathcal{L}_1 \times^! \mathcal{L}_2) = \mathfrak{B}_{\Phi_1}(\mathcal{L}_1) \times^! \mathfrak{B}_{\Phi_2}(\mathcal{L}_2).$$

PROOF: Let us first discuss what we mean by the map $\Phi_1 \times^! \Phi_2 \colon L^\times_N \to L^\times$: it is the unique arrow making

$$\begin{tikzcd}
L_N^\times \arrow[rr, "P_{N_2}"] \arrow[dr, dashed, "\exists!"] \arrow[dd, "P_{N_1}"'] & & L_{N_2} \arrow[d, "\Phi_2"] \\
& L^\times \arrow[r, "P_2"] \arrow[d, "P_1"] & L_2 \\
L_{N_1} \arrow[r, "\Phi_1"] & L_1 &
\end{tikzcd}$$

commute. Mote that we cannot use the universal property of the product in the category of Dirac-Jacobi bundles, since backward Dirac-Jacobi maps are not morphisms in this category. So let $(\Delta, D(\Phi_1 \times^! \Phi_2)^*\psi) \in \mathfrak{B}_{\Phi_1\times^!\Phi_2}(\mathcal{L}_1 \times^! \mathcal{L}_2)$ then we know that $(D(\Phi_1 \times^! \Phi_2)(\Delta), \psi) \in \mathcal{L}_1 \times^! \mathcal{L}_2$. On the other hand, this implies that there are $\psi_i \in J^1 L_i$, such that $\psi = DP_1^*\psi_1 + DP_2^*\psi_2$ and $(D(\Phi_1 \times^! \Phi_2)(\Delta), DP_i^*\psi_i) \in \mathfrak{B}_{P_i}(\mathcal{L}_i)$. Summarizing, we get

$$\begin{aligned}
(\Delta, D(\Phi_1 \times^! \Phi_2)^*\psi) &= (\Delta, D(\Phi_1 \times^! \Phi_2)^*(DP_1^*\psi_1 + DP_2^*\psi_2)) \\
&= (\Delta, DP_{N_1}^*\psi_1 + DP_{N_2}^*\psi_2).
\end{aligned}$$

Now we have that $(\Delta, DP_{N_i}^*\psi_i) \in \mathfrak{B}_{P_{N_i}}(\mathfrak{B}_{\Phi_i}(\mathcal{L}_i))$ by construction and hence $\mathfrak{B}_{\Phi_1\times^!\Phi_2}(\mathcal{L}_1 \times^! \mathcal{L}_2) \subseteq \mathfrak{B}_{\Phi_1}(\mathcal{L}_1) \times^! \mathfrak{B}_{\Phi_2}(\mathcal{L}_2)$. Both bundles are maximally isotropic and thus equal.XΞΣ

Remark 1.3.10 Note that we actually proved that, for two Dirac-Jacobi structures $\mathcal{L}_i \subseteq \mathbb{D}L_2$ $i = 1, 2$ and a regular line bundle morphism $P\colon L_1 \to L_2$,

$$\mathfrak{B}_P(\mathcal{L}_1 \star \mathcal{L}_2) = \mathfrak{B}_P(\mathcal{L}_1) \star \mathfrak{B}_P(\mathcal{L}_2)$$

holds.

The next lemma shows the interplay between forward Dirac-Jacobi maps, backward Dirac-Jacobi maps and (fibered) products.

Lemma 1.3.11 *Let $(L_i \to M_i, \mathcal{L}_i)$ be two Dirac-Jacobi bundles, let $\Phi\colon L_1 \to L_2$ be a forward Dirac-Jacobi map covering $\phi\colon M_1 \to M_2$ and let $\Psi\colon L_N \to L_2$ be a regular line bundle morphism covering $\psi\colon N \to M_2$ transverse to $\mathcal{L}_2$. Then the pull-back*

$$\begin{tikzcd}
L^\times \arrow[r, dashed, "P_N"] \arrow[d, dashed, "P_M"'] & L_N \arrow[d, "\Psi"] \\
L_1 \arrow[r, "\Phi"] & L_2
\end{tikzcd}$$

exists and P_N is a forward Dirac-Jacobi map for the Dirac-Jacobi structures $\mathfrak{B}_{P_M}(\mathcal{L}_1)$ and $\mathfrak{B}_\Psi(\mathcal{L})$.

PROOF: Recall that $\Psi\colon L_N \to L_2$ is called transverse to $\mathcal{L}_2 \subseteq \mathbb{D}L_2$, if

$$D\Psi(DL_N) + \mathrm{pr}_D(\mathcal{L}_2) = DL_2\big|_{\psi(N)}$$

and after applying the anchor we get in particular:

$$T\psi(TN) + \sigma(\mathrm{pr}_D(\mathcal{L}_2)) = TM_2\big|_{\psi(N)}.$$

Let us first prove that the pull-back exists, but using Theorem 1.1.2 we just have to show that the product

$$\begin{tikzcd} M_1 \times_{M_2} N \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] & N \arrow[d, "\psi"] \\ M_1 \arrow[r, "\phi"] & M_2 \end{tikzcd}$$

exists in $\mathfrak{Man}$. A sufficient criterion for the existence, is that

$$T\phi(TM_1) + T\psi(TN) = TM_2\big|_{\phi(M_1)\cap\psi(N)}$$

Since Φ is a forward map, we have that $D\Phi(DL_1) \supset \mathrm{pr}_D\mathcal{L}_2$. Moreover, since Ψ is transversal, we have

$$DL_2\big|_{\phi(M_1)\cap\psi(N)} = \mathrm{pr}_D\mathcal{L} + D\Psi(DL_N) \subseteq D\Phi(DL_1) + D\Psi(DL_N)$$

and hence equality. Applying the symbol $\sigma\colon DL_2 \to TM_2$ we get the required equality $T\phi(TM_1) + T\psi(TN) = TM_2\big|_{\phi(M_1)\cap\psi(N)}$. It is easy to show that P_M is transverse to $\mathcal{L}_1$ and hence $\mathfrak{B}_{P_M}(\mathcal{L}_1)$ is a Dirac-Jacobi structure by Corollary 1.2.18. The property that P_N is a forward Dirac-Jacobi map follows by direct computation and the interplay of the D-functor and the products, see 1.1.5. ΧΞΣ

Chapter 2

Normal Form Theorems

Since the work of Weinstein [51], in which he proved his famous local splitting theorem for Poisson manifolds, many works appeared concerning different viewpoints on the proof and even give more general statements, namely normal form theorems. Frejlich and Mărcuț proved a normal form theorem around Poisson (cosymplectic) transversals of Poisson manifolds in [20]. In [21] they used the techniques of Dual Pairs to prove a similar statement for Dirac structures. And finally, there is a unified approach by Bursztyn, Lima and Meinrenken in [12] to prove normal forms for Poisson related structures.

This chapter reformulates these techniques and results in the Jacobi setting in order discuss a proof of normal forms in Jacobi geometry, i.e. for Dirac-Jacobi bundles, Jacobi structures and generalized contact bundles. We follow [12] as a guideline throughout this chapter. This Chapter is based on [38] and generalizes [41].

2.1 Submanifolds and Euler-like Vector Fields

In this subsection we want to discuss Euler-like vector fields with respect to submanifolds. These vector fields, in particular, induce a homogeneity structure on the manifold around the given submanifold, which is equivalent, under some additional conditions, that the manifold is total space of a vector bundle, see e.g. [24]. This total space turns out to be the normal bundle of the submanifold. Nevertheless, we will not go more in details with these features, since we work directly with tubular neighborhoods. We will begin collecting facts about tubular neighborhoods, submanifolds, corresponding mappings and describe afterwards the notion of Euler-like vector fields and extend this notion the derivations of a line bundle.

2.1.1 Normal Bundles and tubular Neighborhoods

The notations and results throughout this section are taken from [12].

For a pair of manifolds (M, N), i.e. a submanifold $N \hookrightarrow M$, we denote

$$\nu(M, N) = \frac{TM\big|_N}{TN}$$

the normal bundle. If it is clear what is the ambient space, we will just write ν_N instead. Given a map of pairs

$$\Phi\colon (M, N) \to (M', N'),$$

i.e. a map $\Phi\colon M \to M$, such that $\Phi(N) \subseteq N'$, we denote by

$$\nu(\Phi)\colon \nu(M, N) \to \nu(M', N')$$

the induced map between the normal bundles. For a vector field X on M tangent to N, we have that the flow Φ_t^X is a map of pairs from (M, N) to itself. Hence we define

$$T\nu(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu(\Phi_t^X) \in \Gamma^\infty(T\nu_N).$$

Moreover, for a vector bundle $E \to M$ and $\sigma \in \Gamma^\infty(E)$, such that $\sigma\big|_N = 0$ for a submanifold $N \hookrightarrow M$, we denote by

$$\mathrm{d}^N\sigma\colon \nu_N \to E\big|_N$$

the map which is $\nu(\sigma)$, for σ seen as a map $\sigma\colon (M, N) \to (E, M)$, followed by the canonical identification $\nu(E, M) = E$, given by

$$C_E\colon E \ni v_p \to [\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} tv_p]_{TM} \in \nu(E, M).$$

The inverse C_E^{-1} is given by

$$C_E^{-1}([\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma(t)]) = \lim_{t\to 0}\frac{\gamma(t)}{t}. \tag{2.1.1}$$

for a curve $\gamma\colon I \to E$ defined in an open interval I containing 0, such that $\gamma(0) = 0_p$ for $p \in M$.

Remark 2.1.1 For a pair of manifolds (M, N), a vector bundle $E \to M$ and a vector bundle morphism $A\colon \nu_N \to E\big|_N$, one can always find a section $\sigma \in \Gamma^\infty(E)$, such that

i.) $\sigma\big|_N = 0$

ii.) $\mathrm{d}^N\sigma = A$.

This follows locally by elementary techniques and one can extend it via a partition of unity.

Proposition 2.1.2 *Let $E_i \to M_i$ be vector bundles for $i = 1, 2$ and let $\Phi\colon E_1 \to E_2$ be a vector bundle morphism. Then, for $\Phi\colon (E_1, M_1) \to (E_2, M_2)$,*

$$C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1} = \Phi.$$

PROOF: Let $v_p \in E_1$, then

$$\begin{aligned}
(C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1})(v_p) &= (C_{E_2}^{-1} \circ \nu(\Phi))([\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} tv_p]_{TM_1}) \\
&= C_{E_2}^{-1}([T\Phi \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} tv_p]_{TM_2}) \\
&= C_{E_2}^{-1}([\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} t\Phi(v_p)]_{TM_2}) \\
&= \Phi(v_p)
\end{aligned}$$

ΧΞΣ

Proposition 2.1.3 *Let $E_i \to M$ be vector bundles,$i = 1, 2$, let $\Phi\colon E_1 \to E_2$ be a vector bundle morphism covering the identity and let (M, N) be a pair of manifolds. Then, for every section $\sigma \in \Gamma^\infty(E_1)$, such that $\sigma\big|_N = 0$,*

$$\mathrm{d}^N \Phi(\sigma) = \Phi(\mathrm{d}^N \sigma)$$

holds.

PROOF: Consider the map $\Phi(\sigma)\colon (M, N) \to (E_2, M)$, then we have

$$\begin{aligned}
C_{E_2}^{-1} \circ \nu(\Phi(\sigma)) &= C_{E_2}^{-1} \circ \nu(\Phi) \circ \nu(\sigma) \\
&= C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1} \circ C_{E_1}^{-1} \circ \nu(\sigma) \\
&= \Phi \circ C_{E_1}^{-1} \circ \nu(\sigma)
\end{aligned}$$

and the claim follows restricting these maps to N. ΧΞΣ

Proposition 2.1.4 *Let (M, N) be a pair of manifolds and let $X \in \Gamma^\infty(TM)$ be a vector field, such that $X\big|_N = 0$. Then*

$$T\Phi_t^X\big|_N = \exp(tD_X)$$

for a unique $D_X \in \Gamma^\infty(\mathsf{End}(TM\big|_N))$, moreover $TN \subseteq \ker(D_X)$ and

$$\begin{tikzcd}
TM\big|_N \arrow[r, "D_X"] \arrow[d] & TM\big|_N \\
\nu_N \arrow[ur, "\mathrm{d}^N X"'] &
\end{tikzcd}$$

commutes.

PROOF: Since $X\big|_N = 0$, its flow fixes all elements of N. This means

$$T\Phi_t^X : T_pM \to T_pM$$

for all $t \in \mathbb{R}$ and $p \in N$. Moreover, it fulfills the property,

$$T\Phi_t^X \circ T\Phi_s^X = T\Phi_{t+s}^X$$

and $T\Phi_0^X = \mathsf{id}$ and hence the claim follows. ΧΞΣ

Definition 2.1.5 *Let (M, N) be a pair of manifolds. A tubular neighborhood of N is an open subset $U \subseteq M$ containing N, together with a diffeomorphism*

$$\psi \colon \nu_N \to U,$$

such that $\psi\big|_N \colon N \to N$ is the identity and for $\psi \colon (\nu_N, N) \to (M, N)$ the map

$$\nu(\psi) \colon \nu(\nu_N, N) \to \nu_N$$

is the inverse of $C_{\nu_N} \colon \nu_N \to \nu(\nu_N, N)$.

Remark 2.1.6 Definition 2.1.5 is not the only definition of tubular neighborhood. In fact, the condition

$$\nu(\psi) \colon \nu(\nu_N, N) \to \nu_N$$

is usually not considered.

2.1.2 Euler-like Vector Fields and Derivations

In this part, we recall the notion of Euler-like vector fields from [12] and extend this notion to derivations of a line bundle.

Definition 2.1.7 *Let (M, N) be a pair of manifolds. A vector field $X \in \Gamma^\infty(TM)$ is called Euler-like, if*

i.) $X\big|_N = 0$,

ii.) *X is complete (i.e. the flow of X is defined for all $t \in \mathbb{R}$),*

iii.) $T\nu(X) = \mathcal{E}$,

where $\mathcal{E}$ is the Euler vector field on $\nu_N \to N$.

Proposition 2.1.8 *Let (M, N) be a pair of manifolds, then there exists an Euler-like vector field around N.*

Proof: Let us choose a tubular neighborhood

$$\psi\colon \nu_N \to U.$$

For the vector field $X = \psi_* \mathcal{E}$ multiplied by a suitable bump function which is 1 in a neighborhood of N, we have

$$\begin{aligned} T\nu(X) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu(\Phi_t^X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu(\psi \circ \Phi_t^{\mathcal{E}} \circ \psi^{-1}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu(\psi) \circ \nu(\Phi_t^{\mathcal{E}}) \circ \nu(\psi^{-1}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t^{\mathcal{E}} = \mathcal{E}. \end{aligned}$$

We used Proposition 2.1.2 and the fact that $\nu(\psi) = C_{\nu_N}^{-1}$. ΧΞΣ

Lemma 2.1.9 *Let M be a manifold, $N \hookrightarrow M$ a submanifold and $X \in \Gamma^\infty(TM)$ be a Euler-like vector field. Then there exists a unique tubular neighborhood embedding*

$$\psi\colon \nu_N \to U,$$

such that $\psi^* X = \mathcal{E}$.

Proof: The proof can be found in [12]. ΧΞΣ

Proposition 2.1.10 *Let (M, N) be a pair of manifolds and let $X \in \Gamma^\infty(TM)$ be a complete vector field such that $X\big|_N = 0$. Then X is Euler-like, if and only if $\mathrm{d}^N X$ followed by the projection $TM\big|_N \to \nu_N$ is the identity.*

Proof: Let $X \in \Gamma^\infty(TM)$ be given as in the statement. According to Proposition 2.1.4, there exists a unique $D_X \in \Gamma^\infty(\mathsf{End}(TM\big|_N))$, such that $T\Phi_t^X\big|_N = \exp(tD_X)$. Let $[X_p] \in \nu_N$ be an equivalence class of tangent vectors, then

$$\nu(\Phi_t^x)([X_p]) = [T\Phi_t^X(X_p)] = [\exp(tD_X)(X_p)].$$

This is nothing but the flow of the Euler vector field, as if $\mathrm{pr}_{\nu_N} \circ D_X(X_p) = [X_p]$. Using Proposition 2.1.4, we have $\mathrm{d}^N X([X_p]) = D_X(X_p)$ for all $[X_p] \in \nu_N$ and hence the claim. ΧΞΣ

Note that, for a pair of manifolds (M, N) and a Euler like vector field $X \in \Gamma^\infty(TM)$, the set

$$\left\{ p \in M \mid \lim_{t \to -\infty} \Phi_t^X(p) \text{ exists and lies in } N \right\}$$

is an open subset in M containing N, and stable under the flow of X. In fact, by Lemma 2.1.9, the unique tubular neighborhood $\psi\colon \nu_N \to U$ with $\psi^* X = \mathcal{E}$, satisfies

$$U = \{p \in M \mid \lim_{t \to -\infty} \Phi_t^X(p) \text{ exists and lies in } N\}.$$

Let us denote $\lambda_s = \Phi^X_{\log(s)}\big|_U$, which is smooth for all $s \in \mathbb{R}^+$ and can be smoothly extended to $s = 0$. Moreover, we have that

$$\psi \circ \kappa_s = \lambda_s \circ \psi, \tag{2.1.2}$$

where we denote by $\kappa_s \colon \nu_N \to \nu_N$ the map $[X_p] \mapsto [sX_p]$. Note that $\kappa_0 \colon \nu_N \to N$ coincides with the bundle projection. To be more precise $\kappa_0 = \mathrm{pr}_\nu \circ j$, where $\mathrm{pr}_\nu \colon \nu_N \to N$ is the bundle projection and the 0 section $j \colon N \to \nu_N$.

Let us add now the line bundle case

Definition 2.1.11 *Let (M, N) be pair of manifolds and let $L \to M$ be a line bundle. A derivation $\Delta \in \Gamma^\infty(DL)$ is called Euler-like, if*

i.) $\Delta\big|_N = 0$,

ii.) *$\sigma(\Delta)$ is an Euler-like vector field.*

This definition turns out to be the correct one for our purposes, since with that we can prove the analogues of basically all results available for Euler-like vector fields. Let us start collecting such analogues.

Proposition 2.1.12 *Let (M, N) be a pair of manifolds, let $L \to M$ be a line bundle and let $\Delta \in \Gamma^\infty(DL)$ be an Euler-like derivation. Then the flow $\Phi^\Delta_t \in \mathsf{Aut}(L)$ of Δ induces the map*

$$\Lambda_s = \Phi^\Delta_{\log(s)},$$

which, restricted to $U = \{p \in M \mid \lim_{t\to-\infty} \Phi^{\sigma(X)}_t(p)$ exists and lies in $N\}$, can be extended smoothly to $s = 0$. Moreover, the map

$$\Lambda_0 \colon L_U \to L_N$$

is a regular line bundle morphism.

Proof: The proof is an easy verification using a tubular neighborhood $\psi \colon \nu_N \to U$, such that $\psi^*\sigma(\Delta) = \mathcal{E}$. ΧΞΣ

Definition 2.1.13 *Let (M, N) be a pair of manifolds and let $L \to M$ be a line bundle. A fat tubular neighborhood is a regular line bundle morphism*

$$\Psi \colon L_\nu \to L_U,$$

where the line bundle L_ν is given by the pull-back

$$\begin{tikzcd} L_\nu \arrow[r] \arrow[d] & L_N \arrow[d] \\ \nu_N \arrow[r] & N \end{tikzcd},$$

covering a tubular neighborhood $\psi\colon \nu_N \to U$, such that $\Psi\big|_N\colon L_N \to L_N$ is the identity.

Lemma 2.1.14 *Let (M, N) be a pair of manifolds, let $L \to M$ be a line bundle and let $\psi\colon \nu_N \to U$ be a tubular neighborhood. Then there exists a fat tubular neighborhood covering ψ.*

PROOF: The proof is an adaption of the proof in [43, Chapter 3]. First we notice that $L_\nu = (\iota \circ p_\nu)^* L$ for the canonical inclusion $\iota\colon N \hookrightarrow M$ and the bundle projection $\mathrm{pr}_\nu\colon \nu_N \to N$. Moreover, if we consider the the pullback bundle

$$\begin{tikzcd} \psi^* L_U \arrow[r, "\tilde{\Psi}"] \arrow[d] & L_U \arrow[d] \\ \nu_N \arrow[r, "\psi"] & U \end{tikzcd},$$

we see that the multiplication by $t \in [0, 1]$, denoted by $\kappa_t\colon \nu_N \to \nu_N$ induces a smooth homotopy $H\colon [0, 1] \times \nu_N \to M$ between ψ and $\mathrm{pr}_\nu \circ \iota$ via

$$H(t, -) = \psi \circ \kappa_t.$$

For pull-back bundles of homotopic maps there exists a (non-canonical) vector bundle isomorphism covering the identity. The claim follows by choosing an isomorphism $\Phi\colon L_\nu \to \psi^* L$ covering the identity and concatenate it with $\tilde{\Psi}$ and hence

$$\begin{tikzcd} L_\nu \arrow[r, "\Psi = \tilde{\Psi} \circ \Phi"] \arrow[d] & L_U \arrow[d] \\ \nu_N \arrow[r, "\psi"] & U \end{tikzcd},$$

is a fat tubular neighborhood. XΞΣ

For a line bundle $L \to N$ and a vector bundle $p\colon E \to N$ there is always a canonical derivation $\Delta_{\mathcal{E}} \in \Gamma^\infty(DL_E)$ where we denote by $L_E = p^* L$, such that $\sigma(\Delta_{\mathcal{E}}) = \mathcal{E}$ constructed as follows: Consider the map

$$\begin{tikzcd} L_E \arrow[r, "P"] \arrow[d] & L \arrow[d] \\ E \arrow[r, "p"] & N \end{tikzcd}$$

and the corresponding map $DP\colon L_E \to L_N$. We have that canonically $\ker(DP) \cong \mathrm{Ver}(E)$ for the Vertical bundle $\mathrm{Ver}(E)$ of $E \to M$, which induces a flat (partial) connection $\nabla\colon \mathrm{Ver}(E) \to DL_\nu$. Since the Euler vector field is vertical, we can define $\Delta_{\mathcal{E}} = \nabla_{\mathcal{E}}$.

Proposition 2.1.15 *Let $L \to N$ be a line bundle and let $E \to N$ be a vector bundle. Then the flow Φ_t of $\Delta_{\mathcal{E}} \in \Gamma^\infty(DL_E)$ is given by*

$$\Phi_t(v_p, l_p) = (\mathrm{e}^t \cdot v_p, l_p)$$

for all $(v_p, l_p) \in L_E$.

Proof: The derivation $\nabla_{\mathcal{E}}$ is by definition in the kernel of DP, it is related to the 0 derivation on $L \to M$ and hence we have for its flow

$$P \circ \Phi_t = P.$$

Since $L_E = E \times_M L$, we have that

$$\Phi_t(v_p, l_p) = (\phi_t(v_p), l_p)$$

where ϕ_t is the flow of the symbol of $\nabla_{\mathcal{E}}$, which is by construction the Euler vector field and hence the claim follows. ΧΞΣ

Note that for the flow Φ_t of the canonical Euler-like derivation $\Delta_{\mathcal{E}} \in \Gamma^\infty(DL_E)$, we have that

$$P_s = \Phi_{\log(s)} \colon L_E \to L_E \tag{2.1.3}$$

is defined for all $s > 0$ and can be extended smoothly to $s = 0$, moreover P_0 coincides with the canonical projection $P \colon L_E \to L$ followed by the canonical inclusion $J \colon L \to L_E$.

Lemma 2.1.16 *Let (M, N) be a pair of manifolds, let $L \to M$ be a line bundle let $\Delta \in \Gamma^\infty(DL)$ be an Euler-like derivation. Then there is a unique fat tubular neighborhood $\Psi \colon L_\nu \to L_U$, such that $\Psi^*\Delta = \Delta_{\mathcal{E}}$.*

Proof: First, we want to prove existence. It is clear that any such Ψ has to cover the unique tubular neighborhood $\psi \colon \nu_N \to U$, such that $\psi^*\sigma(\Delta) = \mathcal{E}$. So let us choose a fat tubular neighborhood $\tilde{\Psi} \colon L_\nu \to L_U$ covering ψ. We consider $\tilde{\Psi}^*\Delta \in \Gamma^\infty(DL_\nu)$. We have $\sigma(\tilde{\Psi}^*\Delta) = \psi^*\sigma(\Delta) = \mathcal{E}$. Hence $\sigma(\Delta_{\mathcal{E}}) = \sigma(\tilde{\Psi}^*\Delta)$. Consider now the derivation $\Box = \Delta_{\mathcal{E}} - \tilde{\Psi}^*\Delta$ and

$$\Box_t = -\frac{1}{t}\Phi^*_{\log(t)}\Box,$$

where Φ_t is the flow of $\Delta_{\mathcal{E}}$. Note that $\Box_t$ can be smoothly extended to $t = 0$, since $\Box\big|_N = 0$. Let us denote the flow of $\Box_t$ by ϕ_t. Note that it is complete, since $\sigma(\Box_t) = 0$, indeed there is even a explicit formula for it, which we do not use. Note however, that $\phi_t \in \mathsf{Aut}(L_\nu)$ covers the identity for all $t \in \mathbb{R}$. Let us compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*(\Delta_{\mathcal{E}} + t\Box_t) = \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] + \frac{\mathrm{d}}{\mathrm{d}t}t\Box_t)$$

$$
\begin{aligned}
&= \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] - \frac{\mathrm{d}}{\mathrm{d}t}\Phi^*_{\log(t)}\Box) \\
&= \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] - \frac{1}{t}[\Delta_{\mathcal{E}}, \Phi^*_{\log(t)}\Box]) \\
&= \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] + [\Delta_{\mathcal{E}}, \Box_t]) \\
&= 0.
\end{aligned}
$$

Hence $\Delta_{\mathcal{E}} = \phi_0^*(\Delta_{\mathcal{E}}) = \phi_1^*(\Delta_{\mathcal{E}} + \Box_1) = \phi_1^*(\tilde{\Psi}^*\Delta)$. Therefore, we have that the map $\Psi = \tilde{\Psi} \circ \phi_1$ will do the job, since obviously $\phi_1\big|_N = \mathsf{id}$.

Let us now assume that we have $\Psi_1, \Psi_2 \colon L_\nu \to L_U$, such that $\Psi_1^*\Delta = \Psi_2^*\Delta = \Delta_{\mathcal{E}}$. Since both have to cover the unique $\psi \colon \nu_N \to U$, the target L_U is the same for both. Let us consider $\Xi := \Psi_1^{-1} \circ \Psi_2 \colon L_\nu \to L_\nu$, which covers the identity, which implies that there is a nowhere vanishing function $f \in \mathscr{C}^\infty(\nu_N)$, such that $\Xi(l_p) = f(p)l_p$ for all $l_p \in L_\nu$. Moreover, we have that $\Xi\big|_N = \mathsf{id}_{L_\nu}\big|_N$, hence $f(0_n) = 1$ for all $n \in N$, and $\Xi^*\Delta_{\mathcal{E}} = \Delta_{\mathcal{E}}$. We consider now an arbitrary section $\lambda \in \Gamma^\infty(L_\nu)$ and compute

$$
\begin{aligned}
\Delta_{\mathcal{E}}(\lambda) &= (\Xi^*\Delta_{\mathcal{E}})(\lambda) \\
&= \Xi^*(\Delta_{\mathcal{E}}(\Xi_*\lambda)) \\
&= \frac{1}{f}(\Delta_{\mathcal{E}}(f\lambda)) \\
&= \frac{\mathcal{E}(f)}{f}\lambda + \Delta_{\mathcal{E}}(\lambda).
\end{aligned}
$$

Hence $\mathcal{E}(f) = 0$, which means that $f = \mathrm{pr}_\nu^* g$ for some function $g \in \mathscr{C}^\infty(N)$, but since $1 = f(0_n) = g(n)$ for all $n \in N$, we have that $\Xi = \mathsf{id}_{L_\nu}$. ΧΞΣ

For a pair of manifolds (M, N), a line bundle $L \to M$ and an Euler-like derivation $\Delta \in \Gamma^\infty(DL)$, we have that

$$
\Lambda_s := \Phi^\Delta_{\log(s)} \colon L_U \to L_U \tag{2.1.4}
$$

is well defined for $s > 0$ and can be extended smoothly to $s = 0$, where L_U is the target of the unique fat tubular neighborhood $\Psi \colon L_\nu \to L_U$, such that $\Psi^*\Delta = \Delta_{\mathcal{E}}$. Moreover, we have that

$$
\Lambda_s \circ \Psi = \Psi \circ P_s \tag{2.1.5}
$$

for all $s \geq 0$. Note that if we project this equation to the manifold level, this simply gives Eq. 2.1.2.

2.2 Normal Forms of Dirac-Jacobi Bundles

Using the techniques of Euler-like derivations, we want to prove a normal form theorem for Dirac-Jacobi bundles around transversals. Roughly speaking, this means in our case classifying all the possible Dirac-Jacobi structures which coincide on a given submanifold and moreover finding a particularly simple representative in this class. We will be more precise, what we exactly mean in the corresponding subsections. In fact, a transversal N allows us to find special Euler like derivations which are, in some sense, controlling the behaviour of the Dirac-Jacobi bundles near N. The aim is now to prove the existence of this special kind of Euler-like derivations and afterwards, we will be able to prove a normal form theorem and derive some corolloraries from it.

Definition 2.2.1 *Let $L \to M$ be a line bundle, let $H \in \Omega^3_L(M)$ be a closed Atiyah 3-form and let $\mathcal{L} \subseteq \mathbb{D}L$ be a H-twisted Dirac-Jacobi structure. A submanifold $N \hookrightarrow M$ is called transversal, if the inclusion map $I\colon L_N \to L$ is transversal to $\mathcal{L}$, i.e.*

$$DL_N + \mathrm{pr}_D\mathcal{L}\big|_N = (DL)\big|_N.$$

Moreover, N is called minimal *transversal at a point $p \in M$, if*

$$T_pN \oplus \sigma(\mathrm{pr}_D\mathcal{L}) = T_pM.$$

Remark 2.2.2 In Definition 2.2.1, we required for a minimal transversal N to a Dirac-Jacobi structure $\mathcal{L} \subseteq \mathbb{D}L$ not explicitly, that it is a transversal. Nevertheless, this is an immediate consequence of the equation

$$T_pN \oplus \sigma(\mathrm{pr}_D\mathcal{L}) = T_pM.$$

Proposition 2.2.3 *Let $L \to M$ be a line bundle, let $H \in \Omega^3_L(M)$ be closed, let $\mathcal{L} \subseteq \mathbb{D}L$ be a H-twisted Dirac-Jacobi bundle and let $N \hookrightarrow M$ be a transversal. Then*

$$\mathfrak{B}_I(\mathfrak{L}) := \{(\Delta_p, (DI)^*\alpha_{\iota(p)}) \in \mathbb{D}L_N \mid (DI(\Delta_p), \alpha_{\phi(p)}) \in \mathcal{L}\}$$

is an I^*H-twisted Dirac-Jacobi bundle, where $I\colon L_N \to L$ is the canonical inclusion.

Proof: This is an easy consequence of Theorem 1.2.17. ΧΞΣ

Lemma 2.2.4 *Let $L \to M$ be a line bundle, let $H \in \Omega^3_L(M)$ be closed, let $\mathcal{L} \subseteq \mathbb{D}L$ be an H-twisted Dirac-Jacobi structure and let $\iota\colon N \hookrightarrow M$ be a transversal. The backwards transformation $\mathfrak{B}_I(\mathfrak{L})$ is canonically isomorphic (as vector bundles over N) to the fibered product $I^!\mathcal{L}$ uniquely determined by*

$$\begin{array}{ccc} I^!\mathcal{L} & \longrightarrow & \mathcal{L} \\ \big\downarrow & & \big\downarrow \mathrm{pr}_D \\ DL_N & \xrightarrow{DI} & DL \end{array} .$$

Proof: We consider the linear map

$$\Xi\colon I^!\mathcal{L}_p \ni (\Delta_p, (\Box_{\iota(p)}, \alpha_{\iota(p)})) \mapsto (\Delta_p, DI^*\alpha_{\iota(p)}) \in \mathfrak{B}_I(\mathfrak{L}),$$

which is well-defined since $DI(\Delta_p) = \Box_{\iota(p)}$. We claim now that this map is injective. So let us consider $(\Delta_p, (\Box_{\iota(p)}, \alpha_{\iota(p)})) \in \ker(\Xi)$. It follows immediately, that $\Delta_p = 0$ and hence $\Box_{\iota(p)} = 0$. If $(0, \alpha_{\iota(p)}) \in \mathcal{L}$ then $\alpha_{\iota_p} \in \operatorname{Ann}(\operatorname{pr}_D L)$. Since $DI^* \alpha_{\iota(p)} = 0$, we have that $\alpha_{\iota(p)} \in \operatorname{Ann}(DL_N)$, hence $\alpha_{\iota(p)} = 0$ and the claim follows.

