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Estimates for the transition kernel for elliptic operators with unbounded coefficients

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Abstract

This manuscript is devoted to the study of the qualitative behaviour of the solutions of evolution equations arising from elliptic and parabolic problems on unbounded domains with unbounded coefficients. In particular, we deal with the elliptic operator of the form

$$A = \mathrm{div}(Q\nabla) + F \cdot \nabla - V,$$

where the matrix $Q(x) = (q_{ij}(x))$ is symmetric and uniformly elliptic and the coefficients q_{ij}, F and V are typically unbounded functions.

Since the classical semigroup theory does not apply in case of unbounded coefficients, in Chapter 1 we illustrate how to construct the minimal semigroup $T(\cdot)$ associated with A in $C_b(\mathbb{R}^d)$. It provides a solution of the corresponding parabolic Cauchy problem

$$\begin{cases} \partial_t u(t,x) = Au(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

for $f \in C_b(\mathbb{R}^d)$, that is given through an integral kernel p as follows

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, dy.$$

Moreover, such solution is unique if a Lyapunov function exists. Since an explicit formula is in general not available, it is interesting to look for pointwise estimates for the integral kernel p.

In Chapter 2 we consider a Schrödinger type operator in divergence form, namely the operator A when $F = 0$. We prove first that the minimal realization $A_{\min}$ of A in $L^2(\mathbb{R}^d)$ with unbounded coefficients generates a symmetric sub-Markovian and ultracontractive semigroup on $L^2(\mathbb{R}^d)$ which coincides on $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ with the minimal semigroup generated by a realization of A on $C_b(\mathbb{R}^d)$. Moreover, using time dependent Lyapunov functions, we show pointwise upper bounds for the heat kernel of A. We then improve such estimates and deduce some spectral properties of $A_{\min}$ in concrete examples, such as in the case of polynomial and exponential diffusion and potential coefficients.

Chapter 3 deals with the whole operator A. With appropriate modifications, similar kernel estimates described above are valid for this operator. In addition, we prove global Sobolev regularity and pointwise upper bounds for the gradient of p. We finally apply such estimates in case of polynomial coefficients.

Sommario

Questa tesi è dedicata allo studio del comportamento qualitativo delle soluzioni di equazioni di evoluzione derivanti da problemi ellittici e parabolici su domini non limitati con coefficienti non limitati. In particolare, ci si occupa dell'operatore ellittico della forma

$$A = \mathrm{div}(Q\nabla) + F \cdot \nabla - V,$$

dove la matrice $Q(x) = (q_{ij}(x))$ è simmetrica e uniformemente ellittica e i coefficienti q_{ij}, F e V sono tipicamente funzioni non limitate.

Poiché la teoria classica dei semigruppi non si applica in caso di coefficienti non limitati, nel Capitolo 1 illustriamo come costruire il semigruppo minimale $T(\cdot)$ associato ad A in $C_b(\mathbb{R}^d)$. Esso fornisce una soluzione del corrispondente problema di Cauchy parabolico

$$\begin{cases} \partial_t u(t,x) = Au(t,x), & t > 0,\, x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

per $f \in C_b(\mathbb{R}^d)$, che è data attravenso un nucleo integrale p da

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, dy.$$

Inoltre, tale soluzione è unica se esiste una funzione di Lyapunov. Poiché in generale non è disponibile una formula esplicita, è interessante cercare stime puntuali per il nucleo integrale p.

Nel Capitolo 2 si considera un operatore di tipo Schrödinger in forma di divergenza, cioè l'operatore A con $F = 0$. Inizialmente si prova che la minima realizzazione $A_{\min}$ di A in $L^2(\mathbb{R}^d)$ con coefficienti non limitati genera un semigruppo simmetrico, sub-Markoviano e ultracontrattivo su $L^2(\mathbb{R}^d)$ che coincide su $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ con il semigruppo minimale generato da una realizzazione di A su $C_b(\mathbb{R}^d)$. Inoltre, usando le funzioni di Lyapunov dipendenti dal tempo, si mostrano stime puntuali dall'alto per il nucleo del calore di A. Quindi si applicano tali stime e si deducono alcune proprietà spettrali di $A_{\min}$ in esempi concreti, come nel caso di coefficienti di diffusione e potenziale di tipo polinomiale ed esponenziale.

