Abstract

Optimal design problems have aroused particular interest in the scientific community over the past thirty years. In physics, for example, they find application in the investigation of the minimal energy configurations of a mixture of two materials in a bounded and connected open set.

The fascination of such problems derives from their variational formulation, which involves not only the state function of a system, but also a *shape*, that is a set. If a penalizing contribution of perimeter form, due to a *surface energy*, is added to the integral *mass energy*, dependent on the configuration state-shape, the problem becomes even more intriguing and inspiring.

It is not straightforward to investigate the regularity of minimizing pairs because the two energies have different dimensions under commong scalings: once a homothety of factor r is applied, the first energy "behaves" as a volume (rescaling with factor r^n), the second as a perimeter (rescaling with factor r^{n-1}). The coexistence of the two types of energies is managed using techniques and tools of both the Calculus of Variations and the Geometric Measure Theory.

In the first part of this thesis we deal with two optimal design problems, in which the integral functions that constitute the mass energy have different growths.

If their growth is at most quadratic, we prove the $C^{1,\mu}$ regularity of the interface of the shape that constitutes the optimal pair, up to a singular set of Hausdorff dimension less than $n-1$. The technique used combines the regularity theories of the Λ-minimizers of the perimeter and the minimizers of the Mumford-Shah functional.

If the integrands have at most a polynomial growth of degree p, the analysis becomes more involved. The $C^{1,\mu}$ regularity of the interface remains an open problem. However, it is proved that the optimal shape of the problem is equivalent to an open set with a topological boundary that differs from its reduced boundary for a set of Hausdorff dimension less than or equal to $n-1$.

In the second part of the thesis we address to a completely different variational problem, involving a frustrated spin system on a (one-dimensional and two-dimensional) lattice confined in two magnetic anisotropy circles.

This topic is of significant scientific interest, as it is useful for understanding the behavior of low-dimensional magnetic structures existing in nature.

The frustration parameter $\alpha > 0$ of the system averages the ferromagnetic and antiferromagnetic interactions that coexist in the energy. The minimal energy state of the system, for $\alpha \leqslant 4$, consists of a spin that "lives" within only one of the two magnetic anisotropy circles and has a positive or negative chirality.

We find the correct rescaling of the functional and prove the energy needed to detect the two phenomena that break the rigid minimal symmetry described. These are chirality transitions and magnetic anisotropy transitions of the spin.