For dimensional reasons we have that Ξ is an isomorphism. XΞΣ

Proposition 2.2.5 *Let $L \to M$ be a line bundle, let $H \in \Omega^3_L(M)$ be closed, let $\mathcal{L} \subseteq \mathbb{D}L$ be an H-twisted Dirac-Jacobi structure and let $N \hookrightarrow M$ be a transversal. Then there exists $\varepsilon \in \Gamma^\infty(\mathcal{L})$, such that $\varepsilon\big|_N = 0$ and $\operatorname{pr}_D(\varepsilon)$ is Euler-like.*

Proof: The proof follows the same lines as [12]. We consider the exact sequence

$$0 \to \mathfrak{B}_I(\mathfrak{L}) \to \mathcal{L}\big|_N \to \nu_N \to 0,$$

where the first arrow is given by the identification $\mathfrak{B}_I(\mathcal{L}) \cong I^!\mathcal{L}$ from Lemma 2.2.4 followed by the canonical map $I^!\mathcal{L} \to \mathcal{L}$. The second arrow is the projection $\operatorname{pr}_D \colon \mathcal{L}\big|_N \to DL\big|_N$ followed by the symbol map $\sigma \colon DL\big|_N \to TM\big|_N$ and the projection to the normal bundle $\operatorname{pr}_{\nu_N} \colon TM\big|_N \to \nu_N$. Let us choose a section $\varepsilon \in \Gamma^\infty(\mathcal{L})$ with $\varepsilon\big|_N = 0$, such that $\mathrm{d}^N\varepsilon \colon \nu_N \to \mathcal{L}\big|_N$, which we always can do according to Remark 2.1.1. So ε defines a splitting of the sequence. We consider now the diagram

$$\begin{tikzcd}
0 \arrow[r] & I^!\mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}\big|_N \arrow[r] \arrow[d] & \nu_N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TN \arrow[r] & TM\big|_N \arrow[r] & \nu_N \arrow[r] & 0
\end{tikzcd}$$

and obtain that if $\mathrm{d}^N\varepsilon$ splits the upper sequence then $(\sigma \circ \operatorname{pr}_D)\,\mathrm{d}^N\varepsilon$ splits the lower sequence. Using Proposition 2.1.3, we see that $(\sigma \circ \operatorname{pr}_D)\,\mathrm{d}^N\varepsilon = \mathrm{d}^N((\sigma \circ \operatorname{pr}_D)(\varepsilon))$ and by Proposition 2.1.10, we see that $T\nu(\sigma \circ \operatorname{pr}_D)(\varepsilon) = \mathcal{E}$. Multiplying ε by a suitable bump function we may arrange that $(\sigma \circ \operatorname{pr}_D)(\varepsilon)$ is complete and hence an Euler-like vector field. By definition $\operatorname{pr}_D(\varepsilon)$ is hence an Euler-like derivation. XΞΣ

Let us fix now an H-twisted Dirac-Jacobi structure $\mathcal{L} \subseteq \mathbb{D}L$ on a line bundle $L \to M$. Additionally, we consider a transversal $\iota \colon N \hookrightarrow M$ and a section $\varepsilon = (\Delta, \alpha) \in \Gamma^\infty(\mathcal{L})$, such that $\varepsilon\big|_N = 0$ and Δ is an Euler-like derivation. Due to Lemma 2.1.16, we find a unique fat tubular neighborhood

$$\begin{tikzcd}
L_\nu \arrow[r, "\Psi"] \arrow[d] & L_U \arrow[d] \\
\nu_N \arrow[r, "\psi"] & U
\end{tikzcd}$$

such that $\Psi^*\Delta = \Delta_{\mathcal{E}}$. We have now two ways to construct a Dirac-Jacobi structure on $L_\nu \to \nu_N$. Namely we can take the backward transformation $\mathfrak{B}_\Psi(\mathcal{L}_U)$ and, if we consider the diagram

$$\begin{tikzcd} L_\nu \arrow[r, "P"] \arrow[d] & L_N \arrow[r, "I"] \arrow[d] & L \arrow[d] \\ \nu_N \arrow[r] & N \arrow[r] & M \end{tikzcd},$$

we can take the backward transformation $\mathfrak{B}_{I\circ P}(\mathcal{L}) = \mathfrak{B}_P(\mathfrak{B}_I(\mathcal{L}))$, note that this is a Dirac-Jacobi bundle, since (1) $\mathfrak{B}_I(\mathcal{L}$ is a Driac-Jacobi bundle by Corollary 1.2.18, because N is a transversal and (2) and thus $\mathfrak{B}_P(\mathfrak{B}_I(\mathcal{L}))$ is a Dirac-Jacobi bundle, because $P\colon L_\nu \to L_N$ is covering a surjective submersion, so P is in particular transversal. Our aim is to compare $\mathfrak{B}_{I\circ P}(\mathcal{L})$ and $\mathfrak{B}_\Psi(\mathcal{L}_U)$. Let us consider the flow of the derivation $[\![(\Delta,\alpha), -]\!]_H$ of $\mathbb{D}L$, which is given by

$$(\gamma_t, \Phi_t^\Delta) \in Z_L^2(M) \rtimes \mathsf{Aut}(L),$$

where Φ_t^Δ is the flow of Δ and $\gamma_t = \int_0^t (\Phi_{-\tau}^\Delta)^*(\mathrm{d}_L\alpha + \iota_\Delta H)\,\mathrm{d}\tau$ like in 1.2.9. Of course, the action of $(\gamma_t, \Phi_t^\Delta)$ preserves $\mathcal{L}$: explicitly

$$\exp(\gamma_t) \circ \mathbb{D}\Phi_t^\Delta(\mathcal{L}) = \mathcal{L}.$$

This leads us to the following

Theorem 2.2.6 (Normal form for Dirac-Jacobi bundles) *Let $L \to M$ be a line bundle, let $H \in \Omega_L^3(M)$ be closed, let $\mathcal{L} \subseteq \mathbb{D}L$ be a H-twisted Dirac-Jacobi structure and let $N \hookrightarrow M$ be a transversal. Then there exists an open neighborhood $U \subseteq M$ of N and fat tubular neighborhood $\Psi\colon L_\nu \to L_U$, such that*

$$\mathfrak{B}_\Psi(\mathcal{L}\big|_U) = (\mathfrak{B}_{I\circ P}(\mathcal{L}))^\omega$$

for an $\omega \in \Omega_{L_\nu}^2(\nu_N)$, such that $\mathrm{d}_L\omega = \Psi^*H - (I\circ P)^*H$.

Proof: According to Proposition 2.2.5, we can find $(\Delta,\alpha) \in \Gamma^\infty(\mathcal{L})$, such that Δ is Euler-like. Then there is a unique fat tubular neighborhood $\Psi\colon L_\nu \to L_U$, such that $\Psi^*\Delta = \Delta_\mathcal{E}$, due to Lemma 2.1.16. Let us denote by $(\gamma_t, \Phi_t^\Delta) \in Z_L^2(M) \rtimes \mathsf{Aut}(L)$ the flow of $[\![(\Delta,\alpha), -]\!]_H$. We know that $(\gamma_t, \Phi_t^\Delta)$ preserves $\mathcal{L}$ for all $t \in \mathbb{R}$ and so will $(\gamma_{-\log(s)}, \Phi_{-\log(s)}^\Delta)$ for all $s > 0$. Let us take a closer look at

$$\begin{aligned} \gamma_{-\log(s)} &= \int_0^{-\log(s)} (\Phi_{-\tau}^\Delta)^*(\mathrm{d}_L\alpha + \iota_\Delta H)\,\mathrm{d}\tau \\ &= \int_{-\log(1)}^{-\log(s)} (\Phi_{-\tau}^\Delta)^*(\mathrm{d}_L\alpha + \iota_\Delta H)\,\mathrm{d}\tau \\ &= \int_s^1 \frac{1}{t}(\Phi_{\log(t)}^\Delta)^*(\mathrm{d}_L\alpha + \iota_\Delta H)\,\mathrm{d}t \end{aligned}$$

which is is smoothly extendable to $s = 0$, since $(\Delta,\alpha)\big|_N = 0$. Let us denote by ω' the limit $s \to 0$ and put $\omega = \Psi^*\omega'$. We have, using the defining equations 2.1.3 and 2.1.4 and the relation 2.1.5,

$$\mathfrak{B}_\Psi(\mathcal{L}\big|_U) = \mathfrak{B}_\Psi(\exp(\gamma_{-\log(s)}) \circ \mathbb{D}\Phi_{-\log(s)}^\Delta(\mathcal{L}))$$

$$
\begin{aligned}
&= \mathfrak{B}_\Psi(\exp(\gamma_{-\log(s)})\mathfrak{B}_{\Phi^\Delta_{\log(s)}}(\mathcal{L})) \\
&= (\mathfrak{B}_\Psi(\mathfrak{B}_{\Lambda_s}(\mathcal{L}))^{\Psi^*\gamma_{-\log(s)}} \\
&= (\mathfrak{B}_{\Lambda_s\circ\Psi}(\mathcal{L}))^{\Psi^*\gamma_{-\log(s)}} \\
&= (\mathfrak{B}_{\Psi\circ P_s}(\mathcal{L}))^{\Psi^*\gamma_{-\log(s)}}.
\end{aligned}
$$

which holds for all $s \geq 0$. For $s = 0$ we have, using that for the canonical inclusion $J\colon L_N \to L_\nu$ we have that $P_0 = J \circ P$ and $\Psi \circ J = I$, that

$$
\mathfrak{B}_\Psi(\mathcal{L}\big|_U) = (\mathfrak{B}_{I\circ P}(\mathcal{L}))^\omega. \qquad \text{ΧΞΣ}
$$

Recall that there are two kinds of leaves in Dirac-Jacobi geometry, see 1.2.2, so there are also two kinds of transversals, which are even more interesting in the Jacobi setting. In the Dirac-Jacobi setting the differences between these two kinds of transversals are not very significant, nevertheless we discuss them here.

Definition 2.2.7 (Cosymplectic Transversal) *Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq \mathbb{D}L$ be a Dirac-Jacobi structure. A transversal $\iota\colon N \hookrightarrow M$ is called cosymplectic, if*

$$
DL_N \cap \mathfrak{B}_I(\mathcal{L}) = \{0\}.
$$

Remark 2.2.8 The terminology *cosymplectic* comes originally from Poisson geometry, namely a transversal N to a symplectic leaf of a Poisson structure is called cosymplectic, since the associated normal bundle is a symplectic vector bundle, i.e. a vector bundle $E \to N$ with a non-degenerate 2-form $\omega \in \Gamma^\infty(\Lambda^2 E)$. In Dirac geometry, the normal bundle of the transversal carries only a pre-symplectic vector bundle, but we prefer not to give them a special name. In the literature *cosymplectic* manifolds are usually defined differently, see e.g. [13], but throughout this thesis a cosymplectic transversal is always in the sense of Definition 2.2.7.

Remark 2.2.9 Note that a cosymplectic transversal always inherts a Dirac-Jacobi bundle coming from a Jacobi tensor by Proposition 1.2.35. So let us denote $\mathcal{L}_{J_N} = \mathfrak{B}_I(\mathcal{L}) \subseteq \mathbb{D}L_N$.

Cosymplectic transversals naturally appear as minimal transversal to locally conformal pre-symplectic leaves:

Lemma 2.2.10 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi bundle and let $p_0 \in M$ be a locally conformal pre-symplectic point. Then every minimal transversal at p_0 is a cosymplectic transversal in a neighborhood of p_0.*

Proof: Let N be a minimal transversal, then we have by definition that

$$
T_pN \oplus \sigma(\mathrm{pr}_D\mathcal{L}) = T_pM.
$$

Now let us assume that $\Delta \in (DL_N \cap \mathfrak{B}_I(\mathcal{L}))\big|_{p_0}$. So in particular, we have that

$$\Delta \in (DL_N \cap \mathrm{pr}_D \mathfrak{B}_I(\mathcal{L}))\big|_{p_0}$$

and hence $\sigma(\Delta) \in T_pN \cap \sigma(\mathrm{pr}_D\mathcal{L}) = \{0\}$. Hence $\Delta = k\mathbb{1}$, but since p_0 is a locally conformal presymplectic point we have that $\mathbb{1} \notin \mathrm{pr}_D(\mathcal{L})$ and thus $k = 0$. This means that $(DL_N \cap \mathfrak{B}_I(\mathcal{L}))\big|_{p_0} = \{0\}$, which has to hold in an open neighborhood of p_0.XΞΣ

Corollary 2.2.11 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi bundle and let $\iota\colon N \hookrightarrow M$ be a minimal transversal to $\mathcal{L}$ at a locally conformal pre-symplectic point p_0. Assume moreover that $\nu_N \cong V \times N$ is trivial. Then locally around p_0 there is a trivialization of L_ν and a fat tubular neighborhood $\Psi\colon L_\nu \to L_U$, such that:*

$$\mathfrak{B}_\Psi(\mathcal{L}\big|_U) = \left(\mathcal{L}_{J_N} \oplus TV\right)^\omega$$

where J_N is the Jacobi structure on the transversal. where we see

$$\mathcal{L}_{J_N} \subseteq \mathbb{D}L_\nu$$

via the canonical identifications $DL_\nu = TV \oplus DL_N$ and $J^1L_\nu = T^*V \oplus J^1L_N$.

The other kind of leaves of a Dirac-Jacobi structure are the so-called pre-contact leaves. Their minimal transversal possess the following structure :

Definition 2.2.12 (Cocontact Transversal) *Let $L \to M$ be a line bundle and let $\mathcal{L} \in \mathbb{D}L$ be a Dirac-Jacobi structure. A transversal $\iota\colon N \hookrightarrow M$ is called cocontact, if*

$$\mathrm{rank}(DL_N \cap \mathfrak{B}_I(\mathcal{L})) = 1.$$

Lemma 2.2.13 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}L$ be a Dirac-Jacobi structure and let $\iota\colon N \hookrightarrow M$ be a minimal transversal to $\mathcal{L}$ at a pre-contact point p_0. Then N is minimal transversal in a neighborhood of p_0.*

Proof: Recall that a minimal transversal at p_0 is a transversal of minimal dimension, which in particular implies that

$$\sigma(\mathrm{pr}_D(\mathcal{L}))\big|_{p_0} \oplus T_{p_0}N = T_{p_0}M.$$

It is easy to see that

$$(DL_N \cap \mathfrak{B}_I(\mathcal{L}))\big|_{p_0} = \langle \mathbb{1}_{p_0} \rangle,$$

which follows because N is minimal and p_0 is a pre-contact point, i.e. $\mathbb{1}_{p_0} \in \mathrm{pr}_D\mathcal{L}$. To be more precise, by using the pre-contact property of p_0 and the minimality of N, we see $(\mathrm{pr}_D\mathcal{L} \cap DL_N)\big|_{p_0} = \langle \mathbb{1}_{p_0} \rangle$ and hence there is $\alpha \in J^1_{p_0}L$, such that $(\mathbb{1}_{p_0}, \alpha) \in \mathcal{L}$. Let us define $\beta \in J^1_{p_0}L$ by

$$\beta(\Delta) = 0 \text{ for } \Delta \in \mathrm{pr}_D\mathcal{L}$$

and

$$\beta(\Delta) = \alpha(\Delta) \text{ for } \Delta \in DL_N.$$

Then β is well-defined, since $\mathrm{pr}_D\mathcal{L} \cap DL_N = \mathbb{1}$ and $\alpha(\mathbb{1}) = 0$ and moreover $(0, \beta) \in \mathcal{L}$, since $\langle\!\langle (0,\beta), \mathcal{L} \rangle\!\rangle = 0$ and $\mathcal{L}$ is maximal isotropic. We consider now the element $(\mathbb{1}_{p_0}, \alpha - \beta) \in \mathcal{L}$, thus $(\mathbb{1}_{p_0}, DI^*(\alpha - \beta)) = (\mathbb{1}_{p_0}, 0) \in \mathfrak{B}_I(\mathcal{L})$. Moreover, since $\mathrm{pr}_D\mathcal{L} \cap DL_N = \mathbb{1}$, we conclude $DL_N \cap \mathfrak{B}_I(\mathcal{L}) = \langle \mathbb{1}_{p_0} \rangle$ and hence $\mathrm{rank}(DL_N \cap \mathfrak{B}_I(\mathcal{L}))\big|_{p_0} = 1$.

Now we want to argue why this holds in a whole neighborhood. Let us therefore consider the sum $DL_N + \mathfrak{B}_I(\mathcal{L}) \subseteq \mathbb{D}L$ and a (local) section $\alpha \in \Omega^1_L(M)$ such that $\alpha(\mathbb{1})\big|_{p_0} \neq 0$. Let $(0, \beta) \in \big(DL_N + \mathfrak{B}_I(\mathcal{L})\big)\big|_{p_0} \cap \langle \alpha \rangle\big|_{p_0}$, then there exists $\Delta \in D_{p_0}L$ such that $(\Delta, \beta) \in \mathfrak{B}_I(\mathcal{L})$, but since $(\mathbb{1}, 0) \in \mathfrak{B}_I(\mathcal{L})$, we have using the isotropy of $\mathfrak{B}_I(\mathcal{L})$,

$$0 = \langle\!\langle (\Delta, \beta), (\mathbb{1}, 0) \rangle\!\rangle = \beta(\mathbb{1}),$$

but $\beta = k\alpha$ for $k \in \mathbb{R}$, we conclude $k = 0$ and thus $\beta = 0$ and therefore $\big(DL_N + \mathfrak{B}_I(\mathcal{L})\big)\big|_{p_0} \cap \langle \alpha \rangle\big|_{p_0} = \{0\}$. For dimensional reasons we conclude $\mathbb{D}L\big|_{p_0} = (DL_N + \mathfrak{B}_I(\mathcal{L}))\big|_{p_0} \oplus \langle \alpha \rangle\big|_{p_0}$. Therefore this equality holds in a whole neighborhood of p_0, so $\mathrm{rank}(DL_N + \mathfrak{B}_I(\mathcal{L})) = 2n + 1$ in this neighborhood, which implies $\mathrm{rank}(DL_N \cap \mathfrak{B}_I(\mathcal{L})) = 1$ around p_0. ΧΞΣ

Definition 2.2.14 *Let $L \to M$ be a line bundle and let $\mathcal{L} \in \mathbb{D}L$ be a Dirac-Jacobi structure. A homogeneous cocontact transversal $\iota \colon N \hookrightarrow M$ is a cocontact transversal together with a flat connection $\nabla \colon TN \to DL_N$, such that*

$$\mathrm{im}(\nabla) \oplus (DL_N \cap \mathfrak{B}_I(\mathcal{L})) = DL_N.$$

Remark 2.2.15 The definition of a homogeneous cocontact transversal seems a bit strange, since it includes a connection. This fact can be explained quite easily using the homogenezation described in [46], which turns a Dirac-Jacobi structure on a line bundle $L \to M$ into a Dirac structure on $\tilde{L} := L^* \backslash \{0_M\}$ which is homogeneous (in the sense of [41]) with respect to the restricted Euler vector field $\mathcal{E}$ on L^*. The pre-symplectic leaves of this Dirac structure have the additional property that $\mathcal{E}$ is either tangential to it or transversal. If $\mathcal{E}$ is tangential, then the leaf corresponds to a pre-contact leaf on the base M. Hence a minimal transversal N to it is transversal to the Euler vector field and defines therefore a horizontal bundle on $L^*_{\mathrm{pr}(N)}$ and hence a connection.

One may wonder what kind of (not so classical) geometric structure a homogeneous cocontact transversal inherits. The answer is given by the following

Lemma 2.2.16 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi bundle and let $\nabla \colon TM \to DL$ be a flat connection, such that*

$$\mathrm{im}(\nabla) \oplus (DL \cap \mathcal{L}) = DL.$$

Then there exists a locally conformal Poisson structure $\pi \in \Gamma^\infty(\Lambda^2(TM \otimes L^*) \otimes L)$ and a vector field $Z \in \Gamma^\infty(TM)$, such that *i.)* $[\![\pi, Z]\!]_{(TM,L)} = \pi$ *(*homogeneous *locally conformal Poisson)*

ii.) $\mathcal{L} = \mathcal{L}_{(\pi,Z)} := \{\big(h(\mathbb{1}-\nabla_Z)+\nabla_{\pi^\sharp\alpha}, \sigma^*\alpha+\mathbb{1}^*\otimes\alpha(Z)\big) \in \mathbb{D}L \mid \alpha \in T^*M\otimes L,\ h \in \mathbb{R}\}$

where we denote $\mathbb{1}^*$ *is the unique element in* DL^* *such that* $\mathbb{1}^*\big|_{\mathrm{im}(\nabla)} = 0$ *and* $\mathbb{1}^*(\mathbb{1}) = 1$*.*

Proof: It is an easy computation that $\mathcal{L}$ has to be of the above form. The fact that π and Z fulfill condition *i.*) as well as that π is locally conformal Poisson follow from the involutivity on $\mathcal{L}$. ΧΞΣ

Corollary 2.2.17 *Let* $L \to M$ *be a line bundle, let* $\mathcal{L} \subseteq \mathbb{D}L$ *be a Dirac-Jacobi structure and let* $\iota\colon N \hookrightarrow M$ *be a minimal transversal to* $\mathcal{L}$ *at a pre-contact point* p_0*. Then every flat connection* ∇ *gives* N *locally the structure of a homogeneous cocontact transversal.*

Proof: In the proof of Lemma 2.2.13, we have seen that

$$(DL_N \cap \mathfrak{B}_I(\mathcal{L}))\big|_{p_0} = \langle \mathbb{1} \rangle$$

and hence for every flat connection ∇, we have that $\mathrm{im}(\nabla)\big|_{p_0} \oplus (DL_N \cap \mathfrak{B}_I(\mathcal{L}))\big|_{p_0} = DL_N$ and hence this decomposition holds in a whole neighborhood of p_0. ΧΞΣ

Remark 2.2.18 Note that this implies that every minimal transversal to a pre-contact leaf has an induced homogeneous locally conformal Poisson structure. Moreover, it is easy to show that

Corollary 2.2.19 *Let* $L \to M$ *be a line bundle, let* $\mathcal{L} \subseteq \mathbb{D}L$ *be a Dirac-Jacobi structure and let* $\iota\colon N \hookrightarrow M$ *be a minimal transversal to a contact point* p_0*. If* $\nu_N \cong V \times N$*, then there exists a local trivialization of* L_ν *and a fat tubular neighborhood* $\Psi\colon L_\nu \to L_U$ *such that,*

$$\mathfrak{B}_\Psi(\mathcal{L}\big|_U) = \big(\mathcal{L}_{(\pi_N,Z_N)} \oplus TV\big)^\omega,$$

where (π_N, Z_N) *is the homogeneous Poisson structure on the transversal from Lemma 2.2.16. Here we see*

$$\mathcal{L}_{(\pi_N,Z_N)} \subseteq \mathbb{D}L_\nu$$

via the canonical identifications $DL_{\nu_N} = TV \oplus DL_N$ *and* $J^1L_\nu = T^*V \oplus J^1L_N$*.*

Corollaries 2.2.11 and 2.2.19 can be seen as the Jacobi-geometric analogue of the results obtained by Blohmann in [8].

2.3 Normal Forms and Splitting Theorems of Jacobi Bundles

As explained in Example 1.2.35, Jacobi bundles are a special kind of Dirac-Jacobi bundles. In addition, we have that Jacobi isomorphism induces an isomorphism of the corresponding Dirac structures (this holds even for morphisms if one considers forward maps of Dirac-Jacobi structures which we will not explain here, see [46]).But the converse is not true: if the Dirac-Jacobi structures of two Jacobi structures are isomorphic, it does not follow in general that the Jacobi structures are isomorphic. What is not "allowed" in Jacobi geometry are B-field transformations. Nevertheless, we can keep track of them, if we make further assumptions on the transversals.

2.3.1 Cosymplectic Transversals

In this section, we use the notion of cosymplectic transversals as explained in the previous section.

Lemma 2.3.1 *Let $L \to M$ be a line bundle, $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi tensor with corresponding Dirac-Jacobi structure $\mathcal{L}_J \in \mathbb{D}L$ and let $\iota\colon N \hookrightarrow M$ be a cosymplectic transversal. Then*

$$J^\sharp(\operatorname{Ann}(DL_N)) \oplus DL_N = DL\big|_N.$$

PROOF: First we prove that $J^\sharp\big|_{\operatorname{Ann}(DL_N)}$ is injective. So let $\alpha \in \operatorname{Ann}(DL_N)$, such that $J^\sharp(\alpha) = 0$. Then for an arbitrary $\beta \in J^1L$, we have that

$$\alpha(J^\sharp(\beta)) = -\beta(J^\sharp(\alpha)) = 0.$$

Hence $\alpha = \operatorname{Ann}(DL_N) \cap \operatorname{Ann}(\operatorname{im}(J^\sharp)) = \operatorname{Ann}(DL_N + \operatorname{im}(J^\sharp)) = \{0\}$, and $J^\sharp\big|_{\operatorname{Ann}(DL_N)}$ is injective. Let $\Delta \in DL_N \cap J^\sharp(\operatorname{Ann}(DL_N))$, then there exists an $\alpha \in \operatorname{Ann}(DL_N)$, such that $J^\sharp(\alpha) = \Delta$. Thus we have that $(\Delta, \alpha) \in \mathcal{L}_J$ and moreover $(\Delta, DI^*\alpha) \in \mathfrak{B}_I(\mathcal{L}_J)$. But, since $\alpha \in \operatorname{Ann}(DL_N)$, we have that $DI^*\alpha = 0$ and hence $\Delta = 0$, since N is cosymplectic. The claim follows by counting dimensions. ΧΞΣ

Let us from now on fix a Jacobi bundle $(L \to M, \{-,-\})$ with Jacobi tensor J and corresponding Dirac-Jacobi structure $\mathcal{L}_J$. Suppose that $\iota\colon N \hookrightarrow M$ is a cosymplectic transversal, then we have that

$$\operatorname{pr}_\nu \circ \sigma \circ J^\sharp\colon \operatorname{Ann}(DL_N) \to \nu_N$$

is an isomorphism. Let us choose $\alpha \in \Gamma^\infty(J^1L)$, such that $\alpha\big|_N = 0$ and such that $\mathrm{d}^N\alpha\colon \nu_N \to \operatorname{Ann}(DL_N) \subseteq J^1L\big|_N$ is a right-inverse to $\operatorname{pr}_\nu \circ \sigma \circ J^\sharp$. Note that this does always exist due to Lemma 2.1.1. Then we have

$$\operatorname{pr}_\nu(\mathrm{d}^N\sigma(J^\sharp(\alpha))) = \operatorname{pr}_\nu(\sigma(J^\sharp(\mathrm{d}^N\alpha))) = \mathsf{id}_{\nu_N}$$

and hence $T\nu(\sigma(J^\sharp(\alpha))) = \mathcal{E}$. Multiplying α by a bump-function which is 1 near N, we may arrange that $\sigma(J^\sharp(\alpha))$ is complete and hence $J^\sharp(\alpha)$ is an Euler-like derivation. By Theorem 2.2.6, we have that

$$\mathfrak{B}_\Psi(\mathcal{L}_J) = \mathfrak{B}_P(\mathcal{L}_{J_N})^\omega,$$

where $\omega = \Psi^* \int_0^1 \frac{1}{t}(\Phi^{J^\sharp(\alpha)}_{\log(t)}))^*(\mathrm{d}_L\alpha)\,\mathrm{d}t$ and $\Psi\colon L_\nu \to L_U$ is the unique tubular neighborhood, such that $\Psi^*(J^\sharp(\alpha)) = \Delta_{\mathcal{E}}$.

Proposition 2.3.2 *The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ restricted to N has kernel DL_N.*

Proof: One can show, in local coordinates, that $\mathrm{d}^N\alpha([\sigma(\Box)\big|_N]_{TN}) = (\mathscr{L}_\Box\alpha)\big|_N$ for all $\Box \in \Gamma^\infty(DL)$. Hence we have trivially $\mathscr{L}_\Delta\alpha\big|_N = 0$ for $\Delta \in \Gamma^\infty(DL)$, such that $\Delta\big|_N \in \Gamma^\infty(DL_N)$. Let now $\Delta, \Box \in \Gamma^\infty(DL)$, such that $\Delta\big|_N \in \Gamma^\infty(DL_N)$, then

$$\begin{aligned}
\mathrm{d}_L\alpha(\Delta,\Box)\big|_N &= -(\mathrm{d}_L\iota_\Delta\alpha)(\Box)\big|_N = -\Box(\alpha(\Delta))\big|_N \\
&= -(\mathscr{L}_\Box\alpha)(\Delta)\big|_N - \alpha([\Box,\Delta])\big|_N = -(\mathscr{L}_\Box\alpha)\big|_N(\Delta) \\
&= \mathrm{d}^N\alpha([\sigma(\Box)\big|_N])(\Delta) \\
&= 0,
\end{aligned}$$

where the last equality follows since $\mathrm{d}^N\alpha$ takes values in $\mathrm{Ann}(DL_N)$. Hence we have that $\ker((\mathrm{d}_L\alpha)^\flat) \supseteq DL_N$, in particular this is true for $\frac{1}{t}(\Phi^\Delta_{\log(t)})^*(\mathrm{d}_L\alpha)$, since $\Phi_{\log(s)}\big|_N$ is a gauge transformation fixing DL_N. Thus it is true also for ω, since $D\Psi\big|_{DL_N} = \mathsf{id}$. Equality follows from the fact that $\mathrm{d}^N\alpha$ is choosen to be injective, since it has a left-inverse. ΧΞΣ

We want to describe the structure of ω at N. Note that for a cosymplectic transversal N, the normal bundle ν_N always comes together with a canonical symplectic (i.e. non-degenerate) L_N-valued 2-form $\Theta \in \Gamma^\infty(\Lambda^2\nu_N^* \otimes L_N)$ defined by

$$\Theta(X,Y) = (\mathrm{pr}_\nu \circ \sigma \circ J^\sharp\big|_{\mathrm{Ann}(DL_N)})^{-1}(X)(Y)$$

Lemma 2.3.3 *The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ coincides, shrinked to $\nu_N \subseteq DL_{\nu_N}$, with Θ.*

Proof: Note that for a cosymplectic transversal, we have

$$DL\big|_N = DL_N \oplus J^\sharp(\mathrm{Ann}(DL_N)) = DL_N \oplus \nu_N$$

Where we used the canonical identification

$$J^\sharp(\mathrm{Ann}(DL_N)) = \frac{DL\big|_N}{DL_N} = \nu_N.$$

Moreover, we have

$$DL_\nu\big|_N = DL_N \oplus \nu_N,$$

where we include ν_N by the following map:

$$\cdot^{\mathrm{ver}}\colon \nu_N \ni v_n \to \left(\lambda \to \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_0 P_t^*(\lambda)(v_p)\right) \in D_n L_\nu$$

for $P_t\colon L_\nu \ni (v_n, l_n) \mapsto (tv_n, l_n) \in L_\nu$. It is clear that $D\Psi$ fixes DL_N, since $\Psi\big|_N\colon L_N \to L_N$ is identity. We want to show that $D\Psi(\nu_N) \subseteq J^\sharp(\mathrm{Ann}(DL_N))$. One can show that by an elementary calculation, that

$$D\Psi(v_n^{\mathrm{ver}}) = \lim_{t\to 0} \frac{\Delta_{\lambda_t(\psi(v_n))}}{t}$$

using Equation 2.1.5. But by defintion, we have that

$$\mathrm{d}^N\Delta(v_n) = \lim_{t\to 0} \frac{\Delta_{\lambda_t(\psi(v_n))}}{t}$$

hence $D\Psi \circ (\cdot)^{\mathrm{ver}} = \mathrm{d}^N\Delta = J^\sharp \circ \mathrm{d}^N\alpha$, but α was chosen in such a way that $\mathrm{d}^N\alpha$ takes values in $\mathrm{Ann}(DL_N)$. Thus $D\Psi\big|_N(\Delta_n, v_n) = (Delta_n, J^\sharp(\mathrm{d}^N\alpha(v_n))$ and thus respects the splittings $DL_v\big|_N = DL_N \oplus \nu_N$ and $DL\big|_N = DL_N \oplus J^\sharp(\mathrm{Ann}(DL_N))$. Using

$$\mathfrak{B}_\Psi(\mathcal{L}_J) = \mathfrak{B}_P(\mathcal{L}_{J_N})^\omega,$$

$\ker(\omega^\flat)\big|_N = DL_N$ and the definition of Θ, we see that along N ω and $Theta$ have to coincide. XΞΣ

This leads us to the first normal form theorem for Jacobi manifolds.

Theorem 2.3.4 (Normal Form for Jacobi Bundles I) *Let $(L \to M, J)$ be a Jacobi bundle, let $N \hookrightarrow M$ be a cosymplectic transversal. For every closed $\omega \in \Omega^2_{L_\nu}(\nu_N)$, such that $\ker(\omega^\flat)\big|_N = DL_N$ and ω coincides with Θ at $\nu_N \subseteq DL_\nu$, the following holds:*

i.) $\mathfrak{B}_P(\mathcal{L}_{J_N})^\omega$ *is the graph of a Jacobi structure near N.*

ii.) there exists a fat tubular neighborhood $\Psi\colon L_\nu \to L_U$ which is a Jacobi map near the zero section.