Nel Capitolo 3 si considera l'intero operatore A. Con opportune modifiche, simili stime del nucleo sopra descritte rimangono valide. Inoltre, si dimostrano risultati di regolarità globale di Sobolev e stime puntuali per il gradiente di p. Infine si applicano tali stime nel caso di coefficienti polinomiali.

Introduction

Starting from the second half of the past century, elliptic operators with bounded coefficients have been investigated extensively both in $\mathbb{R}^d$ and in open subsets of $\mathbb{R}^d$. Nowadays we have a wide knowledge on this subject.

In recent years, the attention turned to operators with unbounded coefficients in $\mathbb{R}^d$. The motivation lies in their applications in many branches of applied science, engineering and economics. For example, in fluid dynamics, the study of the Navier-Stokes equations with rotating obstacle involves a change of variables which transform operators with bounded coefficients into operators with unbounded ones (see [24, 27]). Equations with unbounded coefficients also arise from stochastic models in mathematical finance, such as the well known Black-Scholes equation in [6]. Moreover, in biology, they play a role in the study of the motion of a particle acting under a force perturbed by noise (see [22]).

The analysis of operators with unbounded coefficients has been developed using several approaches, with methods and ideas from partial differential equations, Dirichlet forms, stochastic processes and stochastic differential equations.

The Ornstein-Uhlenbeck operator represents one of the most famous examples of an operator with unbounded coefficients in $\mathbb{R}^d$. It is defined on smooth functions by

$$A\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d} q_{ij} D_{ij}\varphi(x) + \sum_{i,j=1}^{d} F_{ij} x_j D_i \varphi(x),$$

for any $x \in \mathbb{R}^d$, where (q_{ij}) is a constant positive definite matrix and (F_{ij}) is a constant real matrix. It exhibits the main features of this class of operators, such as the fact that the associated semigroup in $C_b(\mathbb{R}^d)$ is neither strongly continuous nor analytic.

One quickly realizes that leaving the bounded coefficients setting for the unbounded one is not merely a generalization: the classical semigroup theory is unfit as well as the classical theory of elliptic differential operators. Moreover, it has considerable consequences. For example, the failure of the maximum principle leads to the nonuniqueness of the continuous bounded solutions of the corresponding parabolic Cauchy problems.

With this in mind, in Chapter 1 we illustrate specific techniques contained in [36, 47, 48] in order to deal with uniformly elliptic operators defined on smooth functions by

$$A\varphi(x) = \sum_{i,j=1}^{d} q_{ij}(x) D_{ij}\varphi(x) + \sum_{i=1}^{d} F_i(x) D_i\varphi(x) - V(x)\varphi(x), \tag{1}$$

for any $x \in \mathbb{R}^d$, where the matrix $Q = (q_{ij})$ is symmetric and uniformly elliptic and the coefficients are locally Hölder continuous and typically unbounded functions. The goal is to study general properties of the semigroup $T(\cdot)$ in spaces of continuous functions $C_b(\mathbb{R}^d)$ such as the existence and uniqueness of solutions to the elliptic and parabolic equation.

More precisely, for $f \in C_b(\mathbb{R}^d)$ we consider the parabolic problem

$$\begin{cases} \partial_t u(t,x) = Au(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d. \end{cases} \tag{2}$$

By mean of an approximation argument with Cauchy-Dirichlet problems in bounded and smooth domains and classical Schauder estimates, we prove that the problem (2) admits a classical solution for every $f \in C_b(\mathbb{R}^d)$. This solution is given by a semigroup $T(\cdot)$, namely $u(t,x) = T(t)f(x)$. Moreover, it admits an integral representation by

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, dy, \tag{3}$$

where p is a positive function called integral kernel. In general such semigroup is neither strongly continuous nor analytic in $C_b(\mathbb{R}^d)$. Hence, the next step is to make clear the meaning of generator. For that, in comparison with the classical concept of infinitesimal generator, we introduce the weak generator of $T(\cdot)$.