Proof: We have proven this theorem for the special ω given by

$$\omega = \int_0^1 \frac{1}{t} (\Phi^{J^\sharp(\alpha)}_{\log(t)})^* \,\mathrm{d}_L\alpha \,\mathrm{d}t.$$

Let ω' be a second Atiyah 2-form fulfilling the requirements of the theorem, then

$$\sigma_t := t(\omega' - \omega)$$

is a (time-dependent) Atiyah 2-form such that $\sigma_0 = 0$ and moreover $\sigma_t\big|_N = 0$. Thus,

$$(\mathfrak{B}_P(\mathcal{L}_{J_N})^\omega)^{\sigma_t} = \mathfrak{B}_P(\mathcal{L}_{J_N})^{\omega+\sigma_t}$$

is a Jacobi structure near N. Now use Appendix A.1 to get the result. XΞΣ

An immediaty consequence of Theorem 2.3.4 is the Splitting for Jacobi manifolds around a locally conformal symplectic leaf, proven by Dazord, Lichnerowicz and Marle in [17].

Theorem 2.3.5 *Let $L \to M$ be a line bundle, let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi tensor and let $p_0 \in M$ be a locally conformal symplectic point such that the leaf through p_0 has dimension $2k$. Then there are a line bundle trivialization $L_U \cong \mathbb{R}_U$ around p_0 and a cosymplectic transversal $N \hookrightarrow U$, such that $U \cong U_{2k} \times N$ for an open subset $0 \in U_{2k} \subseteq \mathbb{R}^{2k}$, such that corresponding Jacobi pair (Λ, E) is transformed (via this trivialization) to*

$$(\Lambda, E) = (\pi_{\mathrm{can}} + \Lambda_N + E_N \wedge Z_{\mathrm{can}}, E_N).$$

(Λ_N, E_N) is the induced Jacobi pair on the transversal N and the canonical stuctures on the U_{2k} are given by $(\pi_{\mathrm{can}}, Z_{\mathrm{can}}) = (\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, p_i \frac{\partial}{\partial p_i})$.

Proof: As the statement is local, we can assume that the line bundle is trivial. Let us choose an arbitrary minimal transversal N at p_0, such that $\nu_N \cong \mathbb{R}^{2k} \times N$ and

$$(DL_N \cap \mathfrak{B}_I(\mathcal{L}_J)) = \{0\}$$

holds in an open neighborhood of p_0. Using Corollary 2.2.11, we find

$$\mathfrak{B}_\Psi(\mathcal{L}\big|_U) = \big(\mathcal{L}_{J_N} \oplus T\mathbb{R}^{2k}\big)^\omega.$$

Since the line bundle and the normal bundle are trivial, we can identify $\Gamma^\infty(\Lambda^2\nu^* \otimes L_\nu)$ by $\Gamma^\infty(\mathbb{R}^{2k}_N)$. And thus Θ is a symplectic structure on the vector bundle $\mathbb{R}^{2k}_N \to N$ and hence we can find a Darboux frame $\{e_i, f^j\}_{i,j=1,\dots,q} \subseteq \Gamma^\infty(\mathbb{R}^{2k}_N)$, i.e.

$$\Theta = \sum_{i=1}^q e^i \wedge f_i$$

for the dual basis $\{e^i, f_j\}_{i,j=1,\dots,2k}$. We define

$$\omega_{\mathrm{can}} = \mathrm{d}q^i \wedge \mathrm{d}p_i + \mathbb{1}^* \wedge p_i \,\mathrm{d}q^i \in \Omega_{\mathbb{R}_U}(U),$$

where the q^i's (resp. p_i's) are the canonical coordinates on $R^{2k} \times N$ induced by the e_i's (resp. f_i's). By definition, we have that $\ker(\omega^\flat_{\mathrm{can}})\big|_N = DL_N$, ω coincides with Θ at $\nu_N \subseteq DL_\nu$ and ω_{can} is closed. Using Theorem 2.3.4, we have that

$$\big(\mathcal{L}_{J_N} \oplus T\mathbb{R}^{2k}\big)^{\omega_{\mathrm{can}}} \cong \mathcal{L}_J$$

near N. An easy computation shows that the Jacobi structure, inducing the Dirac-Jacobi structure on the right, is exactly the one from the theorem. ΧΞΣ

2.3.2 Cocontact Transversals

The second kind of transversals we want to discuss in the context of Jacobi geometry are cocontact transversals, which were also introduced before in Definition 2.2.12. In fact this notion is not enough for our purposes and we need to assume more information on the structure of the transversal, which is precisely the notion of homogeneous cocontact transversal from Definition 2.2.12.

Lemma 2.3.6 *Let $L \to M$ be a line bundle, $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi tensor with corresponding Dirac- Jacobi structure $\mathcal{L}_J \in \mathbb{D}L$ and let $\iota\colon N \hookrightarrow M$ be a homogeneous cocontact transversal with connection $\nabla\colon TN \to DL_N$. Then*

$$J^\sharp(\operatorname{Ann}(\operatorname{im}(\nabla))) \oplus \operatorname{im}(\nabla) = DL\big|_N.$$

Moreover, $J^\sharp\big|_{\operatorname{Ann}(\operatorname{im}(\nabla))}\colon \operatorname{Ann}(\operatorname{im}(\nabla)) \to DL\big|_N$ is injective.

PROOF: The proof follows the same lines as that of Lemma 2.3.1. ΧΞΣ

Now we pick as in the cosymplectic case, an $\alpha \in \Gamma^\infty(J^1L)$, such that $\alpha\big|_N = 0$ and

$$\mathrm{d}^N\alpha\colon \nu_N \to \operatorname{Ann}(\operatorname{im}(\nabla)) \subseteq J^1L\big|_N$$

defines a splitting of $I^!\mathcal{L} \to \mathcal{L}\big|_N \to \nu_N$, i.e. $\mathrm{pr}_\nu \circ \sigma \circ J^\sharp \circ \mathrm{d}^N\alpha = \mathsf{id}_{\nu_N}$. Hence we have that $J^\sharp(\alpha)$, multiplied by a suitable bump function which is 1 close to N, is an Euler-like derivation. By Theorem 2.2.6, we have that

$$\mathfrak{B}_\Psi(\mathcal{L}_J) = \mathfrak{B}_P(\mathfrak{B}_I(\mathcal{L}))^\omega,$$

where $\omega = \Psi^* \int_0^1 \frac{1}{t}(\Phi^{J^\sharp(\alpha)}_{\log(t)})^*(\mathrm{d}_L\alpha)\,\mathrm{d}t$ and $\Psi\colon L_\nu \to L_U$ is the unique tubular neighborhood, such that $\Psi^*(J^\sharp(\alpha)) = \Delta_{\mathcal{E}}$. We can prove, as before, the following

Proposition 2.3.7 *The Atiyah 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ restricted to N has kernel* $\operatorname{im}(\nabla)$.

PROOF: This proof follows the same lines as the proof of Proposition 2.3.2. ΧΞΣ

As in the cosymplectic transversal case, we can define a skew symmetric 2-form

$$\Theta \in \Gamma^\infty(\Lambda^2 J^\sharp(\operatorname{Ann}(\operatorname{im}(\nabla)) \otimes L_N)$$

by

$$\Theta(X, Y) = (J^\sharp\big|_{\operatorname{Ann}(\operatorname{im}(\nabla))})^{-1}(X)(Y)$$

since $J^\sharp\big|_{\operatorname{Ann}(\operatorname{im}(\nabla))}\colon \operatorname{Ann}(\operatorname{im}(\nabla)) \to J^\sharp(\operatorname{Ann}(\operatorname{im}(\nabla))$ is a bijection. It is easy to see that Θ is non-degenerate. Moreover, we have

Lemma 2.3.8 *The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$, restricted to $\nu_N \oplus K \subseteq DL_{\nu_N}$, concides with Θ, where we denote $K := (DL_N \cap \mathfrak{B}_I(\mathcal{L}_J))$.*

PROOF: Using the ideas of the proof of Lemma 2.3.3, we can show that the fat tubular neighborhood transports $J^\sharp(\mathrm{Ann}(\mathrm{im}(\nabla))$ to $\nu_N \oplus K$, hence the proof is an easy adaption of the proof of Lemma 2.3.3. $\mathrm{X}\Xi\Sigma$

Theorem 2.3.9 (Normal Form for Jacobi bundles II) *Let $L \to M$ be a line bundle, let J be a Jacobi structure and let $N \to M$ be a homogeneous cocontact transversal with connection $\nabla\colon TN \to DL_N$. For every closed 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$, such that $\ker(\omega^\flat)\big|_N = \mathrm{im}(\nabla)$ and ω coincides with Θ at $\nu_N \oplus (\mathfrak{B}_I(\mathcal{L}_J) \cap DL_N) \subseteq DL_\nu$ the following holds*

i.) $\mathfrak{B}_P(\mathcal{L}_{J_N})^\omega$ is the graph of a Jacobi structure near N.

ii.) there exists a fat tubular neighborhood $\Psi\colon L_\nu \to L_U$ which is a Jacobi map near the zero section.

PROOF: The proof follows the lines of Theorem 2.3.4 with the obvious adaptions. $\mathrm{X}\Xi\Sigma$

Now we want to prove the second splitting Theorem of Dazord and Lichnerowicz and Marle in [17], namely the splitting of Jacobi manifolds around contact leaves.

Theorem 2.3.10 *Let $L \to M$ be a line bundle, let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi tensor and let $p_0 \in M$ be a contact point, such that the leaf through p_0 has rank $2k+1$. Then there are a line bundle trivialization $L_U \cong U \times \mathbb{R}$ around p_0 and a homogeneous cocontact transversal $N \hookrightarrow U$, such that $U \cong U_{2k+1} \times N$ for an open subset $0 \in U_{2k+1} \subseteq \mathbb{R}^{2k+1}$ and the corresponding Jacobi pair (Λ, E) is transformed (via this trivialization) to*

$$(\Lambda, E) = (\Lambda_{\mathrm{can}} + \pi_N + E_{\mathrm{can}} \wedge Z_N, E_{\mathrm{can}}),$$

where (π_N, Z_N) is the induced homogeneous Poisson structure on the transversal N and the contact structure on the U_{2k+1} is given by $(\Lambda_{\mathrm{can}}, E_{\mathrm{can}}) = ((\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial u}) \wedge \frac{\partial}{\partial p_i}, \frac{\partial}{\partial u})$.

PROOF: Let $p_0 \in M$ be a contact point and let $N \subseteq M$ be a transversal, such that

$$\sigma(\mathrm{im}\, J^\sharp)\big|_{p_0} \oplus T_{p_0}N = T_{p_0}M.$$

We can again assume that the line bundle $L \to M$ is trivial, since we want to prove a local statement. In a possibly smaller neighborhood, we can assume that $\nu_N = V \times N \to N$. We want to show that there is a trivialization of ν_N, such that Θ looks trivial, where we specialize along the way through the proof what we mean by trivial. So let us denote by λ the local trivializing section of L_N, thus we can write

$$\Theta(\Delta, \Box) = \Omega(\Delta, \Box) \cdot \lambda$$

for $\Delta, \Box \in \nu_N \oplus K$, for a unique $\Omega\Gamma^\infty(\Lambda^2(\nu_N \oplus K)^*)$. Since $L_N \to N$ is trivial, we identify $DL_N = TN \oplus \mathbb{R}_N$ and choose the trivial connection ∇. Hence, we can find a (local) nowhere vanishing section of K of the form $\mathbb{1}-Z$ for a unique $Z \in Secinfty(TN)$. Let us now shrink

$$\Theta\big|_{\nu_N} : \nu_N \times \nu_N \to L_N.$$

Since ν_N has odd dimensional rank and Θ is skew-symmetric, locally we can find a local non-vanishing $X \in \Gamma^\infty(\nu_N)$, such that $\Theta(X, \cdot) = 0$, moreover, since Θ is non-degenerate, we can choose X so that

$$\Omega(\mathbb{1} - Z, X) = 1.$$

It is now easy to see that the symplectic complement $S := \langle \mathbb{1} - Z, X\rangle^{\perp_\omega} \subseteq \nu_N$. Finally, we find a trivialization of S such that $\Omega\big|_S$ is the trivial symplectic form with Darboux frame $\{e_2, e_{k+2}, \dots\}$. Hence, by extending this trivialization to $\nu_N = V \times N$ by using the section X as b, we find that $\{b, \mathbb{1} - Z, e_1, f^1, e_2, f^2, \dots\}$ is a Darboux frame of Ω in this trivialization. Using the dual basis $\{b^*, e^i, f_j\}$ to the Darboux frame we get (linear) funtions on ν_N denoted by (u, q^i, p_j). With the decomposition $DL_\nu = TV \oplus TN \oplus \mathbb{R}_{\nu_N}$ we can choose

$$\omega = -\big(\mathrm{d}q^i \wedge \mathrm{d}p_i + \mathbb{1}^* \wedge (\mathrm{d}u - p_i\, \mathrm{d}q^i)\big)$$

which coincides with Θ on $\nu_N \oplus K$ and is d_L-closed. By applying Theorem 2.3.9, since N together with ∇ is a homogeneous cocontact transversal, we find a Jacobi morphism

$$\mathfrak{B}_P(\mathcal{L}_N)^\omega \cong \mathcal{L}_J.$$

An easy computation shows that $\mathfrak{B}_P(\mathcal{L}_N)^\omega$ is the graph of $(\Lambda_{\mathrm{can}} + \pi_N + E_{\mathrm{can}} \wedge Z_N, E_{\mathrm{can}})$. ΧΞΣ

2.4 Generalized Contact Bundles

The last two sections gave us the methods to attack the local structure of generalized contact bundles. But as for Jacobi structures and Dirac-Jacobi bundles, we need to discuss transversals of generalized contact structures.

Definition 2.4.1 *Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ be a generalized contact structure. A submanifold $N \subseteq M$ is called transversal, if it is a transversal of the corresponding Jacobi tensor.*

As in the Jacobi setting, this isnot enough to ensure a reasonable structure induced on the transversal, but nevertheless we have

Lemma 2.4.2 *Let $(L \to M, \mathcal{L})$ be a generalized contact bundle and let $\iota\colon N \subseteq M$ be a transversal. Then*

$$\mathfrak{B}_I(\mathcal{L}) \subseteq \mathbb{D}_{\mathbb{C}}L_N$$

is a (complex) Dirac-Jacobi bundle.

Let us distinguish the same cases as in the previous sections.

2.4.1 Cosymplectic Transversals

Lemma 2.4.3 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ be a generalized contact structure and let submanifold $\iota\colon N \hookrightarrow M$ be a cosymplectic transversal to the corresponding Jacobi tensor J. Then*

$$\mathfrak{B}_I(\mathcal{L}) \subseteq \mathbb{D}_{\mathbb{C}}L_N$$

is a generalized contact structure. Moreover, its Jacobi tensor agrees with the Jacobi tensor induced by J.

Proof: Recall from Lemma 2.3.1 that a cosymplectic transversal to a Jacobi structure always fulfills

$$J^\sharp(\mathrm{Ann}(DL_N)) \oplus DL_N = DL\big|_N$$

and $J^\sharp\big|_{\mathrm{Ann}(DL_N)}\colon \mathrm{Ann}(DL_N) \to DL_N$ is injective. Let $(\Delta, \psi) \in \mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}})$ be real. Then there exists $\chi \in J^1_{\mathbb{C}}L$, such that $(\Delta, \chi) \in \mathcal{L}$. If we denote by ϕ, J, α the components of the endomorphism $\mathbb{K}$ inducing $\mathcal{L}$,

$$\begin{aligned}(J^\sharp(\mathsf{Im}\chi), -\phi^*\mathsf{Im}\chi) = \mathbb{K}(0, \mathsf{Im}\chi) &= \frac{1}{2\mathrm{i}}\mathbb{K}((\Delta, \chi) - (\overline{\Delta}, \overline{\chi}))\\ &= \frac{1}{2\mathrm{i}}(\mathrm{i}(\Delta, \chi) + \mathrm{i}(\overline{\Delta}, \overline{\chi}))\\ &= (\Delta, \mathsf{Re}\chi)\end{aligned}$$

and we conclude $(\Delta, \chi) = (J^\sharp(\mathsf{Im}\chi), \mathsf{iIm}\chi - \phi^*\mathsf{Im}\chi)$. On the other hand we have that $\chi\big|_{DL_N} = \psi$ and hence it is real, which means that $\mathsf{Im}\chi \in \mathrm{Ann}(DL_N)$. Now we use that N is a cosymplectic transversal: $J^\sharp(\mathsf{Im}\chi) \in DL_N$ if and only if $\mathsf{Im}\chi = 0$. This means that $(\Delta, \chi) \in \mathcal{L}$ is real and hence has to be zero, which implies also $(\Delta, \psi) = 0$. We can also use the above computation to show that for $\psi \in J^1L$, such that $J^\sharp(\psi) \in DL_N$, we have $J^\sharp(\psi) = J^\sharp_N(\psi\big|_{DL_N})$, where we denote by J_N the Jacobi structure induced by $\mathfrak{B}_I(\mathcal{L})$. This property is also shared by the Jacobi tensor induced by J on N and determines it completely. ΧΞΣ

Now, we want to show that a generalized contact structure is uniquely determined by its backwards transform on a cosymplectic transversal, up to a B-field transformation. For the following Theorem, we use the notation of the previous section, more precisely Subsection 2.3.1.

Theorem 2.4.4 (Normal Forms for Generalized Contact Bundles I) *Let $(L \to M, \mathcal{L})$ be a generalized contact bundle and let $\iota\colon N \hookrightarrow M$ be a cosymplectic transversal. Then for every closed $\omega' \in \Omega^2_L(M)$, such that $\ker((\omega')^\flat)\big|_N = DL_N$ and ω' coincides with Θ at $\nu_N \subseteq DL_\nu$, the complex Dirac-Jacobi structure*

$$\mathfrak{B}_P(\mathcal{L}_N)^{\mathrm{i}\omega'}$$

is a generalized contact structure near the zero section and there exists a fat tubular neighborhood $\Psi\colon L_\nu \to L_U$ *which is an isomorphism of generalized contact bundles near the zero section (up to a B-field transformation).*

PROOF: We follow the lines of the proof of Theorem 2.3.4. Let us choose $\alpha \in \Omega^1_L(M)$, such that $J^\sharp(\alpha)$ is Euler-like (we denote by J the Jacobi structure of the generalized contact structure). Since $(J^\sharp(\alpha), \mathrm{i}\alpha - \phi^*\alpha) \in \Gamma^\infty(\mathcal{L})$, we can apply Theorem 2.2.6 to get that

$$\mathfrak{B}_\Psi(\mathcal{L}\big|_U) = (\mathfrak{B}_{I\circ P}(\mathcal{L}))^{\mathrm{i}\omega + B}$$

for the unique tubular neighborhood $\Psi\colon L_\nu \to L_U$, such that $\Psi^*(J^\sharp\alpha) = \Delta_{\mathcal{E}}$, where $\mathrm{i}\omega + B = \int_0^1 \frac{1}{t}(\Phi^{J^\sharp(\alpha)}_{\log(t)})^*(\mathrm{d}_L(\mathrm{i}\alpha - \Phi^*\alpha))\,\mathrm{d}t$. So the claim is proven for ω. Using Appendix A.1, we can argue like in Theorem 2.3.4 to get the claim. ΧΞΣ

Of course we can use this normal form theorem together with the normal form theorem for Jacobi structures 2.3.4 to give a local structure result for generalized contact bundles as well.

Theorem 2.4.5 (Splitting Theorem for Generalized Contact Bundles I)
Let $(L \to M, \mathcal{L})$ *be a generalized contact bundle and let* $p_0 \in M$ *be a locally conformal symplectic point to its Jacobi structure. Then there are a line bundle trivialization* $L_U \cong U \times \mathbb{R}$ *around* p_0 *and a minimal cosymplectic transversal* $N \hookrightarrow U$*, such that* $U \cong U_{2k} \times N$ *for an open subset* $0 \in U_{2k} \subseteq \mathbb{R}^{2k}$ *with coordinates* $\{q^1, p_1, \dots, q^k, p_k\}$*, such that* $\mathcal{L}\big|_U$ *is given by the endomorphism* $\mathbb{K}$ *on*

$$D\mathbb{R}_{U_{2k}\times N} \cong TU_{2k} \oplus \overbrace{TN \oplus \mathbb{R}_{U_{2k}\times N}}^{DL_N},$$

given by its entries

i.) $\phi = (\mathsf{id} - Z_{\mathrm{can}} \otimes \mathbb{1}^*) \circ \phi_N$

ii.) $J = \pi_{\mathrm{can}} + \Lambda_N + (\mathbb{1} - Z_{\mathrm{can}}) \wedge E_N$

iii.) $\alpha = -\,\mathrm{d}x^i \wedge \mathrm{d}p_i + \mathbb{1}^* \wedge p_i\,\mathrm{d}x^i + \alpha_N$

for the structures ϕ_N, $J_N = (\Lambda_N, E_N)$ *and* α_N *inducing the generalized contact bundle on* N *and* $(\pi_{\mathrm{can}}, Z_{\mathrm{can}}) = (\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, p_i \frac{\partial}{\partial p_i})$.

PROOF: Using the the Theorems 2.4.4 and 2.3.5 as well as Corollary 2.2.11, we can see that, up to a B-field,

$$\mathcal{L}\big|_U = \big(T_{\mathbb{C}}U_{2k} \oplus \mathcal{L}_N\big)^{\mathrm{i}\omega}$$

for $\omega = \mathrm{d}x^i \wedge \mathrm{d}p_i - \mathbb{1}^* \wedge p_i\,\mathrm{d}x^i$. The rest is just a computation using that $\big(T_{\mathbb{C}}U_{2k} \oplus \mathcal{L}_N\big)^{\mathrm{i}\omega}$ is the i-eigenbundle of $\mathbb{K}$. ΧΞΣ

The exact same result is proved in [41], using different notation and slightly different techniques.

As a final remark of this subsection, we want to mention that there is a refinement of Theorem 2.4.5, if the Jacobi structure is locally regular around a locally conformal symplectic point p, i.e. its foliation is locally regular around p. In this case the transversal can be described in a better way:

Lemma 2.4.6 *Let $(L \to M, \mathcal{L} \subseteq D_{\mathbb{C}}L)$ a generalized contact bundle and let $\iota\colon N \hookrightarrow M$ be a minimal transversal through the regular locally conformal symplectic point $p_0 \in M$. Then the induced generalized contact structure on N is (locally) a B-field transformation of a complex structure on DL_N.*

Proof: If the foliation induced by the Jacobi structure is regular around p_0 it is easy to see that the Jacobi structure on a minimal transversal is trivial and hence using Lemma 2.4.3, we conclude that $\mathfrak{B}_I(\mathcal{L})$ is induced by an endomorphism of the form

$$\mathbb{K} = \begin{pmatrix} \phi & 0 \\ \alpha^\flat & -\phi^* \end{pmatrix}$$

for a gauge complex structure $\phi \in \Gamma^\infty(\mathsf{End}(DL))$ and $\beta \in \Omega^2_L(M)$. From now on, we use the notation and results from Appendix A.3. Recall that $D_{\mathbb{C}}L_N = DL_N^{(1,0)} \oplus DL_N^{(0,1)}$, where $DL_N^{(1,0)}$ (resp. $DL_N^{(0,1)}$) is the $+\mathrm{i}$-Eigenbundle (resp. $-\mathrm{i}$) of ϕ. Define $\gamma \in \Omega^2_L(M)$ by

$$\gamma(\Delta, \Box) = \frac{1}{2\mathrm{i}}\alpha(\Delta, \Box) \text{ for } \Delta, \Box \in DL_N^{(1,0)} \text{ and } \gamma^\flat(DL_N^{(0,1)}) = 0.$$

Hence we have that $\gamma \in \Omega_L^{(2,0)}(M)$, furthermore, one can show that for $\Delta \in DL_N^{(1,0)}$, we have that $(\Delta, \iota_\Delta \gamma) \in \mathfrak{B}_I(\mathcal{L})$, which implies that $\partial_L \gamma = 0$. Shrinking to a small enough open neighborhood U, where the Atiyah-Dolbeault cohomology is trivial, we can find $\rho \in \Omega_L^{(1,0)}(M)$, such that $\gamma = \partial_L \rho$. Choosing the B-field

$$B = \mathsf{Re}(\gamma + \overline{\partial}_L \rho)$$

the claim is just a computation. ΧΞΣ

2.4.2 Cocontact Transversals

Lemma 2.4.7 *Let $(L \to M, \mathcal{L} \subseteq D_{\mathbb{C}}L)$ be a generalized contact structure and let $\iota\colon N \hookrightarrow M$ be a cocontact transversal for its Jacobi structure J. Then*

i.) $\operatorname{rank}(\mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}})) = 1$

ii.) $\operatorname{pr}_D \mathfrak{B}_I(\mathcal{L}) + \operatorname{pr}_D \mathfrak{B}_I(\overline{\mathcal{L}}) = D_{\mathbb{C}}L$

iii.) $\operatorname{pr}_D(\mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}})) = DL_N \cap \operatorname{pr}_D(\mathfrak{B}_I(\mathcal{L}_J))$.

Proof: Recall that a cocontact transversal for a Jacobi structure J is a transversal N such that

$$\operatorname{rank}(DL_N \cap \mathfrak{B}_I(\mathcal{L}_J)) = 1.$$

Let us choose $0 \neq (\Delta, 0) \in DL_N \cap \mathfrak{B}_I(\mathcal{L}_J)$, this means we can find $\alpha \in \Gamma^\infty(J^1L)$, such that $(J^\sharp(\alpha), \alpha) = (\Delta, \alpha) \in \mathcal{L}_J$, such that $DI^*\alpha = 0$. Moreover, we have

$$(J^\sharp(\alpha), \mathrm{i}\alpha - \phi^*\alpha) \in \mathcal{L}$$

where $\mathcal{L}$ is realized as the $+\mathrm{i}$-eigenbundle of

$$\begin{pmatrix} \phi & J^\sharp \\ \beta^\flat & -\phi^* \end{pmatrix}.$$

In conclusion, $(J^\sharp(\alpha), -DI^*\phi^*\alpha) \in \mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}})$, since it is real. In the same way, one can show that $\mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}})$ has rank 1 and hence every element has to be of that form. Point *ii.)* now follows by counting dimensions.

Let us prove that $\mathrm{pr}_D \colon \mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}}) \to D_\mathbb{C}L$ is injective. Let $(0, \alpha) \in \mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}})$ be real, then there exist $(0, \beta) \in \mathcal{L}$, such that $DI^*\beta = \alpha$. This further means that $\beta \in \operatorname{Ann}(\mathrm{pr}_D(\mathcal{L})) \subseteq \operatorname{Ann}(\mathrm{pr}_D(\mathcal{L}) \cap \mathrm{pr}_D(\overline{\mathcal{L}})) = \operatorname{Ann}(\operatorname{im}(J^\sharp)_\mathbb{C})$. But this implies that β is real, because N is a transversal of J, i.e. $D_\mathbb{C}L_N + \operatorname{im} J^\sharp{}_\mathbb{C} = D_\mathbb{C}L$ and $DI^*\beta = \alpha$. Every real element in $\mathcal{L}$ vanishes identically and thus $\beta = 0$. Let $(\Delta, \alpha) \in \mathrm{pr}_D(\mathfrak{B}_I(\mathcal{L}) \cap \mathfrak{B}_I(\overline{\mathcal{L}}))$, then by *i.)* there exists $\beta \in \operatorname{Ann}(DL_N)$, such that $(J^\sharp(\beta), \mathrm{i}\beta - \phi^*\beta) \in \mathcal{L})$ with $J^\sharp(\beta) = \Delta$. So we conclude $\Delta \in DL_N \cap \operatorname{im}(J^\sharp)$. ΧΞΣ

In Section 2.3, we discussed homogeneous cocontact trasnversals, i.e. cocontact transversals with a special connection (see 2.2.14) in order to obtain a homogeneous locally conformal Poisson structure on the transversal. In the case of generalized contact structures we obtain something very similar.

Lemma 2.4.8 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}_\mathbb{C}L$ be a complex Dirac-Jacobi structure and let $\nabla \colon TM \to DL$ be a flat connection, such that*

i.) $\mathrm{pr}_D\mathcal{L} + \mathrm{pr}_D\overline{\mathcal{L}} = D_\mathbb{C}L$

ii.) $\operatorname{rank}(\mathcal{L} \cap \overline{\mathcal{L}}) = 1$

iii.) $\operatorname{im}(\nabla)_\mathbb{C} \oplus \mathrm{pr}_D(\mathcal{L} \cap \overline{\mathcal{L}}) = D_\mathbb{C}L$

Then there exists a locally conformal generalized complex structure $\mathcal{D} \subseteq \mathbb{T}^L_\mathbb{C}M$ and a section $(Z, \zeta) \in \Gamma^\infty(\mathbb{T}^LM)$, such that

i.) $(X, \alpha) \in \Gamma^\infty(\mathcal{D}) \implies ([Z, X], \alpha - \mathscr{L}^\nabla_X \alpha + \iota_X \mathrm{d}^\nabla \zeta) \in \Gamma^\infty(D)$

ii.) $\mathcal{L} = \{(\nabla_X, \sigma^*\alpha + \mathbb{1}^* \otimes (\alpha(Z) - \zeta(X))) \in \mathbb{T}^L_\mathbb{C}M \mid (X, \alpha) \in \mathcal{D}\} \oplus$
$\langle(\mathbb{1} - \nabla_Z, \sigma^*\zeta + \mathbb{1}^* \otimes \zeta(Z))\rangle_\mathbb{C}$

PROOF: It is an easy verification that $\mathcal{L}$ looks like in *ii.*) for a subbundle $\mathcal{D} \subset \mathbb{T}^L_{\mathbb{C}}M$. The rest follows by the involutivity of $\mathcal{L}$. ΧΞΣ

Corollary 2.4.9 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq D_{\mathbb{C}}L$ be a generalized contact structure and let $\iota\colon N \hookrightarrow M$ be a homogeneous cocontact transversal with flat connection $\nabla\colon TN \to DL_N$ for the induced Jacobi structure J. Then there exists a locally conformal generalized complex structure $\mathcal{D} \subseteq \mathbb{T}^{L_N}_{\mathbb{C}}N$ and a section $(Z,\zeta) \in \Gamma^\infty(\mathbb{T}^{L_N}N)$, such that*

i.) $(X,\alpha) \in \Gamma^\infty(\mathcal{D}) \implies ([Z,X], \alpha - \mathscr{L}^\nabla_X \alpha + \iota_X \mathrm{d}^\nabla \zeta) \in \Gamma^\infty(\mathcal{D})$

ii.) $\mathcal{L} = \{(\nabla_X, \sigma^*\alpha + \mathbb{1}^* \otimes (\alpha(Z) - \zeta(X))) \in \mathbb{T}^{L_N}_{\mathbb{C}}N \mid (X,\alpha) \in \mathcal{D}\} \oplus \langle(\mathbb{1} - \nabla_Z, \sigma^*\zeta + \mathbb{1}^* \otimes \zeta(Z))\rangle_{\mathbb{C}}$.

Moreover, the homogeneous locally conformal Poisson structure induced by $\mathcal{D}$ coincides with the one induced by J (as in Definition 2.2.14 and Lemma 2.2.16).

PROOF: The existence of $\mathcal{D}$ and the form of $\mathcal{L}$ is an easy consequence of Lemmas 2.4.7 and 2.4.8. And the fact that the induced homogeneous locally conformal Poisson structures coincide follows the same lines as Lemma 2.4.3. Note that condition *i.*) implies that the induced locally conformal Poisson structure is indeed homogeneous with respect to Z. ΧΞΣ

We can again collect the results of all the consideration in Jacobi and related geometries to get, using the notation of Section 2.3,

Theorem 2.4.10 (Normal Forms for Generalized Contact Bundles II)
Let $(L \to M, \mathcal{L})$ be a generalized contact bundle and let $N \to M$ be a homogeneous cocontact transversal with connection $\nabla\colon TN \to DL_N$. For every closed Atiyah 2-form $\omega' \in \Omega^2_{L_\nu}(\nu_N)$, such that $\ker((\omega')^\flat)\big|_N = \mathrm{im}(\nabla)$ and ω' coincides with Θ at $\nu_N \oplus (\mathfrak{B}_I(\mathcal{L}_J) \cap DL_N) \subseteq DL_\nu$,

$$\mathfrak{B}_P(\mathcal{L}_N)^{\mathrm{i}\omega'}$$

is a generalized contact structure near the zero section and there exists a fat tubular neighborhood $\Psi\colon L_\nu \to L_U$, which is an isomorphism of generalized contact bundles near the zero section (up to a B-field transformation).

PROOF: The proof follows the same lines of Theorem 2.4.4 and the techniques of Theorem 2.3.9 and by using Lemma 2.4.7. ΧΞΣ

As a last step of this chapter, we give now a local splitting for generalized contact structures, which is just a corollary of the previous considerations.

Theorem 2.4.11 (Splitting Theorem for Generalized Contact Bundles II)
Let $(L \to M, \mathcal{L})$ be a generalized contact structure and let $p_0 \in M$ be a contact point. Then there are a line bundle trivialization $L_U \cong U \times \mathbb{R}$ around p_0 and a minimal

homogeneous cocontact transversal $N \hookrightarrow U$*, such that* $U \cong U_{2k} \times N$ *for an open subset* $0 \in U_{2k+1} \subseteq \mathbb{R}^{2k+1}$*. Additionally there are coordinates* $\{u, q^1, p_1, \ldots, q^k, p_k\}$*, such that* $\mathcal{L}\big|_U$ *is given by the endomorphism* $\mathbb{K}$ *on*

$$D\mathbb{R}_{U_{2k+1}\times N} \cong TU_{2k} \oplus \overbrace{TN \oplus \mathbb{R}_{U_{2k+1}\times N}}^{DL_N},$$

given by its entries

i.) $\phi = \phi_N + \big(\phi_N(Z_N) - \pi_N^\sharp(\xi_N) + \xi_N(Z_N)\frac{\partial}{\partial u}\big) \otimes \mathbb{1}^* + \frac{\partial}{\partial u} \otimes \xi_N$

ii.) $J = \Lambda_{\mathrm{can}} + \pi_N + (\mathbb{1} - Z_N) \wedge E_{\mathrm{can}}$

iii.) $\alpha = \Omega_{\mathrm{can}} + \alpha_N - \mathbb{1}^* \wedge (\iota_{Z_N}\alpha_N - \phi_N^*\xi_N)$

for the structures ϕ_N*,* π_N *and* α_N *inducing the generalized complex structure on* N *together with the section* (Z_N, ξ_N) *and* $(\Lambda_{\mathrm{can}}, E_{\mathrm{can}}) = ((\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial u}) \wedge \frac{\partial}{\partial p_i}, \frac{\partial}{\partial u})$ *and* $\Omega_{\mathrm{can}} = \mathrm{d}q^i \wedge \mathrm{d}p_i + \mathbb{1} \wedge (\mathrm{d}u - p_i\,\mathrm{d}q^i)$.

As a result this is contained already in [41], but obtained, again, by slightly different techniques. The last part is dedicated to a special case of the Theorems 2.4.11 and 2.4.10. As in the previous subsection, we want to discuss the structure of a transversal to a regular contact point.

Lemma 2.4.12 *Let* $L \to M$ *be a line bundle,let* $\mathcal{L} \subseteq D_{\mathbb{C}}L$ *be a generalized contact structure and let* $p_0 \in M$ *be a regular contact point. For a minimal homogeneous cocontact transversal* $\iota\colon N \hookrightarrow M$ *at* p*, the locally conformal generalized complex structure on* N *is a B-field transformation of a complex structure.*

Proof: It is easy to see that, if the Jacobi structure is regular around p, then the induced homogeneous locally conformal Poisson structure induced by it is vanishing and also the homogeneity vector field Z. So the endomorphism $\mathbb{I}$ realizing the locally conformal generalized complex structure is of the form

$$\mathbb{I} = \begin{pmatrix} \phi & 0 \\ \beta^\flat & -\phi^* \end{pmatrix}$$

for a complex structure $\phi \in \Gamma^\infty(\mathsf{End}(TM))$ and $\beta \in \Gamma^\infty(\Lambda^2 T^*M \otimes L)$. Not that, due to Corollary 2.4.9, we have that there is a $\zeta \in \Gamma^\infty(T^*M \otimes L)$, such that

$$(X, \alpha) \in \Gamma^\infty(\mathcal{D}) \implies (0, \alpha + \iota_X\,\mathrm{d}^\nabla\zeta).$$

Note that this implies, shown by an easy computation, that $\beta = \iota_\phi\,\mathrm{d}^\nabla\zeta$ for

$$\iota_\phi\,\mathrm{d}^\nabla\zeta(X, Y) = \mathrm{d}^\nabla\zeta(\phi(X), Y) + \mathrm{d}^\nabla\zeta(X, \phi(Y)).$$

Now we apply the B-field $-\mathrm{d}^\nabla\zeta$ to the endomorphism and see that

$$\exp(-\mathrm{d}^\nabla\zeta)\mathbb{I}\exp(\mathrm{d}^\nabla\zeta) = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix}.$$

ΧΞΣ

Thus is the case of a regular foliation also the transversal inherts an extreme case as a structure and by Theorem 2.4.11, the generalized contact structure looks locally like a "product" of a contact structure and a complex structure.

Remark 2.4.13 Theorems 2.4.5 and 2.4.11 can be seen as the generalized contact bundle analogue of the Abouzaid-Boyarchenko splitting theorem for generalized complex manifolds (see [1]). Note that, eventhough we are obtaining the analogue of their result, we are not using their techniques. Instead, as mentioned before, we are using the techniques of [12].

2.5 Application: Splitting Theorem for homogeneous Poisson Structures

Using the homogenezation scheme from [10], see also Appendix A.2, one can see that Jacobi bundles are nothing else but special kinds of homogeneous Poisson manifolds. Moreover, the two most important examples of Poisson manifolds are of this kind: the cotangent bundle and the dual of a Lie algebra. Using this insight, it is easy to see that proving something for Jacobi structures gives a proof for something in homogeneous Poisson Geometry. We want to apply this philosophy to give a splitting theorem for homogeneous Poisson manifolds. The first appearance of such a theorem was [17, Theorem 5.5] in order to prove the local splitting of Jacobi pairs. Here we want to attack the problem from the other side: we use the splitting of Jacobi manifolds to prove the splitting of homogeneous Poisson structures. Recall that a homogeneous Poisson structure is a pair $(\pi, Z) \in \Gamma^\infty(\Lambda^2 TM \oplus TM)$ for some manifolds, such that π is a Poisson tensor and

$$\mathscr{L}_Z \pi = -\pi.$$

Note that the leaves of π have the property, that Z is either tangential or transversal to a whole leaf.

Theorem 2.5.1 *Let (π, Z) be a homogeneous Poisson structure on a manifold M and let $p_0 \in M$ be a point such that $Z_{p_0} \neq 0$ and $\operatorname{rank}(\pi) = 2k$. Then there exist an open neighborhood U of p_0, an open neighborhood U_{2k} of $0 \in \mathbb{R}^{2k}$, a manifold N with a homogeneous Poisson structure (π_N, Z_N) with $\pi_N\big|_{p_0} = 0$ and a diffeomorphism $\psi\colon U \to U_{2k} \times N$, such that*

$$\psi_* \pi = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \pi_N.$$

Additionally,

i.) if $Z_{p_0} \in \operatorname{im}(\pi^\sharp)$, then $\psi_* Z = p_i \frac{\partial}{\partial p_i} + \frac{\partial}{\partial p_k} + Z_N$.

ii.) if $Z_{p_0} \notin \operatorname{im}(\pi^\sharp)$, then $\psi_* Z = p_i \frac{\partial}{\partial p_i} + Z_N$.

PROOF: Note that since $Z_{p_0} \neq 0$, we find coordinates $\{u, x^1, \ldots, x^q\}$ with $p_0 = (1, 0, \ldots, 0)$, such that $Z = u\frac{\partial}{\partial u}$. In this chart, we have, using $\mathscr{L}_Z \pi = -\pi$,

$$\pi = \frac{1}{u}(\Lambda + u\frac{\partial}{\partial u} \wedge E)$$

for unique $\Lambda \in \Gamma^\infty(\Lambda^2 TM)$ and $E \in \Gamma^\infty(TM)$ which do not depend on u. It is easy to see, that we have

$$[\Lambda, \Lambda] = -E \wedge \Lambda \text{ and } \mathscr{L}_E \Lambda = 0,$$

which means that (Λ, E) is a Jacobi pair. This allows us to use Theorem 2.3.5 and Theorem 2.3.10 to prove the result. We will do it just for the case where p_0 is a contact point, which means, translated to Jacobi pairs, that E_{p_0} is transversal to $\operatorname{im}(\Lambda^\sharp)\big|_{p_0}$ and thus $Z_{p_0} \in \operatorname{im}(\pi^\sharp)$, since the other case is very similar. Note that, we can apply Theorem 2.3.10: there exists coordinates $\{x, q^i, p_i, y^j\}$ and a local non-vanishing function $a \in \mathscr{C}^\infty(M)$ (which is basically the line bundle trivialization), such that

$$\Lambda = \frac{1}{a}(\Lambda_{\mathrm{can}} + \pi_N + E_{\mathrm{can}} \wedge Z_N) \text{ and } E = \frac{1}{a}(E_{\mathrm{can}} + \Lambda^\sharp(\mathrm{d}a)),$$

where $(\Lambda_{\mathrm{can}}, E_{\mathrm{can}}) = ((\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial u}) \wedge \frac{\partial}{\partial p_i}, \frac{\partial}{\partial u})$ and (ϕ_N, Z_N) is a homogeneous Poisson structure just depending on y^j-coordinates.

Applying the diffeomorphism $(u, x^1, \ldots, x^q) \mapsto (a \cdot u, x^1, \ldots, x^q)$, we have

$$\pi = \frac{1}{u}(\Lambda_{\mathrm{can}} + \pi_N + E_{\mathrm{can}} \wedge Z_N + u\frac{\partial}{\partial u} \wedge E_{\mathrm{can}}).$$

A (quite) long and not very insightful computation shows that the diffeomorphism

$$\Phi(u, x^1, \ldots, x^q) = (u, \Phi^{Z_N}_{\log(u)}(\Phi^{E_{\mathrm{can}}}_{-log(u)}(x^1, \ldots, x^q))),$$

where $\Phi^{Z_N}_t$ (resp. $\Phi^{E_{\mathrm{can}}}_t$) is the flow uf Z_N (resp. E_{can}), gives us

$$\pi = \frac{1}{u}(\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}) + \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} + \pi_N \text{ and } Z = u\frac{\partial}{\partial u} + p_i\frac{\partial}{\partial p_i} + Z_N.$$

Renaming coordinates of π we get the result. ΧΞΣ

This Application shows us that, eventhough we can see Poisson structures as Jacobi manifolds, which suggests that they are more general objects than Poisson structures, the splitting theorems (of Jacobi pairs) are a refinement of the known splitting theorems for Poisson structures.

Chapter 3

Dual Pairs in Dirac-Jacobi Geometry

The concept of dual pairs in Poisson Geometry was introduced in [51] and is deeply connected to the concepts of symplectic realizations, symplectic groupoids [51] and Morita equivalence [52]. Recently, dual pairs have also been considered in Jacobi geometry [6]. The aim of this chapter is to introduce the Dirac-Jacobi analogue of them and discuss some properties. The main part of this chapter is the proof of the existence of self-dual pairs and an alternative proof of the normal form theorems from Chapter 2. This whole chapter uses techniques from [21], where one can find the mirror results in Dirac geometry. Throughout this chapter we use the notation

$$\mathcal{L}^{\mathrm{opp}} := \{(\Delta, -\psi) \in \mathbb{D}L \mid (\Delta, \psi) \in \mathcal{L}\}$$

for the opposite Dirac-Jacobi structure of $\mathcal{L} \subseteq \mathbb{D}L$.

3.1 Dual Pairs and weak Dual Pairs

Let us begin this section by simply giving the definition of a weak dual pair.

Definition 3.1.1 *A weak dual pair is a triplet of Dirac-Jacobi bundles* $(L_0 \to M_0, \mathcal{L}_0)$, $(L_1 \to M_1, \mathcal{L}_1)$ *and* $(L \to M, \mathcal{L})$ *together with forward Dirac-Jacobi maps* S, T *covering surjective submersions*

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}}),$$

such that $\mathcal{L} = \mathcal{L}_\omega$ *for a closed Atiyah 2-form* $\omega \in \Omega^2_L(M)$. *Additionally:*

i.) $\omega(\ker DS, \ker DT) = 0$

ii.) $\operatorname{rank}(\ker DS \cap \ker DT \cap \ker \omega^\flat) = \dim(M) - \dim(M_0) - \dim(M_1) - 1$

If $\dim(M) = \dim(M_0) + \dim(M_1) + 1$*, then we say that*

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$$

is a dual pair. In both cases we call $(L_0 \to M_0, \mathcal{L}_0)$ *and* $(L_1 \to M_1, \mathcal{L}_1)$ *the* legs *of the dual pair.*

Remark 3.1.2 If we consider a dual pair, such that both of the legs are Jacobi structures, it is easy to see that the Atiyah 2-form has to be non-degenerate and hence has to be a contact 2-form. In this case we talk about contact dual pairs, see [6] for more details.

A first consequence of the definition of contact dual pairs is

Corollary 3.1.3 *Let*

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$$

be a weak dual pair. Then $\ker DS \cap \ker DT \cap \ker \omega^\flat$ *is a smooth subbundle.*

Proof: Define the map

$$K\colon \ker DS \oplus \ker DT \ni (\Delta, \Box) \mapsto (\Delta + \Box, \iota_\Box \omega) \in \ker DS + \ker DT^\omega \subseteq \mathbb{D}L.$$

Note that we wrote $\ker DT^\omega := \exp(\omega) \ker DT$, even though $\ker DT^\omega$ is not a Dirac-Jacobi bundle. Because of the rank condition, we have

$$\begin{aligned}
\operatorname{rank}(\ker DS + \ker DT^\omega) &= \operatorname{rank}(\ker DS) + \operatorname{rank}(\ker DT) - \operatorname{rank}(\ker DS \cap \ker DT^\omega) \\
&= \operatorname{rank}(\ker DS) + \operatorname{rank}(\ker DT) \\
&\quad - \operatorname{rank}(\ker DS \cap \ker DT \cap \ker \omega^\flat) \\
&= const.
\end{aligned}$$

and hence the target of K is constant. Moreover we can interpret

$$\ker DS \cap \ker DT \cap \ker \omega^\flat \cong \ker K.$$ ΧΞΣ

Note that not every two Dirac-Jacobi bundles admit a (weak) dual pair between them, later on we will find consequences of the the existence of a weak dual pair between two Dirac-Jacobi bundles which makes it easy to construct examples of Dirac-Jacobi bundles which cannot appear as the legs of a weak dual pair. Nevertheless, we want show that two Dirac-Jacobi bundles which are graphs of Atiyah 2-forms always admit a dual pair, and hence a weak dual pair, between them.

Corollary 3.1.4 *Let* $(L_i \to M_i, \mathcal{L}_{\omega_i})$ *be Dirac-Jacobi bundles for* $i = 1, 2$ *coming from two closed Atiyah 2-forms* $\omega_i \in \Omega^2_{L_i}(M_i)$*. Then the product of* $(L_1 \to M_1, \mathcal{L}_{\omega_1})$ *and* $(L_2 \to M_2, \mathcal{L}_{-\omega_2})$ *in the sense of Section 1.3 is a dual pair.*

Let us now continue examining the structures of weak dual pairs and give some equivalent descriptions. We start with a general statement about Dirac-Jacobi structures and maps covering surjective submersions.

Lemma 3.1.5 *Let $(L \to M, \mathcal{L}_\omega)$ be a Dirac bundle induced by a closed 2-form $\omega \in \Omega^2_L(M)$ and let $P\colon L \to L_0$ be a regular line bundle covering a surjective submersion $p\colon M \to M_0$. Then*

$$\mathfrak{B}_P(\mathfrak{F}_P(\mathcal{L})) = \ker DP + (\ker(DP)^{\perp_\omega})^\omega$$

and it is smooth if and only if $\mathfrak{F}_P(\mathcal{L})$ is smooth.

Proof: The proof of the first part of the statement is an easy computation and can be found in the Dirac case in [21]. The smoothness equivalence follows from the fact that P covers a surjective submersion and Theorem 1.2.17. ΧΞΣ

The next statement is a tool in order to prove certain properties of (weak) dual pairs. The exact same statement, with some obvious replacements, can be found in [21, Prop. 6].

Lemma 3.1.6 *Let $(L \to M, \mathcal{L}_\omega)$, $(L_i \to M_i, \mathcal{L}_i)$ be Dirac-Jacobi bundles for $i = 0, 1$ and $\omega \in \Omega^2_L(M)$ closed, together with regular line bundle morphisms $S\colon L \to L_0$ and $T\colon L \to L_1$, then the following statements are equivalent:*

i.) $(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$ *is a weak dual pair.*

ii.) $\mathfrak{B}_S(\mathcal{L}_0) = \mathfrak{B}_T(\mathcal{L}_1)^\omega$ *and*

$$\operatorname{rank}(\ker DS \cap \ker DT \cap \ker \omega^\flat) = \dim(M) - \dim(M_0) - \dim(M_1) - 1$$

iii.) $S \times^! T\colon (L \to M, \mathcal{L}) \to (L^\times \to M^\times, \mathcal{L}_0 \times^! \mathcal{L}_1^{\mathrm{opp}})$ *is a forward Dirac-Jacobi map and one of the following properties hold:*

- $\omega(\ker DS, \ker DT) = 0$
- $\operatorname{rank}(\ker DS \cap \ker DT \cap \ker \omega^\flat) = \dim(M) - \dim(M_0) - \dim(M_1) - 1$

iv.) $\mathfrak{B}_S(\mathcal{L}_0) = \mathfrak{B}_T(\mathcal{L}_1)^\omega$ *and* $S \times^! T\colon (L \to M, \mathcal{L}) \to (L^\times \to M^\times, \mathcal{L}_0 \times^! \mathcal{L}_1^{\mathrm{opp}})$ *is a forward Dirac-Jacobi map*

Proof: Before we start we want to mention, that $\operatorname{rank}(\ker DS \cap \ker DT \cap \ker \omega^\flat) = \operatorname{rank}(\ker DS \cap \ker DT^\omega)$ by an easy computation. In the following we will use this observation usually without further comment. Let us start with *i.*) $\implies$ *ii.*). Since S is a forward Dirac map, we have

$$\begin{aligned} \mathfrak{B}_S(\mathcal{L}_0) &= \mathfrak{B}_S(\mathfrak{F}_S(\mathcal{L}_\omega)) \\ &= \ker DS + (\ker DS^{\perp_\omega})^\omega \end{aligned}$$

by Lemma 3.1.5. Since $\ker DT \subseteq \ker DS^{\perp_\omega}$, we conclude that

$$\mathfrak{B}_S(\mathcal{L}_0) \supseteq \ker DS + (\ker DT)^\omega.$$

Using that $\ker DS \cap \ker DT \cap \ker \omega^\flat = \ker DS \cap (\ker DT)^\omega$, we deduce the equality by the condition on the rank $\ker DS \cap \ker DT \cap \ker \omega^\flat = \dim M - \dim M_1 - \dim M_2 - 1$ of a dual pair. Since T is also a forward map one can show, using the same argument, that

$$\mathfrak{B}_T(\mathcal{L}_1) = \ker DT + (\ker DS)^{-\omega},$$

and hence the claim follows. Let us now assume *ii.)* and let $\Delta \in \ker DS$, then we have that $\Delta \in \mathfrak{B}_S(\mathcal{L}_0) = \mathfrak{B}_T(\mathcal{L}_1)^\omega$, so there is an element $\alpha \in J^1L_1$, such that $(\Delta, DT^*\alpha) \in \mathfrak{B}_T(\mathcal{L}_1)$ with $(\Delta, DT^*\alpha + \iota_\Delta\omega) = (\Delta, 0)$. Therefore, we have that $\iota_\Delta\omega = -DT^*\alpha \in \operatorname{Ann}(\ker DT)$ and hence $\omega(\ker DS, \ker DT) = 0$. Note that having $\mathfrak{B}_S(\mathcal{L}_0) = \mathfrak{B}_T(\mathcal{L}_1)^\omega$, implies

$$\ker(DS) + \ker DT^\omega \subseteq \ker DS + (\ker(DS)^{\perp_\omega})^\omega = \mathfrak{B}_S(\mathfrak{F}_S(\mathcal{L}_\omega)) \subseteq \mathfrak{B}_S(\mathcal{L}_0)$$

and by the rank condition $\ker DS \cap \ker DT \cap \ker \omega^\flat = \dim M - \dim M_1 - \dim M_2 - 1$, we even have a series of equalities. Therefore,

$$\begin{aligned}\mathcal{L}_0 &= \mathfrak{F}_S(\mathfrak{B}_S(\mathcal{L}_0)) \\ &= \mathfrak{F}_S(\mathfrak{B}_S(\mathfrak{F}_S(\mathcal{L}_\omega))) \\ &= \mathfrak{F}_S(\mathcal{L}_\omega),\end{aligned}$$

where we used that, for a regular line bundle morphism S covering a surjective submersion, we have the identity $\mathcal{L} = \mathfrak{F}_S(\mathfrak{B}_S(\mathcal{L}))$ for every Dirac-Jacobi structure $\mathcal{L}$. The same arguments can be used to obtain that T is also a forward map and *i.)* follows. As a next step, we want to show that $S \times^! T \colon L \to L^\times$ being a forward Dirac-Jacobi map is equivalent to the inclusions

$$\mathfrak{B}_S(\mathcal{L}_0) \subseteq \ker DS + \ker DT^\omega \quad \text{and} \quad \mathfrak{B}_T(\mathcal{L}_1) \subseteq \ker DT + \ker DS^{-\omega}. \tag{3.1.1}$$

Recall that the product of L_0 and L_1 is defined in Subsection 1.1.1, where also the notation

$$\begin{tikzcd} L \arrow[dr, dashed, "S\times^! T"] \arrow[drr, bend left, "T"] \arrow[ddr, bend right, "S"'] & & \\ & L^\times \arrow[r, "P_1"] \arrow[d, "P_0"] & L_1 \\ & L_0 & \end{tikzcd}$$

is introduced. We first assume that $S \times^! T$ is a forward Dirac-Jacobi map. So let $(\Delta, DS^*\psi) \in \mathfrak{B}_S(\mathcal{L}_0)$. Then $(DS(\Delta), \psi) \in \mathcal{L}_0$ and, using the splitting from Lemma 1.3.4,

$$DL^\times = \ker DP_0 \oplus \ker DP_1,$$

Hence there is a unique $\tilde{\Delta} \in \ker DP_1$ with $DP_0(\tilde{\Delta}) = DS(\Delta)$ and $(\tilde{\Delta}, DP_0^*\psi) \in \mathcal{L}_0 \times^! \mathcal{L}_1^{\mathrm{opp}}$. Since $S \times^! T$ is a forward Dirac-Jacobi map, we find a $\Box \in DL$, such that $(\Box, DS^*\psi) = (\Box, D(S \times^! T)^* DP_0^*\psi) \in \mathcal{L}_\omega$, which means that $DS^*\psi = \iota_\Box \omega$. Note that we have $\Box = \ker DT$, since $DT(\Box) = DP_1(D(S \times^! T)(\Box)) = DP_1(\tilde{\Delta}) = 0$ and similarly $\Delta - \Box \in \ker DS$. We conclude that

$$(\Delta, DS^*\psi) = (\Delta - \Box, 0) + (\Box, \iota_\Box\omega) \in \ker DS + \ker DT^\omega.$$

The second inclusion can be obtained in the exact same way and the reverse implication is an easy computation. Now let us assume that *ii.*) holds. Then, by the same arguments, we have of *ii.*) $\implies$ *i.*), we conclude

$$\mathfrak{B}_S(\mathcal{L}_0) = \ker DS + \ker DT^\omega \text{ and } \mathfrak{B}_T(\mathcal{L}_1) = \ker DT + \ker DS^{-\omega}$$

and $\omega(\ker DS, \ker DT) = 0$. By the previous consideration, this implies that $S \times^! T$ is a forward Dirac-Jacobi map and hence we have *iii.*).

Now we assume *iii.*) and notice that the rank condition, together with $S \times^! T$ being a forward map, implies equality in the inequalities 3.1.1 and hence

$$\mathfrak{B}_S(\mathcal{L}_0) = \ker DS + \ker DT^\omega = \mathfrak{B}_T(\mathcal{L}_1)^\omega,$$

which implies *iv.*). Moreover, $\omega(\ker DS, \ker DT) = 0$ implies that both $\ker DS + \ker DT^\omega$ and $\ker DT + \ker DS^{-\omega}$ are isotropic and hence their rank has to be $\leq \operatorname{rank}(\mathcal{L}\omega)$, wich also implies equality in 3.1.1 and thus also *iv.*).

The last step is to assume *iv.*). From $\mathfrak{B}_S(\mathcal{L}_0) = \mathfrak{B}_T(\mathcal{L}_1)^\omega$, we get $\omega(\ker DS, \ker DT) = 0$ and hence by the same argument as before $\mathfrak{B}_S(\mathcal{L}_0) = \ker DS + \ker DT^\omega$, which implies immediately

$$\operatorname{rank}(\ker DS \cap \ker DT \cap \ker \omega^\flat) = \dim(M) - \dim(M_0) - \dim(M_1) - 1$$

and hence *ii.*). ΧΞΣ

The relation "being connected by a dual pair" does not form an *equivalence* in the category of Dirac-Jacobi bundles. This is actually the reason to introduce weak dual pairs, since the relation "being connected by a weak dual pair" is an equivalence. The idea and the proofs of this claim can be found in [21] in the Dirac geometric setting. We want to start proving transitivity. (Note that symmetry is obvious). In Subsection 3.3, we will discuss reflexivity. Before we start, we want to stress that, what we claimed before, transitivity does not work with dual pairs, but reflexivity does (consistently with what we claimed).

Lemma 3.1.7 (Transitivity) *Let*

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S_{01}} (L_{01} \to M_{01}, \mathcal{L}_{\omega_{01}}) \xrightarrow{T_{01}} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$$

and

$$(L_1 \to M_1, \mathcal{L}_1) \xleftarrow{S_{12}} (L_{12} \to M_{12}, \mathcal{L}_{\omega_{12}}) \xrightarrow{T_{12}} (L_2 \to M_2, \mathcal{L}_2^{\mathrm{opp}})$$

be weak dual pairs. Then also $(L_0 \to M_0, \mathcal{L}_0)$ *and* $(L_2 \to M_2, \mathcal{L}_2)$ *fit into a weak dual pair.*

Proof: Let us consider the pull-back (in $\mathfrak{Line}$, see Corollary 1.3.8)

$$\begin{tikzcd} L_{02} \arrow[r, "P_2"] \arrow[d, "P_1"] & L_{12} \arrow[d, "S_{12}"] \\ L_{01} \arrow[r, "T_{01}"] & L_1 \end{tikzcd}$$

the 2-form $\omega_{02} = P_1^*\omega_{01} + P_2^*\omega_{12}$ and the maps $S_{02} = S_{01} \circ P_1$ as well as $T_{02} = T_{12} \circ P_2$. Pictorially, we have

$$\begin{tikzcd} & & L_{02} \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & & \\ & L_{01} \arrow[dl, "S_{01}"'] \arrow[dr, "T_{01}"] & & L_{12} \arrow[dl, "S_{12}"'] \arrow[dr, "T_{12}"] & \\ L_0 & & L_1 & & L_2 \end{tikzcd}$$

The fact that T_{01} and S_{12} covering surjective submersions implies that so do P_1 and P_2. Moreover, so do S_{02} and T_{02} and in addition they are forward Dirac-Jacobi maps as a concatenation of forward Dirac-Jacobi maps. Additionally, we have

$$\begin{aligned} \mathfrak{B}_{S_{02}}(\mathcal{L}_0) &= \mathfrak{B}_{P_1}(\mathfrak{B}_{S_{01}}(\mathcal{L}_0)) \\ &= \mathfrak{B}_{P_1}(\mathfrak{B}_{T_{01}}(\mathcal{L}_1)^{\omega_{01}}) \\ &= \mathfrak{B}_{P_1}(\mathfrak{B}_{T_{01}}(\mathcal{L}_1))^{P_1^*\omega_{01}} \\ &= \mathfrak{B}_{P_2}(\mathfrak{B}_{S_{12}}(\mathcal{L}_1))^{P_1^*\omega_{01}} \\ &= \mathfrak{B}_{P_2}(\mathfrak{B}_{T_{12}}(\mathcal{L}_2)^{\omega_{12}})^{P_1^*\omega_{01}} \\ &= \mathfrak{B}_{P_2}(\mathfrak{B}_{T_{12}}(\mathcal{L}_2))^{P_1^*\omega_{01}+\omega_{12}} \\ &= \mathfrak{B}_{T_{02}}(\mathcal{L}_2)^{\omega_{02}}. \end{aligned}$$

Thus by Lemma 3.1.6 part *iv.*), we get that

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S_{02}} (L_{02} \to M_{02}, \mathcal{L}_{\omega_{02}}) \xrightarrow{T_{02}} (L_2 \to M_2, \mathcal{L}_2^{\mathrm{opp}})$$

is a weak dual pair. ΧΞΣ

We discuss next operations with (weak) dual pairs. Again transversals play an important role, namely there is a notion of "pulling back" (weak) dual pairs to transversals.

Lemma 3.1.8 *Let*

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$$

be a (weak) dual pair and let $\Phi_i \colon L_{N_i} \hookrightarrow L_i$ *be transversals to* $\mathcal{L}_i$. *Then*

$$(L_{N_0} \to N_0, \mathfrak{B}_{\Phi_0}(\mathcal{L}_0)) \xleftarrow{S_\Sigma} (L_\Sigma \to \Sigma, \mathcal{L}_{\omega_\Sigma}) \xrightarrow{T_\Sigma} (L_{N_1} \to N_1, \mathfrak{B}_{\Phi_1}(\mathcal{L}_1))$$

is a (weak) dual pair if and only if S_Σ *and* T_Σ *are surjective, where*

i.) $\Sigma = N_0 \times_{M_0} M \times_{M_1} N_1$ *and* $P_2 \colon L_\Sigma = \mathrm{pr}_2^* L \to L$

ii.) $\mathcal{L}_{\omega_\Sigma} = \mathcal{L}_{P_2^*\omega} = \mathfrak{B}_{P_2}(\mathcal{L}_\omega)$

iii.) $S_\Sigma \colon L_\Sigma \ni ((x_0, m, x_1), \lambda_m) \mapsto (x_0, S(\lambda_m)) \in \phi_0^* L$

iv.) $T_\Sigma \colon L_\Sigma \ni ((x_0, m, x_1), \lambda_m) \mapsto (x_1, T(\lambda_m)) \in \phi_1^* L$

Proof: Note that L_Σ fits in the following diagram

$$\begin{tikzcd} L_\Sigma \arrow[r, "\Phi"] \arrow[d, "P_2"] & L_N^\times \arrow[d, "\Phi_0 \times^! \Phi_1"] \\ L \arrow[r, "S \times^! T"] & L^\times \end{tikzcd} \tag{3.1.2}$$

with

$$\Phi((n_0, m, n_1), \lambda_m) = (\Phi_{1,n_1}^{-1} \circ T_m \circ S_m^{-1} \circ \Phi_{0,n_0}, \Phi_{0,n_0}^{-1}(S(\lambda_m)))$$

and one can show that diagram (3.1.2) is a pull-back diagram in $\mathfrak{Line}$ as in Subsection 1.1.1. Thus $\Phi \colon L_\Sigma \to L_N^\times$ is a forward Dirac-Jacobi map for the Dirac-Jacobi structures $\mathcal{L}_{\omega_\Sigma}$ and $\mathfrak{B}_{\Phi_0 \times^! \Phi_1}(\mathcal{L}_0 \times^! \mathcal{L}_1) = \mathfrak{B}_{\Phi_0}(\mathcal{L}_0) \times^! \mathfrak{B}_{\Phi_1}(\mathcal{L}_1)$ by Lemma 1.3.9.

Note that we have the following commutative diagram of regular line bundle morphisms

$$\begin{tikzcd} & & L_\Sigma \arrow[dll, "S_\Sigma"'] \arrow[d, "P_2"] \arrow[drr, "T_\Sigma"] & & \\ & & L \arrow[dl, "S"'] \arrow[dr, "T"] & & \\ L_{N_0} \arrow[r, "\Phi_0"] & L_0 & & L_1 & L_{N_1} \arrow[l, "\Phi_1"'] \end{tikzcd}$$

and hence we can compute

$$\begin{aligned}
\mathfrak{B}_{S_\Sigma}(\mathfrak{B}_{\Phi_0}(\mathcal{L}_0)) &= \mathfrak{B}_{\Phi_0\circ S_\Sigma}(\mathcal{L}_0) = \mathfrak{B}_{S\circ P_2}(\mathcal{L}_0)\\
&= \mathfrak{B}_{P_2}(\mathfrak{B}_S(\mathcal{L}_0)) = \mathfrak{B}_{P_2}(\mathfrak{B}_T(\mathcal{L}_1)^\omega)\\
&= \mathfrak{B}_{T\circ P_2}(\mathcal{L}_1)^{P_2^*\omega} = \mathfrak{B}_{\Phi_1\circ T_\Sigma}(\mathcal{L}_1)^{P_2^*\omega}\\
&= \mathfrak{B}_{T_\Sigma}(\mathfrak{B}_{\Phi_1}(\mathcal{L}_1))^{P_2^*\omega},
\end{aligned}$$

where we used basic properties of the backwards transform and the fact that

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$$

is a (weak) dual pair. By part *iv.*) of Lemma 3.1.6 we get the claim. ΧΞΣ

3.2 Why Dual Pairs?

After having discussed the main properties of (weak) dual pairs, we want to discuss what kind of impact their existence has.