Finally, one can ask for the uniqueness of the solution to problem (2). As anticipated above, the answer is negative: unlike the case when the coefficients are bounded, the classical maximum principle may fail. This is the reason why, in general, the parabolic problem (2) admits more than one solution. Hence, to prove uniqueness results some additional assumptions on the operator A need to be imposed. The typical assumption which we assume is the existence of a so-called Lyapunov function, i.e. a function $0 \le Z \in C^{2+\zeta}(\mathbb{R}^d)$ such that $\lim_{|x|\to\infty} Z(x) = \infty$ and

$$AZ(x) \le \lambda Z(x),$$

for some $\lambda \ge 0$. Furthermore, we introduce time dependent Lyapunov functions for the operator $\partial_t + A$. These are functions $W \in C^{1,2+\zeta}((0,T)\times\mathbb{R}^d) \cap C([0,T]\times\mathbb{R}^d)$ such that $\lim_{|x|\to\infty} W(t,x) = \infty$ uniformly for t in compact subsets of $(0,T]$, $W \le Z$ and there exists $0 \le h \in L^1((0,T))$ such that

$$\partial_t W(t,x) + AW(t,x) \le h(t)W(t,x),$$

for all $(t,x) \in (0,T)\times\mathbb{R}^d$. Finally, we prove that both Z and W are integrable with respect to the measure pdy.

In Chapter 2 we present the work in [17]. We consider the Schrödinger type operator defined on smooth functions φ by

$$A\varphi = \operatorname{div}(Q\nabla\varphi) - V\varphi, \tag{4}$$

where the matrix $Q = (q_{ij})$ is symmetric and uniformly elliptic and the coefficients are typically unbounded functions. If $q_{ij} \in C^{1+\zeta}_{\rm loc}(\mathbb{R}^d)$ and $0 \le V \in C^{\zeta}_{\rm loc}(\mathbb{R}^d)$ for some $\zeta \in (0,1)$, then we can associate the semigroup $T(\cdot)$ in $C_b(\mathbb{R}^d)$ constructed in Chapter 1.

At this point one may wonder if it is possible to obtain generation results also in the space $L^2(\mathbb{R}^d)$. The answer is positive in the sense that the minimal realization of A in $L^2(\mathbb{R}^d)$ generates a positive symmetric C_0-semigroup $T_2(\cdot)$ on $L^2(\mathbb{R}^d)$ which is also sub-Markovian and ultracontractive. In the first part of Chapter 2, we see that the idea behind the construction of the semigroup $T_2(\cdot)$ relies again on an approximation argument and makes use of sesquilinear forms. For this reason, $T_2(\cdot)$ is consistent with $T(\cdot)$, namely it coincides with $T(\cdot)$ in the intersection $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. Actually, even more is achieved: $T_2(\cdot)$ extends to a positive C_0-semigroup of contractions $T_p(\cdot)$ on $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty)$. They are compact and the spectrum of their corresponding generators is independent of p.

Let us now assume that there exists a Lyapunov function Z for the Schrödinger type operator A, i.e. $0 \le Z \in C^{2+\zeta}(\mathbb{R}^d)$ such that $\lim_{|x|\to\infty} Z(x) = \infty$, $AZ(x) \le M$ and $\eta\Delta Z(x) - V(x)Z(x) \le M$ for all $x \in \mathbb{R}^d$ and some constant $M \ge 0$. Then, as mentioned above for the more general elliptic operator (1), for every $f \in C_b(\mathbb{R}^d)$ the semigroup $T(t)$ applied to the initial datum f gives the unique solution (3) to the parabolic problem (2). As usually happens, in general an explicit formula for this solution is not available, thus one tries to find pointwise estimates. Since the semigroup is given through an integral kernel $p(t,x,y)$, this translates in looking for pointwise kernel estimates.

This is the reason why an important aspect in the study of elliptic operators is to have estimates for the kernel p and, consequently, this question has received a lot of attention in the literature. We mention here [1, 4, 9, 10, 11, 30, 31, 34, 41, 50], where specific operators were considered. In particular, in these last years second order elliptic operators with polynomially growing coefficients and their associated semigroups have been widely studied (see for example [12, 13, 14, 15, 16, 21, 36, 37, 43, 44, 45, 46]). For instance, the case of the (non-divergence type) Schrödinger operator

$$(1 + |x|^m)\Delta - |x|^s \tag{5}$$

was discussed extensively in the literature. Kernel estimates were obtained in [16] assuming that $m > 2$, $s > m - 2$ and in [37] when $m \in [0,2)$ and $s > 2$. Furthermore, if $m > 2$ and $s > m - 2$, kernel estimates for the corresponding divergence form operators

$$(1 + |x|^m)\Delta + b|x|^{m-1}\frac{x}{|x|} \cdot \nabla - c|x|^s \tag{6}$$

are proved in [13] and for $c = 0$ in [45]. In the papers mentioned above regarding the operators (5) and (6), the authors used a technique based on the relationship between the log-Sobolev inequality and the ultracontractivity of a suitable semigroup in a weighted space. Let us also mention that for $m = 0$ and $s > 0$ both upper and lower estimates were established in [42].