Let us start with a first lemma concering the relation between the characteristic foliation of a Dirac-Jacobi structure and its backwards transform.

Lemma 3.2.1 *Let $P\colon L_0 \to L_1$ be a regular line bundle morphism covering a surjective submersion $p\colon M_0 \to M_1$ with connected fibers and let $\mathcal{L} \subseteq \mathbb{D}L_1$ be a Dirac-Jacobi structure. Then the pre-image of p establishes a one-to-one between the characteristic leaves of $\mathcal{L}$ and of $\mathfrak{B}_P(\mathcal{L})$. Moreover, this correspondence respects the type of the leaves, i.e. a locally conformal pre-symplectic leaves correspond to locally conformal pre-symplectic leaves and pre-contact leaves correspond to pre-contact leaves.*

Proof: First of all, let us prove that if $C \hookrightarrow M$ is a characteristic leaf of $\mathcal{L}$, then $p^{-1}(C)$ is a characteristic leaf of $\mathfrak{B}_P(\mathcal{L})$. Clearly $p^{-1}(C)$ is a connected submanifold. Let $v_m \in \sigma(\mathrm{pr}_D\mathfrak{B}_P(\mathcal{L}))\big|_{p^{-1}(C)}$, then there exists a $(\Delta, DP^*\psi) \in \mathfrak{B}_P(\mathcal{L})$, such that $\sigma(\Delta) = v_m$. This implies that $(DP(\Delta), \psi) \in \mathcal{L}$ and hence $Tp(v_m) = \sigma(DP(\Delta)) \in \sigma(\mathrm{pr}_D\mathcal{L})\big|_{p(m)} = T_{p(m)}C$, which shows that $v_m \in Tp^{-1}(C)$ and thus $\sigma(\mathrm{pr}_D\mathfrak{B}_P(\mathcal{L}))\big|_{p^{-1}(C)} \subseteq Tp^{-1}(C)$. With a similar argument, starting with a vector $v_m \in p^{-1}(C)$, we see that $Tp(v_m) \in T_{p(m)}C$ and hence there exists a $(\Delta, \psi) \in \mathcal{L}$, such that $\sigma(\Delta) = Tp(v_m)$. Let us choose $\Box \in DL_0$, such that $\sigma(\Box) = v_m$ and $DP(\Box) = \Delta$. This implies that $(\Box, DP^*\psi) \in \mathfrak{B}_P(\mathcal{L})$ and hence $v_m \in \sigma(\mathrm{pr}_D\mathfrak{B}_P(\mathcal{L}))$,so the equality $\sigma(\mathrm{pr}_D\mathfrak{B}_P(\mathcal{L}))\big|_{p^{-1}(C)} = Tp^{-1}(C)$ holds. This proves that $p^{-1}(C)$ is a integral submanifold of the characteristic distribution. One can show, in a similar way, that a leaf of $\mathfrak{B}_P(\mathcal{L})$ projects via p to a leaf of $\mathcal{L}$, since for a leaf $C \hookrightarrow M_1$ we have that $\ker(TP)\big|_C \subseteq TC$. Now we discuss why this correspondence respects the type of the leaves. So consider a pre-contact leaf $\iota\colon C \hookrightarrow M$. This means by definition that $\mathrm{rank}(\mathrm{pr}_D\mathfrak{B}_I(\mathcal{L})\big|_C) = \dim C + 1$, which is if and only if the case if $\mathbb{1} \in \mathrm{pr}_D\mathfrak{B}_I(\mathcal{L})$. Hence there exists a $\psi \in J^1L$, such that $(\mathbb{1}, DI^*\psi) \in \mathfrak{B}_I(\mathcal{L}$ and hence $(\mathbb{1}, \psi) \in \mathcal{L}\big|_C$. Thus we have also $(\mathbb{1}, DP^*\psi) \in \mathfrak{B}_P(\mathcal{L})\big|_{p^{-1}(C)}$, which implies that $\mathrm{pr}_D(\mathfrak{B}_P(\mathcal{L})\big|_{p^{-1}(C)})$ and hence $p^{-1}(C)$ is a pre-contact leaf. Moreover, one can prove, in the similar way, that a locally conformal pre-symplectic leaf in M_0 corresponds to a locally conformal pre-symplectic leaf in M_1. Since we only have these two kinds of leaves, we get the claim. ΧΞΣ

Using Lemma 3.2.1 we want to prove a correspondence of the leaves of the legs of a weak dual pair.

Theorem 3.2.2 *Let*

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$$

be a weak dual pair, then there is a one-to-one correspondence between the leaves of $\mathcal{L}_0$ and $\mathcal{L}_1$ given by $M_0 \supseteq C \to t(s^{-1}(C)) \subseteq M_1$. This correspondence respects the type of the leaves. Moreover, we have that if we have two leaves $\iota_0 \colon C_0 \hookrightarrow M_0$ and $\iota \colon C_1 \hookrightarrow M_1$, which are in correspondence via the leaf $\iota \colon C \hookrightarrow M$, i.e. $s^{-1}(C_0) = C = t^{-1}(C_1)$, then C is a characteristic leaf of the Dirac Jacobi structure $\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})$, which is given by

$$\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}}) = \big((\ker DS + \ker DT) \oplus \mathrm{Ann}(\ker DS + \ker DT)\big)^{\omega}.$$

Moreover, its induced structure can be computed by

$$\mathfrak{B}_I(\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}}))) = \mathfrak{B}_{S_C}(\mathcal{L}_{C_0}) \star \mathfrak{B}_{T_C}(\mathcal{L}_{C_1}^{\mathrm{opp}}) \tag{3.2.1}$$

and all the involved bundles and products are smooth Dirac-Jacobi bundles.

PROOF: Note that by Lemma 3.2.1 the only thing we have to prove for the first part of the statement is the equality

$$\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}}) = \big((\ker DS + \ker DT) \oplus \mathrm{Ann}(\ker DS + \ker DT)\big)^{\omega}$$

and that its characteristic foliation is give by pre-images of leaves of the legs. Using Lemma 3.1.6, we see that

$$\mathfrak{B}_S(\mathcal{L}_0) = \ker DS + \ker DT^{\omega} = \mathfrak{B}_T(\mathcal{L})^{\omega},$$

since we have a dual pair. This implies that

$$\mathrm{pr}_D \mathfrak{B}_S(\mathcal{L}_0) = \mathrm{pr}_D \mathfrak{B}_T(\mathcal{L}_1),$$

and hence the characteristic distributions of $\mathfrak{B}_S(\mathcal{L}_0)$ and $\mathfrak{B}_T(\mathcal{L}_1)$ coincide and thus also the characteristic distribution of $\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})$ since

$$\mathrm{pr}_D\big(\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})\big) = \mathrm{pr}_D\big(\mathfrak{B}_S(\mathcal{L}_0) \cap \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})\big).$$

Using now Lemma 3.2.1, we get the claim and moreover we see that the correspondence respects the type of the leaves. Let us now prove Equation 3.2.1. Let us first remark, that

$$\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})$$

is smooth since we have from our previous considerations

$$\mathrm{pr}_D \mathfrak{B}_S(\mathcal{L}_0) = \mathrm{pr}_D \mathfrak{B}_T(\mathcal{L})$$

and we can use Remark 1.3.3. Moreover, we have that

$$\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}}) = \big((\ker DS + \ker DT) \oplus \mathrm{Ann}(\ker DS + \ker DT)\big)^{\omega}$$

using the equations

$$\mathfrak{B}_S(\mathcal{L}_0) = \ker DS + \ker DT^{\omega} = \mathfrak{B}_T(\mathcal{L})^{\omega},$$

and the fact that $\omega(\ker DS, \ker DT) = 0$.

The second part is now just a matter of computation, using the diagram

$$\begin{tikzcd}
 & & L_C \arrow[ddll, "S_C"'] \arrow[d, "I"] \arrow[ddrr, "T_C"] & & \\
 & & L \arrow[dl, "S"'] \arrow[dr, "T"] & & \\
L_{C_0} \arrow[r, "I_0"] & L_0 & & L_1 & L_{C_1} \arrow[l, "I_1"']
\end{tikzcd},$$

where we used the subscripts S_C, T_C as a short notation for $T_C = T\big|_{L_C}$ and $S_C = S\big|_{L_C}$. Namely

$$\begin{aligned}
\mathfrak{B}_I(\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})) &= \mathfrak{B}_I(\mathfrak{B}_S(\mathcal{L}_0)) \star \mathfrak{B}_I(\mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})) \\
&= \mathfrak{B}_{S\circ I}(\mathcal{L}_0) \star \mathfrak{B}_{T\circ I}(\mathcal{L}_1^{\mathrm{opp}}) \\
&= \mathfrak{B}_{I_0\circ S_C}(\mathcal{L}_0) \star \mathfrak{B}_{I_1\circ T_C}(\mathcal{L}_1^{\mathrm{opp}}) \\
&= \mathfrak{B}_{S_C}(\mathcal{L}_{C_0}) \star \mathfrak{B}_{T_C}(\mathcal{L}_{C_1}^{\mathrm{opp}}),
\end{aligned}$$

where we used Remark 1.3.10 in the second step and $\mathfrak{B}_{I_i}(\mathcal{L}_i) = \mathcal{L}_{C_i}$ by definition.XΞΣ

From Theorem 3.2.2, we can conclude for a weak dual pair

$$L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$$

we can consider the Dirac-Jacobi structure $\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})$, whose characteristic leaves are in one-to-one correspondence with the leaves of $\mathcal{L}_0$ and with the leaves of $\mathcal{L}_1$, but also the induced structures on all the leaves can be compared, which we can see in the following

Proposition 3.2.3 *Let* $(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}_1^{\mathrm{opp}})$ *be a weak dual pair and let* $s^{-1}(C_0) = C = t^{-1}(C_1)$ *be in correspondence. If*

i.) C_0 *and* C_1 *are pre-contact leaves with pre-contact forms* $\omega_i \in \Omega_{L_i}(M_i)$ *for* $i = 0, 1$*, then*

$$I^*\omega = S_C^*\omega_1 - T_C^*\omega_2.$$

ii.) C_0 *and* C_1 *are locally conformal pre-symplectic leaves with locally conformal pre-symplectic forms* $\omega_i \in \Gamma^\infty(\Lambda^2 T^*C_i \otimes L_{C_i})$ *and connections* ∇^i*, then*

(a) $DS \circ \nabla = \nabla^0 \circ Ts$ *and* $DT \circ \nabla = \nabla^1 \circ Tt$

(b) $\omega_C = S_C^*\omega_1 - T_C^*\omega_2$

for the connection $\nabla \colon TC \to DL_C$ *on the locally conformal presymplectic leaf* C *with 2 form* $\omega_C \in \Gamma^\infty(\Lambda^2 T^*C \otimes L_C)$*, which is given by*

$$\omega_C(X, Y) = I^*\omega(\nabla_X, \nabla_Y).$$

PROOF: Let us first prove property *i.*). So let C_0, and hence C_1 and C, be pre-contact leaves. We have by defintion of a contact leaf that $\mathfrak{B}_{I_i}(\mathcal{L}_i) = \mathcal{L}_{\omega_i}$ for $i = 0, 1$ and the pre-contact forms on $L_{C_i} \to C_i$. More over we have that

$$DL_C = \mathrm{pr}_D\big(\big((\ker DS + \ker DT) \oplus \mathrm{Ann}(\ker DS + \ker DT)\big)^\omega\big)\big|_C = (\ker DS + \ker DT)\big|_C$$

and thus

$$\mathfrak{B}_I\big(\big((\ker DS + \ker DT) \oplus \mathrm{Ann}(\ker DS + \ker DT)\big)^\omega\big) = \mathcal{L}_{I^*\omega}. \qquad (*)$$

Using Theorem 3.2.2 and Equation $(*)$, we get

$$\begin{aligned} \mathcal{L}_{I^*\omega} &= \mathfrak{B}_{S_C}(\mathcal{L}_{\omega_0}) \star \mathfrak{B}_{T_C}(\mathcal{L}_{-\omega_1}) \\ &= \mathcal{L}_{S_C^*\omega_0 - T_C^*\omega_1} \end{aligned}$$

where the last step involves a small and straightforward computation.

Now let us in order to prove *ii.*). So assume that C_0, C_1 and C are locally conformal pre-symplectic leaves. Note that by assumption C is a locally conformal pre-symplectic leaf of $\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})$ Let us first prove point (a). Recall that the connection on a locally conformal pre-symplectic leaf is defined as the unique inverse of

$$\sigma\big|_{\mathrm{pr}_D \mathfrak{B}_{I_i}(\mathcal{L}_{C_i})} \colon \mathrm{pr}_D \mathfrak{B}_{I_i}(\mathcal{L}_{C_i}) \to TC_i$$

for the locally conformal pre-symplectic leaves C_i of $\mathcal{L}_i$ and

$$\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}}) = \big((\ker DS + \ker DT) \oplus \mathrm{Ann}(\ker DS + \ker DT)\big)^\omega,$$

respectively. Let $v_m \in T_mC$, then we have that

$$\sigma(DS(\nabla_{v_m})) = Ts(v_m) = \sigma(\nabla^0_{Ts(v_m)}).$$

and thus since ∇ and ∇^0 are the unique inverses of the symbol maps σ we get $DS(\nabla_{v_m}) = \nabla^0_{Ts(v_m)}$. Similarly we obtain $DT \circ \nabla = \nabla^1 \circ Tt$.

Let us now stick to point (b). It is easy to see that the locally conformal pre-symplectic structure on

$$\mathfrak{B}_I(\big((\ker DS + \ker DT) \oplus \operatorname{Ann}(\ker DS + \ker DT)\big)^\omega)$$

is given by ω_C. So let us compute the left-hand side. From Theorem 3.2.2, we know that

$$\mathfrak{B}_I(\big((\ker DS + \ker DT) \oplus \operatorname{Ann}(\ker DS + \ker DT)\big)^\omega = \mathfrak{B}_{S_C}(\mathcal{L}_{C_0}) \star \mathfrak{B}_{T_C}(\mathcal{L}^{\text{opp}}_{C_1})$$

Notice that ω_i is uniquely determined by

$$\omega_i(X, Y) = \psi_i(\nabla^i_Y)$$

for $(\nabla_X, \psi_i) \in \mathcal{L}_{C_i}$ and $Y \in TC_i$ and similarly for ω_C. Let us choose

$$(\nabla_{X_p}, DS^*\psi_0 + DT^*\psi_1) \in \mathfrak{B}_{S_C}(\mathcal{L}_{C_0}) \star \mathfrak{B}_{T_C}(\mathcal{L}^{\text{opp}}_{C_1})$$

and let $Y_p \in TC$. Then

$$\begin{aligned}
\omega_C(X_p, Y_p) &= DS^*\psi_0(\nabla_{Y_p}) + DT^*\psi_1(\nabla_{Y_p}) \\
&= S_p^{-1}\psi_0(DS(\nabla_{Y_p})) + T_p^{-1}\psi_1(DT(\nabla_{Y_p})) \\
&= S_p^{-1}\psi_0(\nabla^0_{Ts(Y_p)}) + T_p^{-1}\psi_1(\nabla^1_{Tt(Y_p)})
\end{aligned}$$

moreover, we have that $(\nabla^0_{Ts(X_p)}, \psi_0) \in \mathcal{L}_{C_0}$ and hence $\psi_0(\nabla^0_{Ts(Y_p)}) = \omega_0(Ts(X_p), Ts(Y_p))$ and similarly $\psi_1(\nabla^0_{Tt(Y_p)}) = -\omega_1(Tt(X_p), Tt(Y_p))$. Continuing the computation we get

$$\omega_C(X_p, Y_p) = S_p^{-1}\omega_0(Ts(X_p), Ts(Y_p)) - T_p^{-1}\omega_1(Tt(X_p), Tt(Y_p)) = S_C^*\omega_1 - T_C^*\omega_2(X_p, Y_p)$$

and the claim follows. ΧΞΣ

Let us now pass to the transverse geometry,i.e. the structures on transversals, in fact we have seen that there is a leaf correspondence, but leaves need not to be isomorphic. Minimal transversals on the other hand are locally isomorphic as Dirac-Jacobi bundles:

Proposition 3.2.4 *Let*

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_1 \to M_1, \mathcal{L}^{\text{opp}}_1)$$

be a weak dual pair, let $m \in M$ and let $\iota_i \colon N_i \hookrightarrow M_i$ minimal transversals at $s(m)$ and $t(m)$ respectively. Then $\mathfrak{B}_{I_0}(\mathcal{L}_0)$ is locally isomorphic to $\mathfrak{B}_{I_1}(\mathcal{L}_1)$ (up to a B-field).

PROOF: First, we remark that the dimensions of the minimal transversals always coincide. Indeed, let C_0, C_1 and C corresponding leaves at $s(m)$ and $t(m)$ respectively, i.e. $s^{-1}(C_0) = C = t^{-1}(C_1)$. Then we have

$$\dim(M) - \dim(N_0) + \dim(C_0) = \dim(M) - \dim(N_1) + \dim(C_1).$$

which means that C_0 and C_1 have the same codimension and hence the dimensions of N_0 and N_1 coincide. We use the pull-back construction of Lemma 3.1.8 in order to consider of the weak dual pair

$$(L_{N_0} \to N_0, \mathfrak{B}_{I_0}(\mathcal{L}_0)) \xleftarrow{S_N} (L_N \to N, \mathcal{L}_{\omega_N}) \xrightarrow{T_N} (L_{N_1} \to N_1, \mathfrak{B}_{I_1}(\mathcal{L}_1)^{\mathrm{opp}})$$

and notice that $s_N(m)$ is a leaf of $\mathfrak{B}_{I_0}(\mathcal{L}_0)$ and $t_N(m)$ is a leaf of $\mathfrak{B}_{I_1}(\mathcal{L}_1)^{\mathrm{opp}}$, since they are minimal transversals. Additionally, the maps S_N and T_N are always surjetive, since N_i are actual submanifolds and not just images of transverse maps as in Lemma 3.2.1. Using the the correspondence of leaves from Theorem 3.2.2, we see that $s_N^{-1}(s_N(m))$ is a leaf of $\mathfrak{B}_S(\mathcal{L}_0) \star \mathfrak{B}_T(\mathcal{L}_1^{\mathrm{opp}})$ as well as $t_N^{-1}(t_N(m))$, but they both contain m and hence they have to coincide, i.e. $s_N^{-1}(s_N(m)) = t_N^{-1}(t_N(m))$. Let us now choose a local right-inverse $\Phi\colon L_{N_0} \to L_N$ of S_N covering $\phi\colon N_0 \to N$ such that $\phi(s_N(m)) = m$. Now we want to prove that

$$k\colon = t \circ \phi\colon N_0 \to N_1$$

is a local diffeomorphism. Let $v_{s_N(m)} \in \ker Tk$, which is eqivalent to $T\phi(s_N(m)) \in Tt_N^{-1}(t_N(m))$, but we have that $s_N^{-1}(s_N(m)) = t_N^{-1}(t_N(m))$ and hence $T\phi(v_{s_N(m)}) \in \ker Ts$, so $0 = Ts(T\phi(v_{s_N(m)})) = T(s \circ \phi)(v_{s_N(m)}) = v_{s_N(m)}$. Thus, since $\dim N_0 = \dim N_1$, we get that k is a local diffeomorphism. So let us choose open neighborhoods of $s_N(m)$ and $t_N(m)$, such that k is actually a diffeomorphism. Note that k is covered by $K = T \circ \Phi$ and we have

$$\begin{aligned}
\mathfrak{B}_K(\mathcal{L}_1) &= \mathfrak{B}_{T\circ\Phi}(\mathcal{L}_1) = \mathfrak{B}_\Phi(\mathfrak{B}_T(\mathcal{L}_1)) \\
&= \mathfrak{B}_\Phi(\mathfrak{B}_S(\mathcal{L}_0)^{-\omega}) = \mathfrak{B}_{S\circ\Phi}(\mathcal{L}_0)^{-\Phi^*\omega} \\
&= \mathcal{L}_0^{-\Phi^*\omega},
\end{aligned}$$

where the third equality uses the fact that we have a dual pair and the last equality that $S \circ \Phi = \mathsf{id}$. ΧΞΣ

3.3 Existence of a Self Dual Pair

The aim of this section is to prove that for every Dirac-Jacobi structure $(L_0 \to M_0, \mathcal{L}_0)$, there is a line bundle $(L \to M, \mathcal{L}_\omega)$ with a closed $\omega \in \Omega^2_L(M)$, together with regular line bundle morphisms $S, T\colon L \to L_0$, such that

$$(L_0 \to M_0, \mathcal{L}_0) \xleftarrow{S} (L \to M, \mathcal{L}_\omega) \xrightarrow{T} (L_0 \to M_0, \mathcal{L}_0^{\mathrm{opp}})$$

is a dual pair, which we call *self dual pair*. Again, this part is a translation of the mirror statement in Poisson geometry from [21]. Let us collect the missing ingredients in order to prove this claim. We start by discussing Dirac-Jacobi sprays.

Definition 3.3.1 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi bundle and let $p^*L \to \mathcal{L}$ be the pull-back line bundle given by the diagram*

$$\begin{tikzcd} p^*L \arrow[r, "P"] \arrow[d] & L \arrow[d] \\ \mathcal{L} \arrow[r, "p"] & M \end{tikzcd}.$$

A derivation $\Sigma \in \Gamma^\infty(Dp^*L)$ is said to be Dirac-Jacobi spray, if

i.) $DP(\Sigma_{(\Delta,\psi)}) = \Delta$

ii.) $M_t^*\Sigma = t\Sigma$ *for* $t > 0$

Where $M_t \colon p^*L \ni ((\Delta, \alpha), \lambda) \to (t(\Delta, \alpha), \lambda) \in p^*L$.

Note that for the flow Φ_ϵ^Σ of a Dirac-Jacobi spray Σ, we have necessarily

$$\Phi_\epsilon^\Sigma \circ M_t = M_t \circ \Phi_{t\epsilon}^\Sigma \tag{3.3.1}$$

As a consequence the equality

$$\phi_\epsilon^\Sigma \circ m_t = m_t \circ \phi_{t\epsilon}^\Sigma, \tag{3.3.2}$$

where ϕ_ϵ^Σ is the map covered by Φ_ϵ^Σ and m_t is the map covered by M_t.

Remark 3.3.2 Note that a Dirac-Jacobi spray for a Dirac-Jacobi bundle $(L \to M, \mathcal{L})$ can be constructed as follows: since the bundle projection $p \colon \mathcal{L} \to M$, the kernel of the map

$$DP \colon Dp^*L \to DL$$

is a regular subbundle of Dp^*L. Let us choose a splitting $Dp^*L \cong \ker(DP) \oplus p^*DL$ and define

$$\Sigma \colon \mathcal{L} \ni (\Delta, \alpha) \to \big((\Delta, \alpha), \Delta\big) \in p^*DL$$

and extend it trivially to Dp^*L. Σ is by construction a Dirac-Jacobi spray.

For a Dirac-Jacobi structure $\mathcal{L} \subseteq \mathbb{D}L$, we use the following notation for maps:

$$\begin{tikzcd} p^*L \arrow[r, "P_J"] \arrow[d] \arrow[rr, bend left, "P"] & \pi^*L \arrow[r, "\Pi"] \arrow[d] & L \arrow[d] \\ \mathcal{L} \arrow[r, "\mathrm{pr}_J"] \arrow[rr, bend right, "p"'] & J^1L \arrow[r, "\pi"] & M \end{tikzcd}.$$

The rest of this section is dedicated to the proof of the following

Theorem 3.3.3 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi structure and let $\Sigma \in \Gamma^\infty(Dp^*L)$ be a Dirac-Jacobi spray. Then there is a open neighborhood $U \subset \mathcal{L}$ containing the zero section, such that*

$$\omega = \int_0^1 (\Phi_t^\Sigma)^* P_J^* \omega_{\mathrm{can}} \, \mathrm{d}t$$

is well-defined and $S := P \colon \mathcal{L}_\omega \to \mathcal{L}$ and $T := P \circ \Phi_1^\Sigma \colon \mathcal{L}_\omega \to \mathcal{L}^{\mathrm{opp}}$ define a dual-pair

$$(L \to M, \mathcal{L}_0) \xleftarrow{S} (p^*L \to U, \mathcal{L}_\omega) \xrightarrow{T} (L \to M, \mathcal{L}^{\mathrm{opp}}).$$

In order to prove this theorem, we proceed as in [21] and prove first a partial result. Let us use the same map as in Section 2.3, defined by

$$\cdot^{\mathrm{ver}} \colon \mathcal{L} \ni e_p \mapsto \left(\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} M_0(M_t^* \lambda(e_p)) \right) \in D_{0_p} p^* L$$

in order to identify $Dp^*L\big|_M = DL \oplus \mathcal{L}$.

Lemma 3.3.4 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi bundle and let $\Sigma \in \Gamma^\infty(Dp^*L)$ be a Dirac-Jacobi spray. Then*

i.) $D\Phi_\epsilon^\Sigma \colon Dp^*L\big|_M \to Dp^*L\big|_M$ *is given by* $(\Delta_p, e_p) \mapsto (\Delta_p + \epsilon \mathrm{pr}_D(e_p), e_p)$

ii.) $(\ker DS \cap \ker DT \cap \ker \omega^\flat)\big|_M = \{0\}$

Proof: Let $e_x \in \mathcal{L}$ and let $\mu \in \Gamma^\infty(L)$, then we have

$$\begin{aligned}
DP(D\Phi_\epsilon^\Sigma(e_x^{\mathrm{ver}}))(\mu) &= D(P \circ \Phi_\epsilon^\Sigma)(e_x^{\mathrm{ver}}))(\mu) = P \circ \Phi_\epsilon^\Sigma(e_x^{\mathrm{ver}}((P \circ \Phi_\epsilon^\Sigma)^* \mu) \\
&= P \circ \Phi_\epsilon^\Sigma(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} M_0\big(M_t^*(P \circ \Phi_\epsilon^\Sigma)^* \mu(e_x)\big)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P \circ \Phi_\epsilon^\Sigma(M_0\big((P \circ \Phi_\epsilon^\Sigma \circ M_t)^* \mu(e_x)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P\big((P \circ \Phi_{t\epsilon}^\Sigma)^* \mu(e_x)\big) \\
&= DP(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_{t\epsilon}^\Sigma)^* P^* \mu(e_x)) \\
&= \epsilon DP(\Sigma_{e_x})(\mu) \\
&= \epsilon \mathrm{pr}_D(e_x)(\mu).
\end{aligned}$$

We used Equations (3.3.1) and (3.3.2), as well as $P \circ M_t = P$ for every t. Finally the last equation is property *i.*) of Dirac-Jacobi sprays. Since Σ vanishes at the zero section M, we conclude that its flow is identity applying to $DL \subseteq Dp^*L\big|_M$. The last non-trivial point to show for part *i.*) is that

$$\mathrm{pr}_{\mathcal{L}} D\Phi_\epsilon^\Sigma(\Delta_p, e_p) = e_p$$

for $p \in M$ and the identification $Dp^*L\big|_M = DL \oplus \mathcal{L}$. This can be achieved by writing down the projection to $\mathcal{L}$ and apply Equation (3.3.1). Let us now prove part *ii.*). A computation in local coordinates shows that $P_J^*\omega_{\mathrm{can}}$ can be expressed at M by

$$P_J^*\omega_{\mathrm{can}}((\Delta_1,(\Box_1,\psi_1)),(\Delta_2,(\Box_2,\psi_2))) = \psi_2(\Delta_1) - \psi_1(\Delta_2),$$

where we used again the splitting $Dp^*L\big|_M = DL \oplus \mathcal{L}$. By using *i.*) and integrating, we get

$$\omega\big((\Delta_1,(\Box_1,\psi_1)),(\Delta_2,(\Box_2,\psi_2))\big) = \psi_2\big(\Delta_1 + \frac{1}{2}\Box_1\big) - \psi_1\big(\Delta_2 + \frac{1}{2}\Box_2\big).$$

Now, let $(\Delta,(\Box,\psi)) \in (\ker DS \cap \ker DT \cap \ker \omega^\flat)\big|_M$. Note that, since we have

$$DS(\Delta,(\Box,\psi)) = \Delta \quad \text{and} \quad DT(\Delta,(\Box,\psi)) = DT\big(D\Phi_1^\Sigma(\Delta,(\Box,\psi))\big) = \Delta + \Box,$$

it follows $\Delta = \Box = 0$. Moreover, we have

$$0 = \omega((\Delta,(\Box,\psi)),(D,0)) = -\psi_1(D),$$

for all $D \in DL$ and hence also $\psi = 0$ and the claim is proven. ΧΞΣ

Now we have the tools to prove Theorem 3.3.3.

Proof (of Theorem 3.3.3): Since $\Sigma\big|_M = 0$ we can find an open neighbourhood of M such that the flow of Σ exists for $t \in [-1,1]$. In a possibly smaller neighborhood we can also assume that $\ker DS \cap \ker DT \cap \ker \omega^\flat = \{0\}$ by Lemma 3.3.4. Let us call this neighborhood $M \subseteq U \subseteq \mathcal{L}$ and denote by $\lambda = P_J^*\lambda_{\mathrm{can}}$, where λ_{can} is the canonical 1-jet from Example 1.2.41. We want to show that $(\Sigma,\lambda) \in \Gamma^\infty(\mathfrak{B}_S(\mathcal{L}))$. So let $(\Delta,\psi) \in \mathcal{L}$. From the universal property of λ_{can}, we get

$$\lambda_{(\Delta,\psi)} = DP^*\psi$$

and hence we have that $(\Sigma_{(\Delta,\psi)},\lambda_{(\Delta,\psi)}) = (\Sigma_{(\Delta,\psi)}, DP^*\psi)$ and moreover, since Σ is a Dirac-Jacobi spray, we get that $DP(\Sigma_{(\Delta,\psi)}) = \Delta$. Hence $(\Sigma,\lambda) \in \Gamma^\infty(\mathfrak{B}_S(\mathcal{L}))$. This means that the flow of (Σ,λ), which is given by

$$\exp\Big(\int_0^t (\Phi_{-\tau}^\Sigma)^*\,\mathrm{d}_L\lambda\,\mathrm{d}\tau\Big) \circ \mathbb{D}\Phi_t^\Sigma,$$

preserves $\mathfrak{B}_S(\mathcal{L})$ whenever it exists. So we have by choosing $t = -1$, that

$$\begin{aligned}
\int_0^{-1} (\Phi_{-\tau}^\Sigma)^*\,\mathrm{d}_L\lambda\,\mathrm{d}\tau &= -\int_0^1 (\Phi_\tau^\Sigma)^*\,\mathrm{d}_L\lambda\,\mathrm{d}\tau \\
&= \int_0^1 (\Phi_\tau^\Sigma)^* P_J^*\omega_{\mathrm{can}}\,\mathrm{d}\tau \\
&= \omega,
\end{aligned}$$

where we used $\mathrm{d}_L\lambda = \mathrm{d}_L P_J^*\lambda_{\mathrm{can}} = -P_J^*\omega_{\mathrm{can}}$. Thus

$$
\begin{aligned}
\mathfrak{B}_S(\mathcal{L}) &= (\mathbb{D}\Phi^{\Sigma}_{-1}(\mathfrak{B}_S(\mathcal{L})))^{\omega} \\
&= \mathfrak{B}_{\Phi^{\Sigma}_1}(\mathfrak{B}_S(\mathcal{L}))^{\omega} \\
&= \mathfrak{B}_T(\mathcal{L})^{\omega}
\end{aligned}
$$

and by point *ii.)* of Lemma 3.1.6 we fianlly get that

$$
(L \to M, \mathcal{L}_0) \xleftarrow{S} (p^*L \to U, \mathcal{L}_\omega) \xrightarrow{T} (L \to M, \mathcal{L}^{\mathrm{opp}})
$$

is a dual pair. ΧΞΣ

As a consequence of the previous Theorem, also a Jacobi bundle, seen as a Dirac-Jacobi bundle, fits into a dual pair. This dual pair has an additional feature: the closed two form is actually non-degenerate and thus contact.

Corollary 3.3.5 *Let $L \to M$ be a line bundle and let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi tensor. Then the self dual pair*

$$
(L \to M, \mathcal{L}_J) \xleftarrow{S} (p^*L \to U, \mathcal{L}_\omega) \xrightarrow{T} (L \to M, \mathcal{L}_J^{\mathrm{opp}}).
$$

constructed in Theorem 3.3.3 can be shrinked to $U' \subseteq U$ such that ω is non-degenerate and hence it is a contact 2-form.

Proof: The claim follows from the fact that in the case of a Jacobi structure $P_J\colon \mathcal{L}_J \to J^1L$ is an isomorphism and hence ω is non-degenerate at the zero section and by upper semi-continuity in a whole neighborhood of it. ΧΞΣ

3.4 Application: Normal Form Theorems

We conclude this chapter, discussing an application of Theorem 3.3.3: we are able to rediscover a variation of the normal form theorems from Chapter 2 up to some technical details. We use the same exact notation as in Subsection 2.1.1.

Theorem 3.4.1 *Let $(L \to M, \mathcal{L})$ be a Dirac-Jacobi bundle and $\iota\colon N \hookrightarrow M$ be a transversal. Then there exist an open neighborhood $U \subseteq \nu_N$ of the zero section, a regular line bundle morphism $\Psi\colon L_\nu\big|_U \to L$ covering a local diffeomorphism and an closed Atiyah 2-form $\alpha \in \Omega^2_{L_\nu}(U)$, such that*

$$
\mathfrak{B}_\Psi(\mathcal{L}) = \mathfrak{B}_{I\circ P}(\mathcal{L})^{\alpha}.
$$

Proof: Let us denote by

$$
(L \to M, \mathcal{L}_0) \xleftarrow{S} (\ell \to P, \mathcal{L}_\omega) \xrightarrow{T} (L \to M, \mathcal{L}^{\mathrm{opp}}),
$$

the self dual pair constructed in Theorem 3.3.3. We denote its pull-back (Lemma 3.1.8) along $\iota\colon N \hookrightarrow M$ by

$$(L_N \to N, \mathfrak{B}_I(\mathcal{L})) \xleftarrow{S_N} (\ell_N \to P_N, \mathcal{L}_{\omega_N}) \xrightarrow{T_N} (L \to M, \mathcal{L}^{\mathrm{opp}}) \tag{3.4.1}$$

with

$$P_N = s^{-1}(N),\ \ell_N = \ell\big|_{P_N},\ \omega_N = \omega\big|_{P_N},\ S_N = S\big|_{P_N} \text{ and } T_N = T\big|_{P_N}.$$

Note that S_N covers a surjective submersion by construction. Moreover, T_N is a submersion, by Lemma 3.1.8, and its image contains N by construction of the self-dual pair. So we may replace M by $t_N(P_N)$, which is an open subset of M containing N. Thus (3.4.1) is a dual pair by Lemma 3.1.8. By the proof of Proposition 2.2.5, we have an exact sequence, this time together with suitable line bundles,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & p^*L\big|_{\mathcal{L}|_N} \arrow[r] \arrow[d] & L_\nu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & i^*\mathcal{L} \arrow[r] & \mathcal{L}\big|_N \arrow[r] & \nu_N \arrow[r] & 0
\end{tikzcd}$$

where K is the suitable pull-back line bundle. Let us choose a splitting $\Xi\colon L_\nu \to p^*L\big|_{\mathcal{L}|_N}$ of this sequence covering a splitting $\xi\colon \nu_N \to \mathcal{L}\big|_N$ of the lower sequence.

By Lemma 3.3.4, we have that

$$DT_N\colon DL_N \oplus \mathcal{L} \ni (\Delta, a) \mapsto \Delta + \mathrm{pr}_D(a) \in DL\big|_N.$$

and hence there exists an open neighborhood $U \subseteq \nu_N$ of the zero section, such that

$$\phi\colon U \to M \text{ with } \phi = t \circ \xi$$

is an embedding covering $\Phi = T \circ \Xi$. Moreover, we have that $S_N \circ \Xi = P$ and thus

$$\begin{aligned}
\mathfrak{B}_{I\circ P}(\mathcal{L}) &= \mathfrak{B}_{I\circ S_N \circ \Xi}(\mathcal{L}) = \mathfrak{B}_\Xi(\mathfrak{B}_{S_N}(\mathfrak{B}_I(\mathcal{L}))) \\
&= \mathfrak{B}_\Xi(\mathfrak{B}_{T_N}(\mathcal{L})^{\omega_N}) \\
&= \mathfrak{B}_\Phi(\mathcal{L})^{\Xi^*\omega_N},
\end{aligned}$$

which proves the claim. ΧΞΣ

Remark 3.4.2 Note that up to some technical details Theorem 2.2.6 and Theorem 3.4.1 coincide. But both of them required a non-trivial choice: for Theorem 2.2.6 we choose a Euler-like derivation and for Theorem 3.4.1 we choose a Dirac-Jacobi spray. It is not clear to the author, how these choices interact with each other.

Chapter 4

Jacobi Structures and Generalized Contact Bundles

After having studied the the local properties of generalized contact bundles in Section 2.4, this chapter is dedicated to the description of their global properties of them. In particular, if the Jacobi bracket of a generalized contact structure is *weakly regular*, a notion we explain throughout this chapter, the generalized contact structure induces a so-called *transversally complex bundle*. It turns out that existence of a given weakly regular Jacobi structure together with a transversally complex bundle is just necessary but not sufficient to construct a generalized contact structure out of. The aim of this chapter is hence to find the precise obstructions on the Jacobi bracket and the transversally complex bundle to be able to find a generalized contact structure.

This chapter is divided as follows: first we introduce weakly regular Jacobi structures and transversally complex bundles, then we discuss a spectral sequence we can attach to these data and show that the obstruction of the existence of a generalized contact structure lives in suitable terms of this spectral sequence. In the section afterwards, we make use of this in order to produce examples: we prove that every 5-dimensional nilpotent Lie group posses an invariant generalized contact bundle. The last part is meant to show that the obstructions might not vanish, i.e. we find a manifold together with a Jacobi structure and a transversally complex bundle, which does not admit a generalized contact bundle.

In [5] the author provides a mirror statement for generalized complex structure with regular Poisson bivectors. There are two main differences of [5] and the following statement, which become clearer throughout the chapter:

i.) In generalized complex geometry the author finds obstructions to the existence of *twisted* generalized complex structures. This is in principle also possible in generalized contact geometry, but does not make too much sense, since the obstructions are exactly the same, which is not true in generalized complex geometry. This is a consequence of the fact that the complex of Atiyah forms is acyclic.

ii.) In [5] the author uses complementary bundles, where we prefer to use spectral sequences. The advantage of our approach is that it is more obvious that the obstructions are canonical, i.e. not dependent on a choice. In fact, our techniques can be applied to the generalized complex case as well in order to find the same exact obstructions as in [5].

Both of the above points are discussed in more detail in the corresponding parts of this chapter.

This chapter is based on [39].

4.1 Tranversally Complex Jacobi Structures

Unlike in Poisson geometry, a Jacobi structure J may also have odd dimensional characteristic leaves, which we discussed already in Chapter 1. This comes from the fact that for a Jacobi tensor J, the characteristic foliation are the integral manifolds of the singular distribution $\operatorname{im}(\sigma \circ J^\sharp)$. Note that the image of the Jacobi tensor is an even dimensional subbundle of DL, but the symbol σ has a one dimensional kernel. Therefore, it seems reasonable to distinguish between regular Jacobi structures, i.e. Jacobi structure inducing a regular distribution, and

Definition 4.1.1 *Let $L \to M$ be a line bundle. A Jacobi tensor $J \in \Gamma^\infty(\Lambda^2 (J^1L)^* \otimes L)$ is said to be weakly regular, if $\operatorname{im}(J^\sharp) \subseteq DL$ is a regular subbundle.*

Remark 4.1.2 To the author's knowledge Definition 4.1.1 does not appear anywhere in the literature, but seems to be very natural. Besides the appearance in generalized contact geometry, these Jacobi structures are interesting objects by themselves and we plan to study them in a separate project.

Remark 4.1.3 A Jacobi structure which is weakly regular is not always regular. To illustrate this, we take for example the canonical Jacobi pair (defined in 2.3.10) $(\Lambda_{can}, E_{can}) \in \Gamma^\infty(\Lambda^2 T\mathbb{R}^{2k+1} \oplus T\mathbb{R}^{2k+1})$ coming from the contact structure and consider $Z \in \Gamma^\infty(T\mathbb{R})$ given by $Z = x\frac{\partial}{\partial x}$. Then $(\Lambda = \Lambda_{can} + E_{can} \wedge Z, E_{can})$ defines a weakly regular Jacobi structure on $\mathbb{R}^{2k+1} \times \mathbb{R}$ where the set of contact points is $\{(v, 0) \in \mathbb{R}^{2k+1} \times \mathbb{R}\}$.

Remark 4.1.4 Let $L \to M$ be a line bundle and let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a weakly regular Jacobi structure. Then by definition $S := \operatorname{im}(J^\sharp) \subseteq DL$ is a regular subbundle. Moreover, one can prove that it is in fact a subalgebroid and there is a canonical form $\omega \in \Gamma^\infty(\Lambda^2 S^* \otimes L)$ defined by

$$\omega(J^\sharp(\alpha), J^\sharp(\beta)) = \alpha(J^\sharp(\beta))$$

for $\alpha, \beta \in J^1L$. Additionally, $\mathrm{d}_{(S,L)}\omega = 0$ and , where $\mathrm{d}_{(S,L)}$ is the de Rham differential with coefficients in the tautological representation $\operatorname{im}(J^\sharp) \to DL$.

We have seen in Lemma 1.2.53, that a generalized contact bundle always comes together with a canonical Jacobi bundle. Assuming that this Jacobi bracket is weakly regular, another structure appears: *transversally complex subbundles.*

Definition 4.1.5 *A transversally complex subbundle on $L \to M$ is a pair (S, K) consisting of two involutive subbundles $S \subseteq DL$ and $K \subseteq D_{\mathbb{C}}L$, such that*

i.) $K + \overline{K} = D_{\mathbb{C}}L$,

ii.) $K \cap \overline{K} = S_{\mathbb{C}}$.

Remark 4.1.6 The name *transversally complex subbundle* comes from the fact, that the decomposition

$$({}^{DL}\!/\!{}_{S})_{\mathbb{C}} = ({}^{K}\!/\!{}_{S_{\mathbb{C}}}) \oplus ({}^{\overline{K}}\!/\!{}_{S_{\mathbb{C}}})$$

defines an almost complex structure on ${}^{DL}\!/\!{}_{S}$.

We are mainly interested in transversally complex structures with an additional Jacobi structure. So let us be precise in the following

Definition 4.1.7 *Let $L \to M$ be a line bundle. A transversally complex Jacobi structure is a pair (J, K) consisting of a weakly regular Jacobi structure $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ and an involutive subbundle $K \subset D_{\mathbb{C}}L$, such that $(\mathrm{im}(J^\sharp), K)$ is transversally complex subbundle.*

This kind of structure appear naturally in generalized contact geometry if one assumes some regularity conditions, to be seen in the next

Proposition 4.1.8 *Let $(L \to M, \mathcal{L})$ be a generalized contact bundle, whose corresponding Jacobi structure J is weakly regular. Then $(J, \mathrm{pr}_D\mathcal{L})$ is a transversally complex Jacobi structure.*

Proof: Let us define $K := \mathrm{pr}_D(\mathcal{L})$. Having in mind that $\mathrm{im}(J^\sharp)_{\mathbb{C}} = \mathrm{pr}_D(\mathcal{L}) \cap \mathrm{pr}_D(\overline{\mathcal{L}})$ and that $\mathrm{pr}_D(\mathcal{L})$ is involutive (due to Lemma 1.2.58), we get the result.

It is now natural to ask which transversally complex Jacobi structure can be induced by a generalized contact structure. To formalize the term "*induced by*", we use the proof of the Proposition 4.1.8.

Definition 4.1.9 *Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ be a a generalized contact structure, and (J, K) be a transversally complex Jacobi structure. We say $\mathcal{L}$ induces (J, K), if J is the Jacobi structure of $\mathcal{L}$ and $K = \mathrm{pr}_D\mathcal{L}$.*

Let us give a first necessary condition of a transversally complex Jacobi structure induced by a generalized contact structure.

Lemma 4.1.10 *Let $L \to M$ be a line bundle and let (J, K) be transversally complex Jacobi structure induced by the generalized contact structure $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$. Then*

i.) for any ω, which is the inverse of J restricted to $\mathrm{pr}_D(\mathcal{L})$, there exists a real $B \in \Omega^2_L(M)$, such that

$$\mathrm{d}_L(\mathrm{i}\omega + B)(\Delta_1, \Delta_2, \Delta_3) = 0 \quad \forall \Delta_i \in K.$$

ii.) $\mathcal{L} = (K \oplus \mathrm{Ann}(K))^{\mathrm{i}\omega + B}$

Proof: Every generalized contact structure $\mathcal{L}$ can be written by a two form $\tilde{\varepsilon} \colon \Lambda^2 \mathrm{pr}_D\mathcal{L} \to L_{\mathbb{C}}$ by

$$\mathcal{L} = \{(\Delta, \alpha) \in \mathbb{D}_{\mathbb{C}}L \mid \alpha\big|_{\mathrm{pr}_D\mathcal{L}} = \iota_\Delta \tilde{\varepsilon}\big|_{\mathrm{pr}_D\mathcal{L}}\},$$

such that $\mathrm{Im}(\tilde{\varepsilon})\big|_{\Lambda^2 S} = (J\big|_S)^{-1}$ for the Jacobi structure J of the generalized contact structure, so in our case with the given weakly regular one. The proof of this can be found in [41, Section 2.2.3]. If the generalized contact structure induces the given transversally complex Jacobi structure, then we have $\mathrm{pr}_D(\mathcal{L}) = K$. Let us now consider an extension of the inverse of J and denote it by ω. Since K is regular, we extend $\tilde{\varepsilon}$ to a Atiyah 2-form ε, such that $\mathsf{Im}(\varepsilon) = \omega$ and get

$$\begin{aligned} \mathcal{L} &= (\mathrm{pr}_D(\mathcal{L}) \oplus \mathrm{Ann}(\mathrm{pr}_D(\mathcal{L})))^{\varepsilon} \\ &= (K \oplus \mathrm{Ann}(K))^{\mathrm{Re}(\varepsilon) + \mathrm{i}\mathrm{Im}(\varepsilon)}, \end{aligned}$$

which is the first statement, since $\mathrm{Im}(\varepsilon)$ extends the inverse of J. The second statement follows directly from the integrability of $\mathcal{L}$. ΧΞΣ

To conclude this section, we collect all the previous results in the following

Corollary 4.1.11 *Let $L \to M$ be a line bundle and let (J, K) be a transversally complex Jacobi structure. These data come from a generalized contact structure, if and only if there exists a real extension ω of the inverse of J and a real $B \in \Omega^2_L(M)$, such that*

$$\mathrm{d}_L(\mathrm{i}\omega + B)(\Delta_1, \Delta_2, \Delta_3) = 0 \; \forall \Delta_i \in K.$$

The condition of Corollary does not seem to be very easy to handle and also involves all extensions of the inverse of the Jacobi structure and the existence of a 2-form B, whcih in practice can be very to check. We will see in the following that the of ω and B existence can be encoded in some properties of J.

4.2 The Spectral Sequence of a Transversally Complex Subbundle

We have seen that every generalized contact structure with weakly regular Jacobi structure induces a transversally complex Jacobi structure. The latter is special case of a transversally complex subbundles. We want to explore these subbundles by means of a canonical spectral sequence attached to them. It turns out that that the existence of a transversally complex Jacobi structure is a necessary (but insufficient) condition for the existence of a generalized contact structure inducing these structures.

Throughout this subsection we will assume the following data: a line bundle $L \to M$ and the subalgebroids $S \subseteq DL$ and $K \subseteq D_{\mathbb{C}}L$, such that $K + \overline{K} = D_{\mathbb{C}}L$ and $K \cap \overline{K} = S_{\mathbb{C}}$, in other words we want to fix a transversally complex subbundle (S, K). Moreover, if not stated otherwise, we see every Atiyah form as complex.

4.2.1 General Statements and Preliminaries

This part is not only meant to fix notation and give a quick reminder on spectral sequences, but also to give a splitting of the zeroth and first page of the spectral sequence induced by a transversally complex subbundle (S, K). Let us begin by showing that (S, K) induces two filtrations of the complex $\Omega_L(M)$.

Lemma 4.2.1 *The subspaces*

$$F_n^m := \{\alpha \in \Omega_L^m(M) \mid \iota_X\alpha = 0 \ \forall X \in \Lambda^{m-n+1} S_{\mathbb{C}}\} \text{ and}$$
$$G_n^m := \{\alpha \in \Omega_L^m(M) \mid \iota_X\alpha = 0 \ \forall X \in \Lambda^{m-n+1} K\}$$

fulfill the following properties

i.) $\Omega_L^m(M) = F_0^m \supseteq F_1^m \supseteq \dots$ *and* $\Omega_L^m(M) = G_0^m \supseteq G_1^m \supseteq \dots,$

ii.) $\mathrm{d}_L(F_n^m) \subseteq F_n^{m+1}$ *and* $\mathrm{d}_L(G_n^m) \subseteq G_n^{m+1}$.

Moreover, we have the following relations between them:

(I) $G_n^m \subseteq F_n^m$

(II) $G_j^m \cap \overline{G}_i^m \subseteq F_{i+j}^m$

(III) $\langle G_j^m \cap \overline{G}_i^m \rangle_{i+j=n} = F_n^m$

(IV) $G_j^m \cap \overline{G}_i^m \cap (\langle G_l^m \cap \overline{G}_k^m \rangle_{i+j=n,(i,j)\neq(k,l)} + F_{i+j+1}^m) \subseteq F_{i+j+1}^m$

(V) $F_{i+j+n}^m \cap G_j^m \cap \overline{G}_i^m = \langle G_{j+k}^m \cap \overline{G}_{i+l}^m \rangle_{k+l=n}$

PROOF: The proof is an easy verification exploiting the involutivity of S and K and the relations $K + \overline{K} = D_{\mathbb{C}}L$ and $K \cap \overline{K} = S_{\mathbb{C}}$. ΧΞΣ

The properties *i.)* and *ii.)* in the previous Lemma show that the subspaces F_n^m and G_n^m induce filtrations of the Der-complex. Note that we did not explicitly introduce the spaces $\overline{G}_i^m$, but from the notation it should be clear that we mean the complex conjugation of the spaces G_i^m or equivalently the filtered complex with respect to $\overline{K}$. Properties *(I)-(V)* will give us a canonical splitting of the spectral sequence and its differentials.

Let us briefly recall the definition of a spectral sequence

Definition 4.2.2 *A spectral sequence is a sequence of bigraded vector spaces $\{E_r^{\bullet,\bullet}\}_{r\geq 0}$, and a sequence of maps $\{\mathrm{d}_r\colon E_r^{\bullet,\bullet}\to E_r^{\bullet+r,\bullet+1-r}\}_{r\geq 0}$, the differentials, such that*

i.) $(\mathrm{d}_r)^2=0$

ii.) $E_r^{\bullet,\bullet}\cong\frac{\ker(\mathrm{d}_{r-1}\colon E_{r-1}^{\bullet,\bullet}\to E_{r-1}^{\bullet+r-1,\bullet-r+2})}{\mathrm{im}(\mathrm{d}^{r-1}\colon E_{r-1}^{\bullet-r+1,\bullet-2+r}\to E_{r-1}^{\bullet,\bullet})}$

There is a canonical way to associate a spectral sequence to a filtered complex. We will define it for the filtered complex $\Omega_L^m(M)=F_0^m\supseteq F_1^m\supseteq\dots$. We consider the quotients

$$E_r^{p,q}=\frac{\{\alpha\in F_p^{q+p}\mid \mathrm{d}_L\alpha\in F_{p+r}^{q+p+1}\}}{F_{p+1}^{q+p}+\mathrm{d}_L(F_{p+1-r}^{q+p-1})}$$

together with the maps

$$\mathrm{d}_r\colon E_r^{p,q}\ni\alpha+F_{p+1}^{q+p}+\mathrm{d}_L(F_{p+1-r})\mapsto \mathrm{d}_L\alpha+F_{p+r+1}^{q+p+1}+\mathrm{d}_L(F_{p+1}^{q+p+1})\in E_r^{p+r,q+1-r}. \tag{4.2.1}$$

Lemma 4.2.3 *The maps $\{\mathrm{d}_r\}_{r\geq 0}$ from Equation 4.2.1 are well-defined and $\{(E_r^{\bullet,\bullet},\mathrm{d}_r)\}_{r\geq 0}$ is a spectral sequence.*

PROOF: The proof is a easy exercise, but can be found in every book treating spectral sequences of filtered complexes, see e.g. [50]. ΧΞΣ

In our case, we do not have only one filtered complex, but two more filtered complexes $\Omega_L^m(M)=G_0^m\supseteq G_1^m\supseteq\dots$ and its complex conjugate. Actually, there is a relation with $\Omega_L^m(M)=F_0^m\supseteq F_1^m\supseteq\dots$, to see this we consider

$$E_r^{(i,j),q}=\frac{\{\alpha\in G_j^{q+i+j}\cap\overline{G}_i^{q+i+j}\mid \mathrm{d}_L\alpha\in F_{i+j+r}^{q+i+j+1}\}}{(F_{i+j+1}^{q+i+j}+\mathrm{d}_L(F_{i+j+1-r}^{q+i+j-1}))\cap G_j^{q+i+j}\cap\overline{G}_i^{q+i+j}}$$

Lemma 4.2.4 *Let $s=0,1$, then the canonical maps*

$$E_s^{(i,j),q}\to E_s^{i+j,q}$$

are injective and moreover $E_s^{p,q}=\bigoplus_{i+j=p}E_s^{(i,j),q}$ for all p,q.

PROOF: Injectivity is straightforward. Let us start with $s = 0$. We have

$$E_0^{(i,j),q} = \frac{G_j^{q+i+j} \cap \overline{G}_i^{q+i+j}}{F_{i+j+1}^{q+i+j} \cap G_j^{q+i+j} \cap \overline{G}_i^{q+i+j}}.$$

By *(III)* of Lemma 4.2.1, we immediately get $E_0^{p,q} = \langle E_0^{(i,j),q} \rangle_{i+j=p}$. Let us now prove that the sum is direct. We consider $\omega_{ij} \in G_j^{q+i+j} \cap \overline{G}_i^{q+i+j}$ for $i + j = p$, such that

$$\sum_{i+j=p} \omega_{ij} \in F_{p+1}^{q+p}.$$

We have that

$$\omega_{kl} \in G_l^{q+p} \cap \overline{G}_k^{q+p} \cap \left(\langle G_j^{q+p} \cap \overline{G}_i^{q+p} \rangle_{k+l=n,(i,j)\neq(k,l)} + F_{p+1}^{q+p} \right)$$

for every choice of $k + l = i + j$ and hence, using *(IV)* of Lemma 4.2.1, $\omega_{kl} \in F_{p+1}^{q+p}$. So $\omega_{kl} = 0$ on the level of equivalence classes for all k, l and we get the result for $s = 0$.

Let us pass to $s = 1$ and let $\omega \in F_{i+j}^{q+i+j}$ such that $\mathrm{d}_L\omega \in F_{i+j+1}^{q+i+j+1}$. Since $F_{i+j}^{q+i+j} = \langle \langle G_l^{q+i+j} \cap \overline{G}_k^{q+i+j} \rangle_{k+l=i+j}$, we can find $\omega_{kl} \in G_l^{q+i+j} \cap \overline{G}_k^{q+i+j}$ for $k+l = i+j$, such that

$$\omega = \sum_{k+l=i+j} \omega_{kl}.$$

We have that $\mathrm{d}_L\omega_{kl} \in G_l^{q+i+j+1} \cap \overline{G}_k^{q+i+j+1}$ similarly as in case $s = 0$, we can prove that actually $\mathrm{d}_L\omega_{kl} \in F_{i+j+1}^{q+i+j+1}$, using $\mathrm{d}_L\omega \in F_{i+j+1}^{q+i+j+1}$. Thus

$$\omega \in \left\langle \{\alpha \in G_k^{q+i+j} \cap \overline{G}_l^{q+i+j} \mid \mathrm{d}_L\alpha \in F_{i+j+r}^{q+i+j+1}\} \right\rangle_{k+l=i+j}$$

and hence $E_1^{i+j,q} = \langle E_1^{(k,l),q} \rangle_{k+l=i+j}$. Let now $\omega_{kl} \in G_l^{q+i+j} \cap \overline{G}_k^{q+i+j}$ for $k + l = i + j$, such that $\mathrm{d}_L\omega_{kl} \in F_{i+j+1}^{q+i+j}$ and $\sum_{k+l=i+j} \omega_{kl} \in F_{i+j+1}^{q+i+j} + \mathrm{d}_L(F_{i+j}^{q+i+j-1})$. Therefore there exists $\alpha \in F_{i+j}^{q+i+j-1}$, such that $\sum_{k+l=i+j} \omega_{kl} + \mathrm{d}_L\alpha \in F_{i+j+1}^{q+i+j}$. Splitting $\alpha = \sum_{k+l=i+j} \alpha_{kl}$ for some $\alpha_{kl} \in G_k^{q+i+j-1} \cap \overline{G}_l^{q+i+j-1}$, we get that

$$\sum_{k+l=i+j} \omega_{kl} + \mathrm{d}_L\alpha_{kl} \in F_{i+j+1}^{q+i+j}.$$

Additionally we have $\omega_{kl} + \mathrm{d}_L\alpha_{kl} \in G_l^{q+i+j} \cap \overline{G}_k^{q+i+j}$. Applying the same argument as in the case $s = 0$, we get $\omega_{kl} + \mathrm{d}_L\alpha_{kl} \in F_{i+j+1}^{q+i+j}$ for all $k + l = i + j$. Passing to equivalence classes, we get the result for $s = 1$. ΧΞΣ

We consider the differentials on the zeroth and first page and use this splitting to decompose them.

Proposition 4.2.5 *For the differentials* $\mathrm{d}_0\colon E_0^{p,q} \to E_0^{p,q+1}$ *and* $\mathrm{d}_1\colon E_0^{p,q} \to E_0^{p+1,q}$, *the following hold*

i.) $\mathrm{d}_0(E_0^{(i,j),q}) \subseteq E_0^{(i,j),q+1}$

ii.) $\mathrm{d}_1(E_1^{(i,j),q}) \subseteq E_1^{(i+1,j),q} \oplus E_1^{(i,j+1),q}$

Hence there is a canonical splitting $\mathrm{d}_1 = \partial_1 + \overline{\partial}_1$, *where* $\partial_1(E_1^{(i,j),q}) \subseteq E_1^{(i+1,j),q}$ *and* $\overline{\partial}_1(E_1^{(i,j),q}) \subseteq E_1^{(i,j+1),q}$. *Finally,* $(\partial_1)^2 = (\overline{\partial}_1)^2 = \partial_1\overline{\partial}_1 + \overline{\partial}_1\partial_1 = 0$.

PROOF: We start with the zeroth page. Let $\omega + F_{i+j+1}^{q+i+j} \in E^{(i,j),q}$, such that $\omega \in G_j^{q+i+j} \cap \overline{G}_i^{q+i+j}$, then

$$\mathrm{d}_0(\omega + F_{i+j+1}^{q+i+j}) = \mathrm{d}_L\omega + F_{i+j+1}^{q+i+j+1}.$$

We have that $\mathrm{d}_L\omega \in G_j^{q+i+j+1} \cap \overline{G}_i^{q+i+j+1}$ and hence $\mathrm{d}_0(\omega + F_{i+j+1}^{q+i+j}) \in E^{(i,j),q+1}$. For the first page let us choose $\omega + F_{i+j+1}^{q+i+j} + \mathrm{d}_L(F_{i+j}^{q+i+j-1})$, with $\omega \in G_j^{q+i+j} \cap \overline{G}_i^{q+i+j}$ and $\mathrm{d}_L\omega \in F_{i+j+1}^{q+i+j+1}$. Then

$$\mathrm{d}_L\omega \in G_j^{q+i+j+1} \cap \overline{G}_i^{q+i+j+1} \cap F_{i+j+1}^{q+i+j+1} = G_{j+1}^{q+i+j+1} \cap \overline{G}_i^{q+i+j+1} + G_j^{q+i+j+1} \cap \overline{G}_{i+1}^{q+i+j+1}$$

and the claim follows by (V) of Lemma 4.2.1. ΧΞΣ

4.2.2 The Obstruction Class of Transversally Complex Subalgebroids

In Section 4.1, we have seen that a transversally complex Jacobi structure (J, K) comes from a generalized contact structure, if and only if there exists an extension of the inverse of J, $\omega \in \Omega_L^2(M)$, and a real 2-form $B \in \Omega_L^2(M)$, such that

$$\mathrm{d}_L(\mathrm{i}\omega + B)(\Delta_1, \Delta_2, \Delta_3) = 0 \ \forall \Delta_i \in K. \tag{$*$}$$

We want to apply the techniques from the previous subsection to obtain a cohomological obstruction for this condition to hold.

Using the formalism of Subsection 4.2.1 and using the notation $\mathrm{im}(J^\sharp) = S$, we see that $(*)$ is equivalent to

$$\mathrm{d}_L(\mathrm{i}\omega + B) \in G_1^3 = G_1^3 \cap \overline{G}_0^3.$$

Using the non-degeneracy of ω on S, we have that $\omega \in F_0^2$ and $\omega \notin F_1^2$. Thus, $\mathrm{i}\omega + B \in G_0^2 \cap \overline{G}_0^2$ and $\mathrm{d}_L\omega, \mathrm{d}_L B \in F_1^3$. Hence both forms define classes in $\in E_0^{(0,0),2}$, denoted by $[\omega]_0$ and $[B]_0$. Note that for two real extensions ω, ω' of the inverse of J^{-1} we have that $[\omega]_0 = [\omega']_0$. So the class $[\omega]_0$ only depends on J. Therefore, we write $J^{-1} \in E_0^{(0,0),2}$ without further comment.

Moreover, $[\omega]_0$ and $[B]_0$ are d_0-closed and hence they define iterated classes in $E_1^{(0,0),2}$, denoted by $[[\omega]_0]_1 = [J^{-1}]_1$ and $[[B]_0]_1$.

Corollary 4.2.6 *The condition* $\mathrm{d}_L(\mathrm{i}\omega + B) \in G_1^3 = G_1^3 \cap \overline{G}_0^3$ *is equivalent to*

$$\partial_1(\mathrm{i}[[\omega]_0]_1 + [[B]_0]_1) = 0.$$

Proof: We have that $\mathrm{d}_L(\mathrm{i}\omega + B) \in G_1^3 \cap \overline{G}_0^3$, which implies

$$\mathrm{d}_1(\mathrm{i}\omega + B + F_1^2 + \mathrm{d}_L(F_0^1)) = \mathrm{d}_L(\mathrm{i}\omega + B) + F_2^3 + \mathrm{d}_L(F_1^2) \in G_1^3 \cap G_0^3 + F_2^3 + \mathrm{d}_L(F_1^2)$$

Hence $\mathrm{d}_1(\mathrm{i}[[\omega]_0]_1 + [[B]_0]_1) \in E_1^{(0,1),2}$. Using the splitting of the differential $\mathrm{d}_1 = \partial_1 + \overline{\partial}_1$, we get that $\partial_1(\mathrm{i}[[\omega]_0]_1 + [[B]_0]_1) = 0$. The converse works by reading the equation from the bottom to the top. ΧΞΣ

We want to go a step further and ask for which ω can we find a B, such that $\mathrm{d}_L(\mathrm{i}\omega + B) \in G_1^3 \cap \overline{G}_0^3$. The answer is contained in the following

Lemma 4.2.7 *Let* $\omega \in \Omega_L^2(M)$ *be real, such that* $\mathrm{d}_L\omega \in F_1^3$. *Then there exists a* $B \in \Omega_L^2(M)$, *such that* $\mathrm{d}_L(\mathrm{i}\omega + B) \in G_1^3 = G_1^3 \cap \overline{G}_0^3$ *if and only if*

i.) $\partial_1\overline{\partial}_1[[\omega]_0]_1 = 0$

ii.) $\overline{\partial}_1[[\omega]_0]_1 - \partial_1[[\omega]_0]_1$ *is* d_1*-exact.*

Remark 4.2.8 Note that *ii.)* can only be fulfilled, if *i.)* is fulfilled, since *i.)* just ensures that $\overline{\partial}_1[[\omega]_0]_1 - \partial_1[[\omega]_0]_1$ is d_1-closed.