On this path, the second part of Chapter 2 is devoted to generalize the above results concerning second order elliptic operators with polynomially growing coefficients to our Schrödinger type operator (4). The starting point is the case of bounded diffusion coefficients, see [1, 10, 34, 41]. These techniques were extended to include also unbounded diffusion coefficients in [30, 31] for nonautonomous operators in non-divergence form. In here we adopt the technique of time dependent Lyapunov functions used in [1, 30, 31, 51] to our divergence form setting. In particular, we deal with time dependent Lyapunov functions W for $\partial_t + A$ and $\partial_t + \eta\Delta - V$, so they satisfy

$$\partial_t W(t,x) + AW(t,x) \le h(t)W(t,x)$$

and

$$\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \le h(t)W(t,x),$$

for any $(t,x) \in (0,T) \times \mathbb{R}^d$, where η is the ellipticity constant of the matrix Q and $T > 0$. This allows for a unified approach to obtain kernel bounds corresponding to [13, 42] in the divergence form setting. As a matter of fact, we can allow even more general conditions on m and s in order to get kernel estimate for our prototype operator

$$\operatorname{div}\big((1 + |x|_*^m)\nabla\big) - |x|^s, \tag{7}$$

where $x \mapsto |x|_*$ is a C^2-function satisfying $|x|_* = |x|$ for $|x| \ge 1$. We require merely that $m > 0$ and $s > |m-2|$; moreover, we can drop the assumption $d \ge 3$ imposed in [13, 42]. We first establish sufficient conditions under which functions like $W(t,x) = e^{\varepsilon t^\alpha |x|_*^\beta}$ are time dependent Lyapunov functions for more general operators with polynomially growing coefficients, namely such that

$$\sum_{i,j=1}^d q_{ij}(x)\xi_i\xi_j \le c_q(1 + |x|^m)|\xi|^2, \tag{8}$$

for some constant $c_q > 0$ and every ξ, $x \in \mathbb{R}^d$. Then, applying the technique mentioned above to the kernel p associated to the operator (7), we get the following inequality

$$p(t,x,y) \le Ct^{1-\frac{\alpha(2m\vee s)}{s-m+2}k} e^{-\frac{\varepsilon}{2}t^\alpha |x|_*^{\frac{s-m+2}{2}}} e^{-\frac{\varepsilon}{2}t^\alpha |y|_*^{\frac{s-m+2}{2}}}, \tag{9}$$

for any $t \in (0,1)$, $x, y \in \mathbb{R}^d$, where $k > d+2$, $\beta = \frac{s-m+2}{2}$, $0 < \varepsilon < 1/\beta$, $\alpha > \frac{\beta}{\beta+m-2}$.

As our approach does not depend on the specific structure of the coefficients, we can establish kernel estimates not only in the case where $Q(x) =$

$(1+|x|_*^m)I$. Indeed, an estimate similar to (8) of the quadratic form associated to Q is enough. In addition, we can even leave the setting of polynomially growing coefficients and consider coefficients of exponential growth; this includes the case $Q(x) = e^{|x|^m} I$ and $V(x) = e^{|x|^s}$ for $d \geq 1$ and $2 \leq m < s$. We can then handle operators of the form

$$\mathrm{div}\big(e^{|x|^m}\nabla u\big) - e^{|x|^s}.$$

Here we would like to mention the paper [20] where pointwise estimates are obtained in the elliptic case for exponentially growing coefficients. We stress that these estimates can be improved by choosing a Lyapunov function like

$$W(t,x) = \exp\left(\varepsilon t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau\right).$$

The kernel estimate obtained in this setting is

$$\begin{aligned} p(t,x,y) \leq & Ct^{1-\frac{k}{2}} \exp(Ct^{-\alpha}) \exp\left(-\frac{\varepsilon}{2}t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau\right) \\ & \times \exp\left(-\frac{\varepsilon}{2}t^\alpha \int_0^{|y|_*} e^{\frac{\tau^\beta}{2}}\, d\tau\right), \end{aligned} \tag{10}$$

for any $t \in (0,1)$, $x, y \in \mathbb{R}^d$, where $k > d+2$, $1 + \frac{m}{2} \leq \beta \leq m$, $\varepsilon > 0$, $\alpha > \frac{2\beta+m-2}{2m}$.