Proof (of Lemma 4.2.7): Let us first assume, that $\mathrm{d}_L(\mathrm{i}\omega + B) \in G_1^3 = G_1^3 \cap \overline{G}_0^3$ for a real B, which is equivalent to $\partial_1(\mathrm{i}[[\omega]_0]_1 + [[B]_0]_1) = 0$ by Proposition 4.2.6. Hence we have

$$\begin{aligned}
2\,\mathrm{d}_1[[B]_0]_1 &= 2\mathrm{Re}(\mathrm{d}_1[[\mathrm{i}\omega + B]_0]_1) \\
&= 2\mathrm{Re}(\overline{\partial}_1[[\mathrm{i}\omega + B]_0]_1) \\
&= (\overline{\partial}_1[[\mathrm{i}\omega + B]_0]_1 + \overline{\overline{\partial}_1[[\mathrm{i}\omega + B]_0]_1}) \\
&= \mathrm{d}_1[[B]_0]_1 + \mathrm{i}(\overline{\partial}^1[[\omega]_0]_1 - \partial_1[[\omega]_0]_1),
\end{aligned}$$

and hence $\overline{\partial}_1[[\omega]_0]_1 - \partial_1[[\omega]_0]_1$ is d_1-exact. Assuming, on the other hand, that $\mathrm{i}(\overline{\partial}_1[[\omega]_0]_1 - \partial_1[[\omega]_0]_1) = \mathrm{d}_1[B]_1$. Note that $\mathrm{i}(\overline{\partial}_1[[\omega]_0]_1 - \partial_1[[\omega]_0]_1)$ is real and thus we can choose a real representant B of $[[B]_0]_1$, then it is easy to see that $\partial_1[[\mathrm{i}\omega + B]_0]_1 = 0$ and the claim follows. ΧΞΣ

Let us conclude this section with the main theorem of this chapter, which is basically just a summary of the previous results. Afterwards we will discuss the connection to generalized complex structures.

Theorem 4.2.9 *Let* $L \to M$ *be a line bundle and let* (J, K) *be a transversally complex Jacobi structure on* L. *These data are induced by a generalized contact structure, if and only if* *i.)* $\partial_1\overline{\partial}_1[J^{-1}]_1 = 0$

ii.) $\overline{\partial}_1[J^{-1}]_1 - \partial_1[J^{-1}]_1$ *is* d^1*-exact,*

where we interpret J^{-1} *as an element in* $E_0^{(0,0),2}$. *Moreover, the generalized contact structure inducing the data is of the form*

$$\mathcal{L} = (K \oplus \mathrm{Ann}(K))^{\mathrm{i}\omega + B}$$

for any choice of $\omega \in J^{-1}$ *real and any real* $B \in \Omega^2_L(M)$ *such that* $[B]_0$ *is closed and* $\mathrm{d}_1[[B]_0]_1 = \mathrm{i}(\overline{\partial}_1[J^{-1}]_1 - \partial_1[J^{-1}]_1)$.

Corollary 4.2.10 *Let* $L \to M$ *be a line bundle and let* (J, K) *be a transversally complex Jacobi structure. If* $[J^{-1}]_1 = 0$*, then the data comes from a generalized contact structure of the form*

$$\mathcal{L} = (K \oplus \mathrm{Ann}(K))^{\mathrm{i}\omega},$$

where $\omega \in J^{-1}$.

Remark 4.2.11 (Generalized Complex Geometry) Let us recall the mirror result in generalized complex geometry. In [5] the author obtains similar results, given a regular Poisson structure $\pi \in \Gamma^\infty(\Lambda^2 TM)$ and a transversally complex distribution, i.e. an involutive subbundle $K \subseteq T_{\mathbb{C}}M$ such that $S_{\mathbb{C}} := \mathrm{im}(\pi^\sharp)_{\mathbb{C}} = K \cap \overline{K}$. Now S and K induce filtrations of the de Rham complex and hence give rise to spectral sequences which are very similar to the ones obtained in Subsection 4.2.1. All the proofs of the splitting of the spectral sequence and the differentials can be obtained in the same exact way as in Subsection 4.2.1. Adapting the notations, we find that the data comes from a generalized complex structure, if and only if

i.) $\partial_1\overline{\partial}_1[\pi^{-1}]_1 = 0$

ii.) $\overline{\partial}_1[\pi^{-1}]_1 - \partial_1[\pi^{-1}]_1$ is d_1-exact.

These obstructions differ quite a lot from those found in [5]. The reason for this is that in [5] the author searches for H-generalized complex structures which is a generalization of generalized complex structure, while our approach gives obstructions to find an honest generalized complex structures. This is not a difference in the case of generalized contact geometry, but in fact it is in generalized complex geometry. It is an easy exercise to see that there is an H-generalized complex structure inducing (π, K), if and only if

i.) $\mathrm{d}_1(\overline{\partial}_1[\pi^{-1}]_1 - \partial_1[\pi^{-1}]_1) = 0$

ii.) $\mathrm{d}_2[\overline{\partial}_1[\pi^{-1}]_1 - \partial_1[\pi^{-1}]_1]_2 = 0$

iii.) $\mathrm{d}_3[[\overline{\partial}_1[\pi^{-1}]_1 - \partial_1[\pi^{-1}]_1]_2]_3 = 0.$

To be more precise *ii.)* is only well-defined, if *i.)* is fulfilled and *iii.)* is only well-defined, if *ii.)* is fulfilled. These obstructions are equivalent to the ones found in [5]. It is a bit of a computational effort to prove this, since the author used a transversal to $\mathrm{im}(\pi^\sharp)$ to obtain his results. We want to stress that, as we work completely within the spectral sequence, this is not really necessary and we prefer not to make this arbitrary choice.

4.3 Examples I: five dimensional Nilmanifolds

Weakly regular Jacobi structures appear as invariant Jacobi structures on Lie groups, which are even canonically regular. We begin this section defining invariant Jacobi structures and invariant generalized contact structures. Afterwards, we will formulate everything at the level of Lie algebras.

Definition 4.3.1 *Let $L \to G$ be a line bundle over a Lie group G and let $\Phi\colon G \to \mathsf{Aut}(L)$ be a smooth Lie group action covering the left multiplication. A Jacobi bracket $\{-,-\}\colon \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$ is said to be invariant, if*

$$\Phi_g^*\{\lambda, \mu\} = \{\Phi_g^*\lambda, \Phi_g^*\mu\} \quad \forall \lambda, \mu \in \Gamma^\infty(L), \ \forall g \in G.$$

Let us put invariant Jacobi structures in the context of invariant generalized contact bundles. Our arena is the omni-Lie algebroid $DL \oplus J^1L$ for a line bundle $L \to G$ over a Lie group. For a Lie group action $\Phi\colon G \to \mathsf{Aut}(L)$ covering the left multiplication, we have the the canonical action

$$\mathbb{D}\Phi_g\colon \mathbb{D}L \ni (\Delta, \psi) \mapsto (D\Phi_g(\Delta), (D\Phi_{g^{-1}})^*\psi) \in \mathbb{D}L.$$

Definition 4.3.2 *Let $L \to G$ be a line bundle over a Lie group G and let $\Phi\colon G \to \mathsf{Aut}(L)$ be a Lie group action covering the left multiplication. A generalized contact structure $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ is said to be G-invariant, if and only if $\mathbb{D}\Phi_g(\mathcal{L}) = \mathcal{L}$ for all $g \in G$.*

Proposition 4.3.3 *Let $L \to G$ be a line bundle over a Lie group G, let $\Phi\colon G \to \mathsf{Aut}(L)$ be a Lie group action covering the left multiplication and let $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ be a G-invariant generalized contact structure, then its Jacobi-structure is G-invariant.*

A Lie group action $\Phi\colon G \to \mathsf{Aut}(L)$ allows us to trivialize the line bundle itself, its derivations and its first jet. Similarly to the tangent bundle of a Lie group, we have

$$L \cong G \times \ell$$

where $\ell = \Gamma^\infty(L)^G$. Note that ℓ is a 1-dimensional vector space over $\mathbb{R}$. Moreover, in this trivialization the action of G looks like

$$\Phi_g(h, l) = (gh, l) \quad \forall (h, l) \in G \times \ell, \ \forall g \in G.$$

Additionally the Atiyah algebroid is also a trivial vector bundle by

$$DL \cong G \times \Gamma^\infty(DL)^G,$$

where $\Gamma^\infty(DL)^G$ is a $(\dim(G)+1)$-dimensional vector space over $\mathbb{R}$. Moreover, since the symbol maps invariant derivations to left-invariant vector fields, we have the *G-invariant Spencer sequence* for $\mathsf{Lie}(G) = \mathfrak{g}$

$$0 \to \mathbb{R} \to \Gamma^\infty(DL)^G \to \mathfrak{g} \to 0,$$

by using the fact that G-invariant endomorphisms are just multiplications by constants. This sequence splits canonically, since, we have $\Gamma^\infty(L) \cong \mathscr{C}^\infty(M) \otimes_{\mathbb{R}} \ell$. Thus

$$\Gamma^\infty(DL)^G \cong \mathfrak{g} \oplus \mathbb{R},$$

with bracket

$$[(\xi, r), (\eta, k)] = ([\xi, \eta], 0) \quad \forall (\xi, r), (\eta, k) \in \mathfrak{g} \oplus \mathbb{R}.$$

Similarly, using $J^1L = (DL)^* \otimes L$, plus the choice of a basis of ℓ, we get

$$\Gamma^\infty(J^1L)^G \cong \mathfrak{g}^* \oplus \mathbb{R}.$$

The differential d_L reduces to

$$\mathrm{d}_L(\alpha + k\mathbb{1}^*) = \delta_{CE}\alpha + \mathbb{1}^* \wedge \alpha,$$

where $\alpha \in \mathfrak{g}^*$ and $\mathbb{1}^*$ is again the projection to the $\mathbb{R}$-component. These are all the ingredients, we need to describe G-invariant generalized contact structures via their infinitesimal data, i.e. in terms of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Using the trivialization, we see immediately that for two invariant sections $(\Delta_i, \psi_i) \in \Gamma^\infty(\mathbb{D}L)^G$

$$[\![(\Delta_1, \psi_1), (\Delta_2, \psi_2)]\!] \in \Gamma^\infty(\mathbb{D}L)^G$$

by naturality of the Dorfman-bracket. It is easy to see that the bracket has the form of the bracket of the following

Definition 4.3.4 *Let $\mathfrak{g}$ be a Lie algebra with the abelian extension $\mathfrak{g}_{\mathbb{R}} := \mathfrak{g} \oplus \mathbb{R}$, where we denote by $\mathbb{1}$ and $\mathbb{1}^*$ the canonical elements in $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}^*_{\mathbb{R}}$, respectively. The omni-Lie algebra of $\mathfrak{g}$ is the vector space $\mathfrak{g}_{\mathbb{R}} \oplus \mathfrak{g}^*_{\mathbb{R}}$ together with*

i.) the (Dorfman-like) bracket

$$[\![(X_1, \psi_1), (X_2, \psi_2)]\!] = ([X_1, X_2], \mathscr{L}_{X_1}\psi_2 - \iota_{X_2}\mathrm{d}\psi_1)$$

ii.) the non-degenerate pairing

$$\langle\!\langle (X_1, \psi_1), (X_2, \psi_2) \rangle\!\rangle := \psi_1(X_2) + \psi_2(X_1)$$

iii.) the canonical projection $\mathrm{pr}_D\colon \mathfrak{g}_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R}}^* \to \mathfrak{g}_{\mathbb{R}}$

Here the differential is given by

$$\mathrm{d}(\alpha + \mathbb{1}^* \wedge \beta) = \delta_{\mathrm{CE}}\alpha + \mathbb{1}^* \wedge (\alpha - \delta_{\mathrm{CE}}\beta)$$

for $\alpha, \beta \in \mathfrak{g}^*$ *and* $\mathscr{L}_X = [\iota_X, \mathrm{d}]$.

Proposition 4.3.5 *Let* $L \to G$ *be a line bundle over a Lie group* G *and let* $\Phi\colon G \to \mathsf{Aut}(L)$ *be a Lie group action. A* G*-invariant generalized contact structure* $\mathcal{L} \subseteq \mathbb{D}_{\mathbb{C}}L$ *is equivalently described by a subspace* $\mathcal{L}^{\mathfrak{g}} \subseteq [(\mathfrak{g} \oplus \mathbb{R}) \oplus (\mathfrak{g}^* \oplus \mathbb{R})]_{\mathbb{C}}$ *defined by its invariant sections, which is maximally isotropic, fulfills* $\mathcal{L}^{\mathfrak{g}} \cap \overline{\mathcal{L}^{\mathfrak{g}}}$ *and is involutive with respect to the Dorfman-bracket.*

PROOF: The proof is based on the fact that an invariant generalized contact structure is completely characterized by its invariant sections. ΧΞΣ

The idea is now to forget about the Lie group and perform every construction directly on the Lie algebra, having in mind, of course, that we can reconstruct a generalized contact structure on the Lie group by translating. Being a bit more precise, we give the following

Definition 4.3.6 *Let* $\mathfrak{g}$ *be a Lie algebra. A generalized contact structure on* $\mathfrak{g}$ *is a subbundle* $\mathcal{L} \in (\mathfrak{g}_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R}}^*)_{\mathbb{C}}$*, which is involutive, maximally isotropic and fulfills* $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$.

From the above discussion, we can immediatly obtain

Lemma 4.3.7 *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$*. The left translations establish a* $1:1$*-correspondence between generalized contact structures on* $\mathfrak{g}$ *and left-invariant generalized contact structures on* $G \times \mathbb{R} \to G$.

As in the geometric setting we have extreme cases

Example 4.3.8 Let $(\mathfrak{g}, \Theta)$ be a $(2n+1)$-dimensional contact Lie algebra, i.e. $\Theta \in \mathfrak{g}^*$, such that $\Theta \wedge (\delta_{CE}\Theta)^n \neq 0$, then we denote by $\Omega = p^*\Theta$ for the projection $p\colon \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}$ and get that

$$\mathcal{L} = \{(X, \mathrm{i}\iota_X \,\mathrm{d}\Omega) \in (\mathfrak{g}_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R}}^*)_{\mathbb{C}} \mid X \in (\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}\}$$

gives $\mathfrak{g}$ the structure of a generalized contact Lie algebra.

Example 4.3.9 Let $\mathfrak{g}$ be a Lie algebra and $\phi \in \mathsf{End}(\mathfrak{g}_{\mathbb{R}})$ be a complex structure, then

$$\mathcal{L} = \mathfrak{g}_{\mathbb{R}}^{(1,0)} \oplus \mathrm{Ann}(\mathfrak{g}_{\mathbb{R}}^{(1,0)})$$

gives $\mathfrak{g}$ the structure of a generalized contact Lie algebra, where $\mathfrak{g}_{\mathbb{R}}^{(1,0)}$ is the $+\mathrm{i}$-Eigenbundle of $\phi_{\mathbb{C}}\colon (\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}} \to (\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}$.

Now, we restrict ourselves to the case of 5-dimensional nilpotent Lie algebras, since we are able to use already existing classification results, which are not available in more general classes of Lie algebras, in order to prove the following

Theorem 4.3.10 *Every five dimensional nilpotent Lie algebra possesses a generalized contact structure.*

From this theorem, we can immediately conclude

Corollary 4.3.11 *Every five dimensional nilmanifold possesses an invariant generalized contact structure.*

To prove Theorem 4.3.10, we will use Section 4.2.2, to be more precise, we will make use of Theorem 4.2.9. In particular, we will find a generalized contact structure on a given Lie algebra by looking for a transversally complex Jacobi structure. Afterwards, we use Theorem 4.2.9 to prove the existence of a generalized contact structure. Note that we did not prove the invariant analogue of Theorem 4.2.9, but as the proof of Theorem 4.2.9 can be performed also in the invariant setting.

A big help in proving Theorem 4.3.10 is the classification of five dimensional nilpotent Lie algebras provided in [18]. In that work the author proved that there are exactly nine (isomorphism classes of) five dimensional nilpotent Lie algebras. Since we want to prove that there are generalized contact structures on all of them, it seems convenient to test first the extreme examples, i.e. integrable complex structures on $\mathfrak{g}_{\mathbb{R}}$ (Example 4.3.9) on the one hand and contact structures on the other hand (Example 4.3.8). For the complex structures we can can use the work of Salamon in [37], where he classified all the complex nilpotent Lie algebras of six dimensions. Of course not every six dimensional nilpotent Lie algebra arises as an abelian extension of a five dimensional one.

In the following, we denote by $\{e_1, \dots, e_5\}$ a given basis of a five dimensional vector space $\mathfrak{g}$. The only 5-dimensional nilpotent Lie algebras, such that $\mathfrak{g}_{\mathbb{R}}$ admits a complex structure are (we use the notation of [18] for the description of 5-dimesional nilpotent Lie algebras):

i.) $\mathfrak{L}_{5,1}$ (abelian)

ii.) $\mathfrak{L}_{5,2} : \; [e_1, e_2] = e_3$

iii.) $\mathfrak{L}_{5,4} : \; [e_1, e_2] = e_5, \; [e_3, e_4] = e_5$

iv.) $\mathfrak{L}_{5,5} : \; [e_1, e_2] = e_3, \; [e_1, e_3] = e_5, \; [e_2, e_4] = e_5$

v.) $\mathfrak{L}_{5,8} : \; [e_1, e_2] = e_4, \; [e_1, e_3] = e_5$

vi.) $\mathfrak{L}_{5,9} : \; [e_1, e_2] = e_3, \; [e_1, e_3] = e_4, \; [e_2, e_3] = e_5$

We have to check that the remaining 5-dimensional nilpotent Lie algebras $\mathfrak{L}_{5,3}$, $\mathfrak{L}_{5,6}$ and $\mathfrak{L}_{5,7}$ are also generalized contact. Let us denote by $\{e^1, \dots, e^5\}$ the dual basis of $\{e_1, \dots, e_5\}$.

4.3.1 $\mathfrak{L}_{5,3} : [e_1, e_2] = e_3,\ [e_1, e_3] = e_4$

It is easy to see that $J = e_3 \wedge e_1 + \mathbb{1} \wedge e_4$ is a Jacobi structure. Additionally

$$K := \langle \mathbb{1}, e_1, e_3, e_4, e_2 - \mathrm{i}e_5 \rangle$$

is a subalgebra of $(\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}$ and that (J, K) is a transversally complex Jacobi structure. Moreover, $\omega = e^1 \wedge e^3 - \mathbb{1}^* \wedge e^4$ is an extension of the inverse of J and we obtain that $\mathrm{d}\omega = \delta_{CE}\omega + \mathbb{1}^* \wedge \omega = 0$, which implies that $[J^{-1}]_1 = [[\omega]_0]_1 = 0$. Using Corollary 4.2.10, we see that there is a generalized contact structure inducing this data, an explicit example is given by

$$(K \oplus \mathrm{Ann}(K))^{\mathrm{i}\omega}.$$

4.3.2 $\mathfrak{L}_{5,6} \colon [e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5,\ [e_2, e_3] = e_5$

This Lie algebra is actually a contact Lie algebra with contact 1-form $\Theta = e^5$.

4.3.3 $\mathfrak{L}_{5,7} \colon [e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5$

It is easy to see that $J = e_1 \wedge e_3 + e_4 \wedge (\mathbb{1} + e_5)$ is a Jacobi structure. Let us define

$$K := \langle \mathbb{1} + e_5, e_1, e_3, e_4, \mathbb{1} + \mathrm{i}e_2 \rangle.$$

We have $[\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}] \subseteq \mathrm{im}(J^\sharp)$ and hence K is integrable. Moreover, we have that $\omega = -(e^1 \wedge e^3 + e^4 \wedge e^5) + (\mathbb{1} - e^5) \wedge e^4$ is an extension of J^{-1} and $\mathrm{d}\omega = 0$. Using Corollary 4.2.10, we find a generalized contact structure given by

$$(K \oplus \mathrm{Ann}(K))^{\mathrm{i}\omega}$$

We have already seen that the Lie algebras $\mathfrak{L}_{5,3}$, $\mathfrak{L}_{5,6}$ and $\mathfrak{L}_{5,7}$ do not admit a complex structure on their one dimensional abelian extension. Moreover, $\mathfrak{L}_{5,6}$ is a contact Lie algebra. In the following we want to show that $\mathfrak{L}_{5,3}$ and $\mathfrak{L}_{5,7}$ do not admit a contact structure, so that there are generalized contact structures on them but not of the extreme types. Let us first collect some basic properties of contact Lie algebras

Theorem 4.3.12 *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\Theta \in \mathfrak{g}^*$ be a contact form. Then the center $Z(\mathfrak{g})$ has dimension one.*

A reference for Theorem 4.3.12 and its proof is [35]. As a first consequence is

Corollary 4.3.13 *The Lie algebra $\mathfrak{L}_{5,3}$ is not contact.*

The only Lie algebra, which is left over is $\mathfrak{L}_{5,7}$. Here we do not have a general statement about contact Lie algebras that we can use, nevertheless we have

Lemma 4.3.14 *The Lie algebra $\mathfrak{L}_{5,7}$ is not contact.*

PROOF: From the commutation relation in Subsection 4.3.3 it is clear that we have $\delta_{CE}(\Lambda^\bullet\mathfrak{g}^*) \subseteq e^1 \wedge \Lambda^\bullet\mathfrak{g}^*$. Hence we have that for all $\alpha \in \mathfrak{g}^*$ $\delta_{CE}\alpha = e^1 \wedge \beta$ for some $\beta \in \mathfrak{g}^*$. As a consequence $\alpha \wedge (\delta_{CE}\alpha)^2 = 0$ for all $\alpha \in \mathfrak{g}^*$, and hence the Lie algebra can not be contact. ΧΞΣ

Remark 4.3.15 To prove that $\mathfrak{L}_{5,1}$, $\mathfrak{L}_{5,2}$, $\mathfrak{L}_{5,8}$ and $\mathfrak{L}_{5,9}$ are not contact one can use Theorem 4.3.12. Finally, for the remaining ones the contact structures are given by e^5.

As a summary we have the following table

	contact	$\mathfrak{g}_{\mathbb{R}}$-complex	generalized contact
$\mathfrak{L}_{5,1}$	×	✓	✓
$\mathfrak{L}_{5,2}$	×	✓	✓
$\mathfrak{L}_{5,3}$	×	×	✓
$\mathfrak{L}_{5,4}$	✓	✓	✓
$\mathfrak{L}_{5,5}$	✓	✓	✓
$\mathfrak{L}_{5,6}$	✓	×	✓
$\mathfrak{L}_{5,7}$	×	×	✓
$\mathfrak{L}_{5,8}$	×	✓	✓
$\mathfrak{L}_{5,9}$	×	✓	✓

We used the term $\mathfrak{g}_{\mathbb{R}}$-complex short for $\mathfrak{g}_{\mathbb{R}}$ admits a complex structure.

4.4 Examples II: Contact Fiber Bundles

The next class of examples are *contact fiber bundles* over a complex base manifold. We begin explaining what we mean by contact fiber bundle. Similarly to symplectic fiber bundles, there is a contact structure on the vertical bundle

$$\mathrm{Ver}_L(P) = \sigma^{-1}(\mathrm{Ver}(P)) \subseteq DL,$$

for a line bundle $L \to P$, such that $P \to M$ is a fiber bundle. More precisely:

Definition 4.4.1 *Let $\pi\colon P \to M$ be a fiber bundle and let $L \to P$ by a line bundle. A smooth family of contact manifolds is the data of $L \to P$ together with a closed non-degenerate 2-form $\omega \in \Gamma^\infty(\Lambda^2(\mathrm{Ver}_L(P))^* \otimes L)$. If additionally the contact structures $(L\big|_{P_m} \to P_m, \omega\big|_{D(L|_{P_m})})$ are contactomorphic, we say that $L \to P$ is a contact fiber bundle.*

Before we come to examples, we want to make some general remarks on smooth families of contact structures and contact fiber bundles, which are more or less known.

Remark 4.4.2 Let $(\pi\colon P \to M, \omega)$ be a smooth family of contact structures. If the fiber is compact and connected and the base is connected, then the data automatically define a contact fiber bundle. This follows from the stability theorem of Gray in [25], which states that two contact forms which are connected by a smooth path of contact structures are contactomorphic.

Remark 4.4.3 As in the setting of symplectic fiber bundles, we can express the data in local terms, namely: the datum of a contact fiber bundle over a manifold M with typical fiber F is equivalent to:

i.) a line bundle $L_F \to F$ and a contact 2-form $\omega \in \Gamma^\infty(\Lambda^2(DL)^* \otimes L)$

ii.) an open cover $\{U_i\}_{i\in I}$

iii.) smooth transition maps $T_{ij}\colon U_i \cap U_j \to \mathsf{Aut}(L_F)$ which are point-wise contactomorphisms

Remark 4.4.4 Obviously, one can define smooth families of contact structures as a Jacobi structure of contact type, such that the characteristic distribution of it is the vertical bundle of a fiber bundle.

Using this remarks, we can show that under certain assumptions on the base, a smooth family of contact structures always induces a generalized contact structure on the total space.

Lemma 4.4.5 *Let $\pi\colon P \to M$ be a fiber bundle with typical fiber F over a complex base M, let $L \to M$ by a line bundle and let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi structure giving P the structure of a smooth family of contact manifolds as in Remark 4.4.4. Then P possesses a generalized contact structure with Jacobi structure J.*

Proof: First of all, we notice that the Jacobi structure is weakly regular, since $\operatorname{im}(J^\sharp) = \operatorname{Ver}_L(P)$. The only thing what we have to show is that the data induce a transversally complex Jacobi structure, since we have that the inverse of the Jacobi structure is leaf-wise exact with canonical primitive $\iota_{\mathbb{1}}\omega$, which implies that $[J^{-1}]_1 = 0$ by Corollary 4.2.10. Let us denote by $T^{(1,0)}M \subseteq T_{\mathbb{C}}M$ the holomorphic tangent bundle induced by the complex structure on M. With this we define

$$K := (\sigma \circ T\pi)^{-1}(T^{(1,0)}M) \subseteq D_{\mathbb{C}}L.$$

It is an easy consequence of the definitions of the bundles that (J, K) is a transversally complex Jacobi structure. This concludes the proof. ΧΞΣ

We see that in this case the existence of a generalized contact structure is unobstructed. Let us show that smooth families of contact structure over a complex base do actually exist.

Example 4.4.6 (Projectivized Vertical Bundle) Let $\pi\colon P \to M$ be a fiber bundle with typical fiber F over a complex base M. Given a set of local trivializations $(U_i, \tau_i)_{i\in I}$

$$\begin{tikzcd} P\big|_{U_i} \arrow[rr, "\tau_i"] \arrow[dr, "\pi"'] & & U_i \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

with transition functions $\tau_{ij}\colon U_i \cap U_j \to \mathrm{Diffeo}(F)$. Let us denote by $T_*\tau_{ij}\colon U_i \cap U_j \to \mathrm{Diffeo}(T^*F)$ there cotangent lifts, which fulfil also the cocycle condition for transition functions and hence they induce a fiber bundle $\tilde{V} \to M$ (actually this is $\mathrm{Ver}^*(P)$) with local trivializations $(U_i, \phi_i)_{i\in I}$, such that the transition functions fulfil $\phi_{ij} = T_*\tau_{ij}$. We consider now the canonical symplectic form $\omega_{can} \in \Gamma^\infty(\Lambda^2 T^*(T^*F))$ on T^*F, note that the functions $T_*\tau_{ij}$ are obviously symplectomorphisms. The next step is to consider the fiber bundle $V \to M$ with typical fiber $T^*F \setminus 0_F$, which we get by the obvious restrictions. Note that on $T^*F \setminus 0_F$ we have a canonical $\mathbb{R}^\times$-action which is free and proper and the for restricted symplectic form ω_{can} we have that $\mathscr{L}_{(1)_{T^*F}}\omega_{can} = \omega_{can}$. Using the results from [10], we conclude that the associated line bundle $L \to \mathbb{R}T^*F := \frac{T^*F\setminus 0_F}{\mathbb{R}^\times}$ carries a contact structure and the transitions functions, which are obviously commuting with the $\mathbb{R}^\times$-action, act as line bundle automorphisms preserving the contact structure. This is exactly the data we need to cook up a contact fiber bundle (4.4.3) and hence its total space possess a generalized contact structure, due to Lemma 4.4.5. Note that here the input was a generic fiber bundle over a complex base and the output is a generalized contact bundle. Moreover, if both the base and the fiber are compact, then the output is also compact. We hence proved the existence of compact examples.

Example 4.4.7 (Principal fiber Bundles) Let $\mathfrak{g}$ be a Lie algebra with a contact 1-form $\Theta \in \mathfrak{g}^*$. Let us consider a Lie group G integrating $\mathfrak{g}$ and a manifold M with a complex structure. Additionally, let $P \to M$ be a G-principal fiber bundle and let $\mathbb{R}_P \to P$ be the trivial line bundle, where we denote by 1_P the generating section. Recall that here the gauge algebroid splits canonically as $D\mathbb{R}_P = TP \oplus \mathbb{R}_P$. Moreover, we have that $\mathrm{Ver}_{\mathbb{R}_P}(P) = \mathrm{Ver}(P) \oplus \mathbb{R}_P$. Thus, a generic derivation $\Delta_p \in \mathrm{Ver}_{\mathbb{R}_P}(P)$ is of the form $\Delta = (\xi_P(p), k) \in \mathrm{Ver}(P) \oplus \mathbb{R}_P$, where ξ_P is the fundamental vector field of a unique $\xi \in \mathfrak{g}$ and $p \in P$. We define $\omega \in \Gamma^\infty(\Lambda^2(\mathrm{Ver}_{\mathbb{R}_P}(P))^* \otimes \mathbb{R}_P)$ by

$$\omega((\xi_P(p), k), (\eta_P(p), r)) = \big((\delta_{CE}\Theta)(\xi, \eta) + k\Theta(\eta) - r\Theta(\xi)\big) \cdot 1_P(p).$$

It is easy to check that ω gives $P \to M$ the structure of a contact fiber bundle. Since M was assumed to be complex, we can apply Lemma 4.4.5 to obtain a generalized contact bundle on P. Note that this notion includes $\mathbb{S}^1$-principal fiber bundles over a complex manifold. Moreover, contact Lie algebras are an active field of research and there are many examples around and even a classification of nilpotent contact Lie algebras in [3].

4.5 A Counterexample

In this last section, we want to construct a transversally complex Jacobi structure which cannot be induced by a generalized contact structure. The remarkable feature of this counterexample is, that it is, as manifolds, a global product of a (locally conformal) symplectic manifold and an Atiyah-complex manifold. Note that in [41] it was proven that every generalized contact bundle is locally isomorphic to a product, however not all generalized contact bundles arise in this way globally.

Let us consider the 2-sphere $\mathbb{S}^2$ and its symplectic form $\omega \in \Gamma^\infty(\Lambda^2 T^*\mathbb{S}^2)$. Its inverse $\pi \in \Gamma^\infty(\Lambda^2 TM)$ is a Poisson structure and hence $\pi + \mathbb{1} \wedge 0 = \pi$ is a Jacobi structure on the trivial line bundle.

The second manifold which is involved is the circle $\mathbb{S}^1$. Our counterexample will live on the trivial line bundle over the product

$$\mathbb{R}_M \to M := \mathbb{S}^2 \times \mathbb{S}^1.$$

Using Remark 1.2.48, we see that

$$D\mathbb{R}_M = TM \oplus \mathbb{R}_M = T\mathbb{S}^2 \oplus T\mathbb{S}^1 \oplus \mathrm{R}_M$$

and we can define a Jacobi structure $J = \pi + \mathbb{1} \wedge 0 = \pi$ on it by "pulling back" the bi vector π by setting it to be constant in $\mathbb{S}^1$ direction. We see that $\operatorname{im}(J^\sharp) = T\mathbb{S}^2 \subseteq D\mathbb{R}_M$.