Chapters 3 deals with the results in paper [32]. We are concerned with the more general operator

$$A = \mathrm{div}(Q\nabla) + F\cdot\nabla - V, \tag{11}$$

where in addition we assume that the drift term F belongs to $C^{1+\zeta}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. For $F = 0$ we obtain the operator (4) studied in Chapter 2. Thanks to the chosen symmetric structure, in the right hand side of (9) and (10) terms involving both x and y appear. However, all the results in Chapter 2 can be refined in order to deal with the more general operator (11).

In this chapter, we are aim to establish not only estimates for p but also for ∇p, the gradient of p. Apart from the existence of time dependent Lyapunov functions, an important tool to obtain such estimates is the square integrability of the logarithmic gradient of p. Such integrability property plays an important role to obtain regularity results for p, cf. [9, Section 7.4]. Moreover, as in [41], once estimates for ∇p are obtained, one can repeat the same procedure to get estimates for $D^2 p$ and hence estimates for $\partial_t p$. This allows us to obtain the differentiability of the semigroup $T(\cdot)$. Estimates for the gradient of p were obtained in [41, Section 5] in the case of bounded diffusion coefficients. As in [30, 31], we use approximation to extend this to unbounded diffusion coefficients. We point out that the constant in the estimate for ∇p obtained in [41, Thm. 5.3] depend on $\|Q\|_\infty$ and thus this estimate cannot be used in an approximation result. Therefore, we first establish an estimate for ∇p in the case of bounded diffusion coefficients where the constant in the estimate does not depend on $\|Q\|_\infty$. With this estimate at hand, we can then tackle the case of unbounded diffusion coefficients by approximating them with bounded ones. In this way, we can prove our main result which provides an estimate of ∇p in the general case.

We illustrate our results by applying them to the prototype operator

$$A = \mathrm{div}((1+|x|_*^m)\nabla) - |x|^{p-1}x\cdot\nabla - |x|^s,$$

with $p > (m-1)\vee 1$, $s > |m-2|$, $m > 0$. Then, we derive that

$$|\nabla p(t,x,y)| \le C(1-\log t)t^{\frac{3}{2}-\frac{3\alpha(m\vee p\vee\frac{s}{2})k+\alpha}{2\beta}}e^{-\varepsilon t^\alpha|y|_*^\beta},$$

for any $t\in(0,1)$, $x,y\in\mathbb{R}^d$, where $\beta = \frac{s-m+2}{2}$, $k > 2(d+2)$, $0 < 2k\varepsilon < \frac{1}{\beta}$ and $\alpha > \frac{\beta}{\beta+m-2}$.

Chapter 1

The minimal semigroup in $C_b(\mathbb{R}^d)$

Elliptic operators with unbounded coefficients have been studied a lot recently since they have applications in many fields of science, economic and engineering. The literature significantly improved and we are now able to deal with second order elliptic partial differential operators A defined by

$$A\varphi(x) = \sum_{i,j=1}^{d} q_{ij}(x)D_{ij}\varphi(x) + \sum_{i=1}^{d} F_i(x)D_i\varphi(x) - V(x)\varphi(x), \quad x \in \mathbb{R}^d,$$

on smooth functions, where the diffusion coefficients Q, the drift F and the potential V are typically unbounded functions. Throughout, we will keep the following assumptions.

Hypothesis 1.0.1. *(a) The coefficients q_{ij}, F_i and $0 \le V$ belong to $C^{\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\zeta \in (0,1)$ and for all $i, j = 1, \ldots, d$.*

(b) The matrix $Q = (q_{ij})_{i,j=1,\ldots,d}$ is symmetric and uniformly elliptic, i.e. there is $\eta > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \ge \eta|\xi|^2 \quad \textit{for all } x,\, \xi \in \mathbb{R}^d.$$

Our main interest is the parabolic problem associated with A

$$\begin{cases} \partial_t u(t,x) = Au(t,x), & t > 0,\, x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

The aim of this first chapter is to show how to construct analytically the semigroup $T(\cdot)$ associated with A in $C_b(\mathbb{R}^d)$.

If the coefficients of A were bounded, then for any $f \in C_b(\mathbb{R}^d)$ and any $t > 0$ we would define $T(t)f$ as the value at t of the classical solution to the Cauchy problem (1.1). We point out that the boundedness of the coefficients leads to the uniqueness results: it is a straightforward consequence of the classical maximum principle (see [38]). Thus, it is not surprising that in case of unbounded coefficients the bounded classical solution to the parabolic problem (1.1) may not be unique. It is all about the failure of the classical maximum principle.

In Section 1.2 the arguments used to prove the existence of a classical solution are based both on an approximation argument with Cauchy-Dirichlet problems in bounded and smooth domains, and classical Schauder estimates. In particular, we consider in a ball B_ρ of fixed radius $\rho > 0$ the problem (1.1) with Dirichlet boundary conditions on ∂B_ρ. Named $T_\rho(\cdot)$ the associated semigroup, we set $T(\cdot)$ as the limit as $\rho \to \infty$ of $T_\rho(\cdot)$. Since $T_\rho(\cdot)$ is well known (see e.g. [38]), it is not hard to investigate the properties of $T(\cdot)$. It turns out that such a limit defines a positive contraction semigroup in $C_b(\mathbb{R}^d)$ which has an integral representation through a kernel and gives a solution to problem (1.1). Actually, if $f \geq 0$, it gives the minimal positive solution. That's why $T(\cdot)$ is called the minimal semigroup associated with A.

In general, such a semigroup is neither strongly continuous nor analytic in $C_b(\mathbb{R}^d)$, so it does not make sense to consider its infinitesimal generator. However, the corresponding concept in the unbounded context is that of weak generator.

In order to introduce it, we need to take a step back and to study the resolvent equation

$$\lambda u - Au = f, \tag{1.2}$$

with $\lambda > 0$ and $f \in C_b(\mathbb{R}^d)$. With a similar approximation procedure, in Section 1.1 we prove the existence of a solution $u \in D_{\max}(A)$ to the elliptic equation (1.2), where

$$D_{\max}(A) = \{u \in C_b(\mathbb{R}^d) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d) \text{ for all } 1 \leq p < \infty \colon Au \in C_b(\mathbb{R}^d)\}. \tag{1.3}$$

Moreover, u is given by $u(x) = (R(\lambda)f)(x)$, where $R(\lambda)$ is the resolvent of a closed linear operator $\hat{A} = (A, \hat{D})$.

Subsequently, in Section 1.3 we show that $\hat{A}$ is the weak generator of the semigroup $T(\cdot)$ in the sense that the resolvent $R(\lambda)f(x)$ is the Laplace transform of $T(t)f(x)$.

In Section 1.4, we investigate the assumptions that allow us to prove the uniqueness of solutions to the parabolic problem (1.1), the existence of the so-called Lyapunov functions. Even though up to now it is already clear that the parabolic problem (1.1) and the elliptic equation (1.2) are not independent of one another, we highlight the connection by proving that there exists a unique solution to the elliptic equation in $D_{\max}(A)$ if and only if there exists a unique classical solution to the parabolic problem, which is bounded in $[0,T] \times \mathbb{R}^d$ for any $T > 0$. In such a situation, the domain $\hat{D}$ of the weak generator $\hat{A}$ coincides with $D_{\max}(A)$.

Then, in Section 1.5, we explore more properties of Lyapunov functions Z

for the following operator in divergence form

$$A\varphi = \operatorname{div}(Q\nabla\varphi) + F\cdot\nabla\varphi - V\varphi.$$

We discuss the integrability of Z with respect to the measure pdy. We find out that if the diffusion coefficients and their spatial derivatives are bounded, then Z satisfies

$$\int_{\mathbb{R}^d} p(t,x,y)Z(y)\,dy \le e^{\lambda t}Z(x),$$

for all $t\ge 0$, $x\in\mathbb{R}^d$ and some $\lambda\ge 0$. Approximating A with a family of operators with bounded diffusion coefficients, we prove that the same inequality holds if we assume that Z is a Lyapunov function not only for A, but also for $\eta\Delta + F\cdot\nabla - V$. This allows us to show the tightness of the family $\{p(t,x,y)dy \mid t\in(0,T)\}$.

Finally, in Section 1.6, we introduce time dependent Lyapunov functions for $\partial_t + A$, with A the operator in divergence form mentioned above. We proceed similarly to prove the integrability of such functions with respect to the measure pdy. This is an important result we will use in the next chapters.