The next step is to choose an everywhere non-vanishing vector field $e \in \Gamma^\infty(T\mathbb{S}^1)$ and define

$$K := T_\mathbb{C}\mathbb{S}^2 \oplus \langle \mathbb{1} - \mathrm{i}e \rangle \subseteq D_\mathbb{C}\mathbb{R}_M.$$

Note that we have

$$D_\mathbb{C}\mathbb{R}_M = T_\mathbb{C}\mathbb{S}^2 \oplus \langle \mathbb{1} - \mathrm{i}e \rangle \oplus \langle \mathbb{1} + \mathrm{i}e \rangle. \tag{4.5.1}$$

An easy computation shows that (J, K) is a transversally complex Jacobi structure. Our claim is now that (J, K) can not be induced by a generalized contact structure. To see this, let us examine the *Der*-complex a bit closer. We have that

$$\Lambda^k(D\mathbb{R}_M)^* \otimes \mathbb{R}_M = \Lambda^k(TM^* \oplus \mathbb{R}_M).$$

Recall the notation of Remark 1.2.48: we obtain that an Atiyah form $\psi \in \Gamma^\infty(\Lambda^k(TM^* \oplus \mathbb{R}_M))$ can be uniquely written as $\psi = \alpha + \mathbb{1}^* \wedge \beta$ for some $(\alpha, \beta) \in \Gamma^\infty(\Lambda^k T^*M \oplus \Lambda^{k-1}T^*M)$. Moreover, we have

$$\mathrm{d}_{\mathbb{R}_M}(\alpha + \mathbb{1}^* \wedge \beta) = \mathrm{d}\alpha + \mathbb{1}^* \wedge (\alpha - \mathrm{d}\beta)$$

where d is the usual de Rham differential. Now we want to pass to the spectral sequence, therefore we split the *Der*-complex according to the splitting of Equation (4.5.1), we have

$$\Omega^{(i,j),q}_{\mathbb{R}_M} = \Gamma^\infty(\Lambda^q T^*\mathbb{S}^2 \otimes \Lambda^i \langle \mathbb{1}^* + \mathrm{i}\alpha \rangle \otimes \Lambda^j \langle \mathbb{1}^* - \mathrm{i}\alpha \rangle),$$

where $\alpha \in \Gamma^\infty(T\mathbb{S}^1)$ such that $\alpha(e) = 1$. Note that here we have the canonical identification $D\mathbb{R}_M \supseteq \operatorname{Ann}(T\mathbb{S}^2) = T^*\mathbb{S}^1 \oplus \mathbb{R}_M$, which allows us to identify

$$E_0^{(i,j),q} = \Omega_{\mathbb{R}_M}^{(i,j),q}.$$

In case of a cartesian product, the differential $\mathrm{d}_{\mathbb{R}_M}$ splits canonically with respect to the bi-grading into

$$\mathrm{d}_{\mathbb{R}_M} = \mathrm{d}_0 + \partial_1 + \overline{\partial}_1,$$

where $\mathrm{d}_0 \colon \Omega_{\mathbb{R}_M}^{(i,j),q} \to \Omega_{\mathbb{R}_M}^{(i,j),q+1}$, $\partial_1 \colon \Omega_{\mathbb{R}_M}^{(i,j),q} \to \Omega_{\mathbb{R}_M}^{(i+1,j),q}$ and $\overline{\partial}_1 \colon \Omega_{\mathbb{R}_M}^{(i,j),q} \to \Omega_{\mathbb{R}_M}^{(i,j+1),q}$. Additionally, all three maps are differentials themselves and anticommute pairwise. Now we want to consider the inverse of J, which is the pullback of ω with respect to the canonical projection $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$. With a tiny abuse of notation we will see ω as an element of $\Gamma^\infty(\Lambda^2 T^*\mathbb{S}^2) \subseteq E_0^{(0,0),2}$. A long and not very enlightening computation shows that

$$\partial_1 \overline{\partial}_1 \omega = \frac{1}{4}((\mathbb{1}^* - \mathrm{i}\alpha) \wedge (\mathbb{1}^* + \mathrm{i}\alpha) \wedge \omega)$$

Hence, we have for the cohomology class

$$\partial_1 \overline{\partial}_1 [[\omega]_0]_1 = [[(\mathbb{1}^* - \mathrm{i}\alpha) \wedge (\mathbb{1}^* + \mathrm{i}\alpha) \wedge \omega]_0]_1.$$

But this cannot vanish, since a d_0-primitive ψ has to be of the form

$$\psi = (\mathbb{1}^* - \mathrm{i}\alpha) \wedge (\mathbb{1}^* + \mathrm{i}\alpha) \wedge \beta$$

for $\beta \in \Gamma^\infty(T^*\mathbb{S}^2)$. This implies that $\mathrm{d}\beta = \omega$, which is an absurd because the symplectic form on the sphere is not exact.

Appendix A

Appendices

This Appendices contain three topics which are relevant for some proofs of this thesis, but are not relevant for the core of it: Jacobi related geometries.

In the first section, we give a proof of a *Moser*-like trick, which allows us to construct isomorphisms of Dirac-Jacobi bundles. The second section discusses the so-call homogenization trick, which establishes, roughly speaking, a one-to-one correspondence between "Jacobi related geometries" and "homogeneous Poisson related geometries". Even though, this trick is very important for the whole theory of Jacobi related geometries, we prefer to shift it to the Appendix, since we want to stress that almost everything in the whole thesis does not use the one-to-one correspondence. The last section discusses *Atiyah* complex structures, which appear as one of the extreme cases in generalized contact geometry. Since they have not been considered so far in literature, the last section can be seen as a short introduction to the geometry of Atiyah complex structures.

A.1 The Moser Trick for Dirac-Jacobi Structures

Let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi structure on a line bundle $L \to M$. Moreover, we assume we have a smooth family of closed Atiyah 2-forms σ_t, such that $\sigma_0 = 0$ and $\mathcal{L}_J^{\sigma_t}$ is a Jacobi structure for all t, denoted by J_t. For

$$\alpha_t := -\frac{\partial}{\partial t}\iota_{\mathbb{1}}\sigma_t$$

the equation

$$\frac{\partial}{\partial t}\sigma_t = -\,\mathrm{d}_L\alpha_t$$

holds. We define the Moser-derivation by

$$\Delta_t := -J_t^\sharp(\alpha_t)$$

and its flow by $\Phi_t \in \mathsf{Aut}(L)$, where we assume it exists for on open subset containing $[0,1]$. Let us compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* J_t &= \Phi_t^*\Big([\Delta_t, J_t] + \frac{\mathrm{d}}{\mathrm{d}t}J_t\Big)\\ &= \Phi_t^*\Big(-[J_t^\sharp(\alpha_t), J_t] + \frac{\mathrm{d}}{\mathrm{d}t}J_t\Big)\\ &= \Phi_t^*\Big(J_t^\sharp(-\mathrm{d}_L\alpha_t) + \frac{\mathrm{d}}{\mathrm{d}t}J_t\Big).\end{aligned} \tag{A.1.1}$$

It is easy to see that

$$J_t^\sharp = J^\sharp \circ (\mathsf{id} + \sigma_t^\flat \circ J^\sharp)^{-1}$$

and hence we can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J_t^\sharp &= \frac{\mathrm{d}}{\mathrm{d}t}J^\sharp \circ (\mathsf{id} + \sigma_t^\flat \circ J^\sharp)^{-1}\\ &= -J^\sharp \circ (\mathsf{id} + \sigma_t^\flat \circ J^\sharp)^{-1} \circ \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathsf{id} + \sigma_t^\flat \circ J^\sharp)\right) \circ (\mathsf{id} + \sigma_t^\flat \circ J^\sharp)^{-1}\\ &= -J^\sharp \circ (\mathsf{id} + \sigma_t^\flat \circ J^\sharp)^{-1} \circ \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathsf{id} + \sigma_t^\flat \circ J^\sharp)\right) \circ (\mathsf{id} + \sigma_t^\flat \circ J^\sharp)^{-1}\\ &= -J_t^\sharp \circ \left(\frac{\partial}{\partial t}\sigma_t\right)^\flat \circ J_t^\sharp\\ &= (-J_t^\sharp\Big(\frac{\partial}{\partial t}\sigma_t)\Big)^\sharp\\ &= \big(J_t^\sharp(\mathrm{d}_L\alpha_t)\big)^\sharp,\end{aligned}$$

and hence $\frac{\mathrm{d}}{\mathrm{d}t}J_t = J_t^\sharp(\mathrm{d}_L\alpha_t)$. If we use this equality in Equation A.1.1, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* J_t = 0,$$

so we finally have $J = \Phi_0^* J_0 = \Phi_1^* J_1$ and hence the two Jacobi structures J_0, J_1 are isomorphic. We want to show that this well-known trick is just a special instance of a Moser-like trick for Dirac-Jacobi structures, which we need in order to discuss the semi-local structure of generalized contact bundles. We assume we have a Dirac-Jacobi structure $\mathcal{L} \subset \mathbb{D}L$ and a smooth family of closed 2-forms σ_t, such that $\sigma_0 = 0$. Moreover, we assume there exists a time-dependent derivation Δ_t, such that $(\Delta_t, \alpha_t - \iota_{\Delta_t}\sigma_t) \in \Gamma^\infty(\mathcal{L})$ for all t, where

$$\alpha_t := -\frac{\partial}{\partial t}\iota_{\mathbb{1}}\sigma_t.$$

The flow of $(\Delta_t, \alpha_t - \iota_{\Delta_t}\sigma_t)$ is given by

$$\exp(\gamma_t) \circ \mathbb{D}\Phi_t,$$

where Φ_t is the flow of Δ_t and

$$\begin{aligned}\gamma_t &= (\Phi_t)_* \int_0^t (\Phi_\tau)^*(\mathrm{d}_L(\alpha_\tau - \iota_{\Delta_\tau}\sigma_\tau))\,\mathrm{d}\tau \\ &= (\Phi_t)_* \int_0^t (\Phi_\tau)^*(-\frac{\partial}{\partial\tau}\sigma_\tau - \mathscr{L}_{\Delta_\tau}\sigma_\tau)\,\mathrm{d}\tau \\ &= (\Phi_t)_* \int_0^t -\frac{\mathrm{d}}{\mathrm{d}\tau}(\Phi_\tau)^*\sigma_\tau\,\mathrm{d}\tau \\ &= -\sigma_t\end{aligned}$$

Since $(\Delta_t, \alpha_t - \iota_{\Delta_t}\sigma_t) \in \Gamma^\infty(\mathcal{L})$, this flow preserves $\mathcal{L}$ and hence

$$\exp(-\sigma_t) \circ \mathbb{D}\Phi_t(\mathcal{L}) = \mathcal{L}$$

which implies

$$\mathbb{D}\Phi_t(\mathcal{L}) = \mathcal{L}^{\sigma_t},$$

showing that $\mathcal{L}$ and $\mathcal{L}^{\sigma_t}$ are isomoprhic for all t.

A.2 Homogenization of Jacobi related Geometries

Homogenization means in the context of Jacobi related geometries, roughly speaking, a functor from a suitable category of Jacobi related geometries, for example Jacobi bundles to a suitable category of homogeneous Poisson related geometries, for example homogeneous Poisson manifolds.

Even though, the homogenization is not needed in this thesis besides Appendix A.3, it provides a powerful tool in order to understand some instances in Jacobi geometry and can even give much simpler proofs, but in all of presented results in this thesis the homogenization did not provide a good framework for the proofs. It first appeared as the so- called symplectization trick in contact geometry, see for example [29] and its references. A unified approach for homogenization is presented in [10] including non-coorientable contact structures and Jacobi brackets, see also [48]. We will recall in a very vague way [10] in order to present the homogenization, but using our notation. Note that this appendix is not meant to present every proof in detail, it is more intended to give a global idea of the homogenization construction.

Let us start considering a line bundle $L \to M$. Its co-frame bundle $p\colon \widetilde{M} := L^*\backslash 0_M \to M$ is a $\mathbb{R}^\times$-principal fiber bundle with principal action

$$P\colon \mathbb{R}^\times \times \widetilde{M} \ni (r, \alpha_p) \mapsto r\alpha_p \in \widetilde{M}.$$

Note that sections of $L \to M$ can be canonically identified with homogeneous functions on $\widetilde{M}$ with respect to the principal action by

$$\widetilde{\cdot} : \Gamma^\infty(L) \ni \lambda \mapsto \widetilde{\lambda} := (\alpha_p \mapsto \alpha_p(\lambda(p))) \in \mathscr{C}^\infty(\widetilde{M}).$$

We have for every $p \in \widetilde{M}$

$$T^*_{\alpha_p}\widetilde{M} = \{\mathrm{d}\widetilde{\lambda}\big|_{\alpha_p} \mid \lambda \in \Gamma^\infty(L)\},$$

which can be shown easily in local coordinates. This allows us to define

$$\widetilde{\cdot} : \Gamma^\infty(\Lambda^k(J^1L)^* \otimes L) \ni \Delta \mapsto \Big((\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_k) \mapsto \Delta(j^1\lambda_1\widetilde{, \dots ,} j^1\lambda_k))\Big) \in \Gamma^\infty(\Lambda^k TM). \tag{$*$}$$

for all $k \geq 0$. Note that this map is injective and moreover the image are exactly the multi-vector fields $X \in \Gamma^\infty(\Lambda^k TM)$, fulfilling

$$P_r^* X = r^{1-k} X,$$

which span the tangent space at each point (which again can be shown easily in coordinates). It is worth mentioning at this moment that

$$\widetilde{\mathbb{1}} = \frac{\mathrm{d}}{\mathrm{d}t} M_{\exp(t)} = \mathcal{E}.$$

is the Euler vector field of $L^* \to M$ restricted to $\widetilde{M}$. The map defined in Equation $(*)$ is by definition also compatible with the Gerstenhaber-Jacobi bracket and the Schouten bracket, i.e. we have

$$\widetilde{[\![\Delta, \Box]\!]}_L = [\widetilde{\Delta}, \widetilde{\Box}]$$

for all $\Delta, \Box \in \Gamma^\infty(\Lambda^\bullet(J^1L)^* \otimes L)$. This means, in particular, that having a Jacobi structure $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$, its homogenization $\widetilde{J}$ is a Poisson structure(J being Jacobi is equivalent to $[\![J, J]\!]_L = 0$).

Let us now discuss the dual picture. The complex relevant for us is the de Rham complex of DL with coefficients in L, denoted by $\Omega^\bullet_L(M)$. We define now

$$\widetilde{\cdot} : \Omega^k_L(M) \ni \psi \mapsto \big((\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_k) \mapsto \psi(\Delta_1\widetilde{, \dots ,}\Delta_k)\big) \in \Gamma^\infty(\Lambda^k T^*M),$$

which is a cochain map intertwining d_L and and the usual de Rham differential d. Note that we have here

$$P_r^* \widetilde{\psi} = r\widetilde{\psi}$$

for $\omega \in \Omega^\bullet_L(M)$. We can now even introduce the map

$$\widetilde{\cdot} : \Gamma^\infty(\Lambda^\bullet(J^1L)^* \otimes \Lambda^\bullet(DL)^* \otimes L)$$

in a similar fashion, i.e. defining them on the image of $\widetilde{\cdot}$. Moreover, we have that

$$P_r^*(\Delta \widetilde{\otimes \psi} \otimes \lambda) = r^{1-k}(\Delta \widetilde{\otimes \psi} \otimes \lambda)$$

for $(\Delta \otimes \psi \otimes \lambda) \in \Gamma^\infty(\Lambda^k(J^1L)^* \otimes \Lambda^\bullet(DL)^* \otimes L)$.

And one can show that almost all the structures appearing in this thesis can be mapped via this map into more classical structure, i.e.

i.) A Jacobi tensor J gets mapped in a Poisson structure $\widetilde{J}$ fulfilling $P_r^* \widetilde{J} = r^{-1}\widetilde{J}$.

ii.) A pre-contact structure ω, i.e. $\omega \in \Omega_L^2(M)$ and closed, gets mapped into a presymplectic structure $\widetilde{\omega} \in \Gamma^\infty(\Lambda^2 T^*\widetilde{M})$ fulfilling $P_r^*\widetilde{\omega} = r\widetilde{\omega}$. Moreover, if ω is non-degenerate, i.e. contact, then $\widetilde{\omega}$ is symplectic.

iii.) An Atiyah complex structure $\varphi \in \Gamma^\infty(\mathsf{End}(DL))$ gets mapped into a complex structure $\widetilde{\varphi}$ fulfilling $P_r^*\widetilde{\varphi} = \widetilde{\varphi}$.

iv.) For a Dirac-Jacobi structure $\mathcal{L} \subseteq \mathbb{D}L$, the subset $(\widetilde{\Delta}, \widetilde{\psi}) \in \Gamma^\infty(TM \oplus T^*M)$ for $(\Delta, \psi) \in \Gamma^\infty(\mathcal{L})$ generates a Dirac structure $\mathcal{D}$ fulfilling: $(X, \alpha) \in \Gamma^\infty(\mathcal{D}) \implies (P_r^* X, \frac{1}{r} P_r^* \alpha) \in \Gamma^\infty(\mathcal{D})$.

v.) A generalized contact structure

$$\mathbb{K} = \begin{pmatrix} \varphi & J^\sharp \\ \beta^\flat & -\varphi^* \end{pmatrix}$$

gets mapped into a generalized complex structure

$$\widetilde{\mathbb{K}} = \begin{pmatrix} \widetilde{\varphi} & \widetilde{J}^\sharp \\ \widetilde{\beta}^\flat & -\widetilde{\varphi}^* \end{pmatrix},$$

such that $P_r^*\widetilde{\varphi} = \widetilde{\varphi}$, $P_r^*\widetilde{J} = r^{-1}\widetilde{J}$ and $P_r^*\widetilde{\beta} = r\widetilde{\beta}$.

We refer to all the structures in $i.) - v.)$ on $\widetilde{M}$ as homogeneous. Moreover, it is clear that having a homogeneous structure on $\widetilde{M}$ with the indicated properties, they are actually coming from their "Jacobi"-version, for example for a Poisson structure π on $\widetilde{M}$ fulfilling $P_r^*\pi = r^{-1}\pi$ there is a unique Jacobi structure J such that $\pi = \widetilde{J}$.

Let us now talk about morphisms. In $\mathfrak{Line}$ morphisms areregular line bundle morphisms $\Phi \colon L_1 \to L_2$. Let us denote by $\phi \colon M_1 \to m_2$ the map Φ is covering. We can define

$$\widetilde{\Phi} \colon \widetilde{M_1} \ni \alpha_p \mapsto \alpha_p \circ \Phi_p^{-1} \in \widetilde{M_2}.$$

Note that $\widetilde{\Phi}$ intertwines the principal actions and is hence a morphism of $\mathbb{R}^\times$-principal fiber bundles. This makes $\widetilde{\cdot}$ a functor $\mathfrak{Line}$ into the category of $\mathbb{R}^\times$-principal fiber bundles.

This functor provides an equivalence of categories. To see this we consider a $\mathbb{R}^\times$-principal fiber bundle $P \to M$, where we denote the principal action by $P \colon \mathbb{R}^\times \times P \to P$. The associated line bundle is the quotient

$$L := \mathbb{R}_P / \mathbb{R}^\times \to P/\mathbb{R}^\times = M$$

with the action given by

$$\mathbb{R}^\times \times \mathbb{R}_P \ni (r, (p, k)) \mapsto (rp, r^{-1}k) \in \mathbb{R}_P.$$

It is now easy to show that $L^* \backslash 0_M \cong P$, this can be found in every classical book treating principal fiber bundles, or equivalently [10].

A.3 Atiyah Complex Structures and their Dolbeault Cohomologies

Complex structures on the Gauge algebroid of a line bundle have not been studied so far, in fact the author is just aware of the references [27] and [41], where this appendix also appeared. Following the homogenezation scheme from Appendix A.2, Atiyah complex structures seem to be natural objects and in the opinion of the author it provides the right framework of complex geometry in odd dimension. Moreover, it includes what is known as *normal almost contact structures* (see below).

A.3.1 Complex Structures on the Gauge Algebroid

Let $L \to M$ be a line bundle. In this appendix we study the local properties of a generalized contact structure of *complex type*, i.e. a generalized contact structure $\mathbb{K}$ on L, of the form

$$\mathbb{K} = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}. \tag{A.3.1}$$

In this case $\varphi \colon DL \to DL$ is a(n integrable) *complex structure* on the gauge algebroid DL, i.e.

i.) φ is *almost complex*, i.e. $\varphi^2 = -\mathsf{id}$,

ii.) φ is *integrable*, i.e. its Lie algebroid *Nijenhuis torsion* $\mathcal{N}_\varphi$ vanishes.

Here $\mathcal{N}_\varphi \in \Gamma^\infty(\Lambda^2(DL) * \otimes DL)$ is the skew-symmetric bilinear map defined by

$$\mathcal{N}_\varphi(\Delta, \Box) = [\varphi\Delta, \varphi\Box] - \varphi([\varphi\Delta, \Box]) + \varphi([\Delta, \varphi\Box]) - [\Delta, \Box], \quad \Delta, \Box \in \Gamma(DL).$$

Conversely, given a complex structure on DL, (A.3.1) defines a generalized contact structure.

Example A.3.1 Consider the cylinder $\mathbb{R} \times \mathbb{C}^n$ over the standard complex space $\mathbb{C}^n$. Let u be the standard real coordinate on the first factor, and let $z^i = x^i + \mathrm{i}y^i$, $i = 1, \dots, n$, be the standard complex coordinates on the second factor. There is a canonical integrable complex structure φ_{can} on the gauge algebroid of the trivial line bundle $\mathbb{R}_{\mathbb{R}\times\mathbb{C}^n}$ defined by

$$\varphi_{can}\mathbb{1} = \frac{\partial}{\partial u}, \quad \text{and} \quad \varphi_{can}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}.$$

Example A.3.2 (Normal Almost Contact Structures) Our main reference for this example is [27], where the reader will find basically all the proofs. We will see in this example and Lemma A.3.3 that almost contact structures (resp. normal almost contact structures) are locally the same as almost complex structures (resp. integrable almost complex structures) on the gauge algebroid of a trivial line bundle $\mathbb{R}_M \to M$.

Recall that an *almost contact structure* on a manifold M is a triple (Φ, ξ, η), where $\Phi\colon TM \to TM$ is a $(1,1)$-tensor, ξ is a vector field, and η is a 1-form on M such that

$$\Phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \text{and} \quad \eta(\xi) = 1.$$

See, e.g., [7] for more details. The idea behind this definition is that *an almost contact structure is the odd-dimensional analogue of an almost complex structure.* We believe that the use of line bundles and their gauge algebroids makes the analogy much more transparent. Namely, recall that the gauge algebroid of the trivial line bundle is $D\mathbb{R}_M \cong TM \oplus \mathbb{R}_M$. Now take a triple (Φ, ξ, η) consisting of an $(1,1)$-tensor, a vector field and a 1-form on M, and let $\varphi\colon D\mathbb{R}_M \to D\mathbb{R}_M$ be the endomorphism given by

$$\varphi(X, r) = (\Phi(X) - r\xi, \eta(X)) \tag{A.3.2}$$

Then (Φ, ξ, η) is an almost contact structure if and only if $\varphi^2 = -\mathsf{id}$ is a complex structure, i.e. $\varphi^2 = -\mathrm{id}$. Additionally, φ is integrable if and only if

$$\mathcal{N}_\Phi + d\eta \otimes \xi = 0, \quad d\eta(\Phi -, -) + d\eta(-, \Phi -) = 0, \quad \mathcal{L}_\xi \Phi = 0, \quad \text{and} \quad \mathcal{L}_\xi \eta = 0, \tag{A.3.3}$$

where $\mathcal{N}_\Phi$ is the Nijenhuis torsion of Φ [27]. One can actually show that the first condition in (A.3.3) implies the other ones [7, Section 6.1] (see also [27]). An almost contact structure (Φ, ξ, η) such that $\mathcal{N}_\Phi + d\eta \otimes \xi = 0$ is called *normal* [7]. So normal almost contact structures provide examples of complex structures on the Atiyah algebroid (of the trivial line bundle), and, in turn, of generalized contact structures of complex type. It turns out that, locally, every generalized contact structure of complex type is of this form (see Lemma A.3.3 below).

Example A.3.2 is special in view of the following

Lemma A.3.3 *Let $L \to M$ be a line bundle and let $\varphi\colon DL \to DL$ be an integrable complex structure. Then, around every point of M, there is a trivialization $L \cong \mathbb{R}_M$ identifying φ with a complex structure of the form (A.3.2) for some normal almost contact structure (Φ, ξ, η).*

Proof: Without loss of generality, we can assume $L = \mathbb{R}_M$, so that $DL \cong TM \oplus \mathbb{R}_M$. It is clear that, under this identification, φ is necessarily of the form

$$\varphi(X, r) = (\Phi(X) - r\xi, \eta(X) + gr) \tag{A.3.4}$$

for some quadruple (Φ, ξ, η, g), where Φ is a $(1,1)$-tensor, ξ is a vector field, η is a 1-form, and g is a smooth function on M. Locally, we can achieve $g = 0$ as follows. First of all, let $f \in C^\infty(M)$. A straightforward computation shows that, under the line bundle automorphism $\mathbb{R}_M \to \mathbb{R}_M$, $(x, r) \mapsto (x, e^{-f(x)} r)$, the quadruple (Φ, ξ, η, g) changes into

$$(\Phi + df \otimes \xi,\ \xi,\ \eta + df \circ \Phi + (\xi(f) - g) df,\ g - \xi(f))$$

Now, from $\varphi^2 = -\mathrm{id}$, we easily find that ξ is everywhere non-zero. Hence, locally, around every point, there exists a function f such that $\xi(f) = g$. This concludes the proof. ΧΞΣ

Remark A.3.4 Not all integrable complex structures on DL are globally of the form (A.3.2), in general, not even when $L = \mathbb{R}_M$ is the trivial line bundle. To see this, let M be a manifold such that $\mathrm{H}^1_{\mathrm{dR}}(M) \neq 0$, and let (Φ', ξ', η') be a normal almost contact structure on M (such manifolds exist, and the 1-dimensional sphere provides the simplest possible example). Now, pick a closed, but not exact, 1-form α on M, and put

$$\Phi = \Phi' + \alpha \otimes \xi', \quad \xi = \xi', \quad \eta = \eta' + \alpha \circ \Phi' + \alpha(\xi')\alpha, \quad g = -\alpha(\xi'). \tag{A.3.5}$$

Then the endomorphism $\varphi \colon D\mathbb{R}_M \to D\mathbb{R}_M$ given by (A.3.4) is an integrable complex structure that cannot be put in the form (A.3.2) by a global line bundle automorphism $\mathbb{R}_M \to \mathbb{R}_M$.

A.3.2 Local normal Form

Theorem A.3.5 *Let $L \to M$ be a line bundle equipped with a complex structure $\varphi \colon DL \to DL$ on the gauge algebroid. Then, locally, around every point of M, there are*

i.) coordinates $(u, x^1, \dots, x^n, y^1, \dots, y^n)$ on M, and

ii.) a flat connection ∇ in L, such that

$$\varphi \mathbb{1} = \nabla_{\partial/\partial u}, \quad \textit{and} \quad \varphi \nabla_{\partial/\partial x^i} = \nabla_{\partial/\partial y^i}. \tag{A.3.6}$$

In other words, locally, around every point of M, there is trivialization $L \cong \mathbb{R}_{\mathbb{R}\times\mathbb{C}^n}$ identifying φ with φ_{can} from Example A.3.1.

PROOF: Let $\varphi : DL \to DL$ be an integrable complex structure. Consider its homogenization $\widetilde{\varphi}$ (see Appendix A.2) as a complex structure on $\widetilde{M}$. As $\mathcal{E}$ is nowhere vanishing, it can be locally completed to a holonomic complex frame, i.e. locally, around every point of $\widetilde{M}$, there are coordinates $(T, U, X^1, \dots, X^n, Y^1, \dots, Y^n)$ such that

$$\mathcal{E} = \frac{\partial}{\partial T}, \quad \widetilde{\varphi}\mathcal{E} = \frac{\partial}{\partial U}, \quad \text{and} \quad \widetilde{\varphi}\frac{\partial}{\partial X^i} = \frac{\partial}{\partial Y^i}.$$

As all coordinate vector fields commute with $\mathcal{E}$, they all come from (commuting) derivations of L. In particular

i.) $(U, X^1, \dots, X^n, Y^1, \dots, Y^n)$, are pull-backs via projection $\widetilde{M} \to M$ of uniquely defined coordinates $(u, x^1, \dots, x^n, y^1, \dots, y^n)$ on M, and

ii.) there exists a unique flat connection ∇ in L such that

$$\frac{\partial}{\partial U} = \nabla_{\partial/\partial u}, \ \dots \ , \ \frac{\partial}{\partial X^i} = \nabla_{\partial/\partial x^i}, \ \dots \ , \ \frac{\partial}{\partial Y^i} = \nabla_{\partial/\partial y^i}, \ \dots$$

Therefore, the coordinates $(u, x^1, \dots, x^n, y^1, \dots, y^n)$ on M and flat connection ∇ possess all the required properties. ΧΞΣ

As an immediate corollary of Theorem A.3.5 and Lemma A.3.3 we get a local normal form for normal almost contact structures.

Corollary A.3.6 *Let (Φ, ξ, η) be a normal almost contact structure on a manifold M. Then, around every point, there exist local coordinates (u, x^i, y^i) and a local function f, such that:*

i.) $\xi = \frac{\partial}{\partial u}$,

ii.) $\eta = du + \frac{\partial f}{\partial y^i} dx^i - \frac{\partial f}{\partial x^i} dy^i$,

iii.) $\Phi = dx^i \otimes \frac{\partial}{\partial y^i} - dy^i \otimes \frac{\partial}{\partial x^i} + df \otimes \frac{\partial}{\partial u}$.

A.3.3 Dolbeault-Atiyah Cohomology

Let $L \to M$ be a line bundle, and let $\varphi \colon DL \to DL$ be an integrable complex structure on the gauge algebroid of L. Similarly as in the case of a complex manifold, there is a cohomology theory attached to φ. Namely, consider $DL_{\mathbb{C}}\otimes$ of the gauge algebroid and denote by $D^{(1,0)}L$ and $D^{(0,1)}L$ the +i and the $-i$-eigenbundles of φ respectively, so that

$$D_{\mathbb{C}}L = D^{(1,0)}L \oplus D^{(0,1)}L,$$

and complex Atiyah forms $\Omega^\bullet_{L,\mathbb{C}}(M)$ splits as

$$\Omega^\bullet_{L,\mathbb{C}}(M) = \bigoplus_{r,s} \Omega^{(r,s)}_{L,\mathbb{C}}(M),$$

where we denoted by $\Omega^{(r,s)}_{L,\mathbb{C}}(M)$ the sections of (complex) vector bundle

$$\Lambda^r(D^{(1,0)}L)^* \otimes \Lambda^s(D^{(0,1)}L)^* \otimes L.$$

de Rham differential d_L splits, in the obvious way, as $\mathrm{d}_L = \partial_L + \overline{\partial}_L$, where

$$\partial_L \colon \Omega^{(\bullet,\bullet)}_{L,\mathbb{C}}(M) \to \Omega^{(\bullet+1,\bullet)}_{L,\mathbb{C}}(M), \quad \text{and} \quad \overline{\partial}_L \colon \Omega^{(\bullet,\bullet)}_{L,\mathbb{C}}(M) \to \Omega^{(\bullet+1,\bullet)}_{L,\mathbb{C}}(M),$$

and the integrability of φ is equivalent to

$$\partial_L^2 = \overline{\partial}_L^2 = \partial_L \overline{\partial}_L + \overline{\partial}_L \partial_L = 0.$$

We call cohomology of $\overline{\partial}_L$ the *Dolbeault-Atiyah cohomology.*

Theorem A.3.7 *The Dolbeault-Atiyah cohomology vanishes locally.*

Proof: In view of Theorem A.3.5, it is enough to work in the case when $M = \mathbb{R} \times \mathbb{C}^n$. Let u be the standard (real) coordinate on the first factor and let $z^i = x^i + \mathrm{i}y^i$, $i = 1, \ldots, n$, be the standard complex coordinates on the second factor. We can also assume that $L = \mathbb{R}_M$ is the trivial line bundle and (A.3.6) holds with ∇ being the canonical flat connection on $\mathbb{R}_M$. In this case $D^{(1,0)}L$ is spanned by complex derivations

$$\square := \frac{1}{2}\big(\mathbb{1} - \mathrm{i}\nabla_{\partial/\partial u}\big), \quad \text{and} \quad \nabla_i = \frac{1}{2}\big(\nabla_{\partial/\partial x^i} - \mathrm{i}\nabla_{\partial/\partial y^i}\big), \quad i = 1, \ldots, n. \tag{A.3.7}$$

Let $\omega \in \Omega^2_{\mathbb{R}_M,\mathbb{C}}(M)$ be arbitrary. Using Subsection 1.1.2, we can write

$$\omega = \omega_0 + \mathbb{1}^* \wedge \omega_1$$

where ω_0, ω_1 are standard complex forms on M. A long but straightforward computation exploiting (A.3.7), shows that

$$\overline{\partial}_L\omega = \overline{\partial}\omega_0 + \mathbb{1}^* \wedge \big(\omega_0 + \mathcal{L}_Y\omega_0 - \overline{\partial}\omega_1\big)$$

where

$$Y := \sigma(\overline{\square}) = \frac{\mathrm{i}}{2}\frac{\partial}{\partial u},$$

and $\overline{\partial}$ is the standard Dolbeault differential on $\mathbb{C}^n$ (acting on forms on $\mathbb{R} \times \mathbb{C}^n$ in the obvious way). So ω is $\overline{\partial}_D$-closed iff

$$\overline{\partial}\omega_0 = \omega_0 + \mathcal{L}_Y\omega_0 - \overline{\partial}\omega_1 = 0.$$

In this case, use the vanishing of standard Dolbeault cohomology (with a real parameter u), to choose a form ρ_0 such that $\overline{\partial}\rho_0 = \omega_0$. As the Lie derivative along Y commutes with $\overline{\partial}$ we find

$$\overline{\partial}(\omega_1 - \rho_0 - \mathcal{L}_Y\rho_0) = 0,$$

and we can choose ρ_1 such that $\overline{\partial}\rho_1 = -(\omega_1 - \rho_0 + \mathcal{L}_Y\rho_0)$. It is now easy to see that

$$\overline{\partial}_L(\rho_0 + \mathbb{1}^* \wedge \rho_1) = \omega.$$

This concludes the proof. ΧΞΣ

Remark A.3.8 It immediately follows from Theorem A.3.7 that the cohomology of ∂_L does also vanish locally.

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