For more details of the results that we present here, we refer the reader to [36, 47, 48] .

1.1 The resolvent equation

This section is devoted to the study of the elliptic equation (1.2), that is

$$\lambda u - Au = f,$$

with $\lambda > 0$ and $f\in C_b(\mathbb{R}^d)$. We aim to prove that it admits a solution in the maximal domain $D_{\max}(A)$ defined in (1.3). We call $A_{\max}$ the realization of A in $C_b(\mathbb{R}^d)$ with domain $D_{\max}(A)$.

First, we prove that $A_{\max}$ is a closed operator.

Lemma 1.1.1. *[48, Lemma 3.1] The operator $A_{\max}$ is closed.*

Proof. Let (u_n) be a sequence in $D_{\max}(A)$ such that u_n converges to $u\in C_b(\mathbb{R}^d)$ and Au_n to $g\in C_b(\mathbb{R}^d)$ uniformly in $\mathbb{R}^d$. Then, by Theorem C.1.1, for any pair of bounded sets $\Omega\subset\subset\Omega'\subset\subset\mathbb{R}^d$ we have

$$\|u_n-u_k\|_{W^{2,p}(\Omega)} \le C[\|Au_n-Au_k\|_{L^p(\Omega')} + \|u_n-u_k\|_{L^p(\Omega')}] < \infty,$$

for some constant C depending on p,Ω,Ω',A and for every $1<p<\infty$. It follows that (u_n) is a Cauchy sequence in $W^{2,p}(\Omega)$, thus $u\in W^{2,p}(\Omega)$. Since Ω is arbitrary, it implies that $u\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$. Finally, since A is continuous from $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ to $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, we conclude that $g = Au$. $\square$

The next step is to construct a solution of (1.2) in $D_{\max}(A)$ with an approximation argument. We fix a ball B_ρ with $\rho > 0$ and for every $f \in C_b(\overline{B}_\rho)$ we consider the Dirichlet problem in $C_0(B_\rho)$

$$\begin{cases} \lambda u(x) - Au(x) = f(x), & x \in B_\rho, \\ u(x) = 0, & x \in \partial B_\rho. \end{cases} \tag{1.4}$$

It admits a unique solution $u_\rho \in W^{2,p}(B_\rho)$ for all $1 \le p < \infty$, according to Proposition C.3.4. Taking into account the realization A_ρ of the operator A with domain

$$D_\rho(A) = \{u \in C_0(B_\rho) \cap W^{2,p}(B_\rho) \text{ for all } 1 \le p < \infty \colon Au \in C(\overline{B}_\rho)\}, \tag{1.5}$$

this means that the operator $\lambda - A$ is bijective from $D_\rho(A)$ onto $C_b(\overline{B}_\rho)$. Thus, we have that

$$u_\rho = R(\lambda, A_\rho) f, \tag{1.6}$$

where $R(\lambda, A_\rho) := (\lambda I - A_\rho)^{-1}$ is the resolvent operator of A_ρ for $\lambda > 0$.

In the following theorem we construct a solution of (1.2) by taking the limit of the sequence (u_ρ). In other words, the operator $\lambda - A$ is surjective from $D_{\max}(A)$ to $C_b(\mathbb{R}^d)$.

Theorem 1.1.2. *[48, Theorem 3.4] For every $f \in C_b(\mathbb{R}^d)$ there exists $u \in D_{\max}(A)$ solving equation (1.2) and satisfying the inequality*

$$\|u\|_\infty \le \frac{\|f\|_\infty}{\lambda}. \tag{1.7}$$

Moreover, if $f \ge 0$, then $u \ge 0$.

Proof. Let $0 \le f \in C_b(\mathbb{R}^d)$. As above, we consider the solution u_ρ to the problem (1.4) in $C_0(B_\rho)$ for all $\rho > 0$. Applying the maximum principle (Theorem C.2.2) to the functions u_ρ and $u_\rho - u_\sigma$ for any $\sigma \le \rho$, we deduce that the sequence (u_ρ) is nonnegative and increasing. Moreover,

$$\|u_\rho\|_\infty \le \frac{\|f\|_\infty}{\lambda}. \tag{1.8}$$

Then, we may define

$$u(x) := \lim_{\rho \to \infty} u_\rho(x).$$

As a result, not only $u \ge 0$, but we also obtain that (1.7) is valid by letting $\rho \to \infty$ in (1.8). In addition, considering that $Au_\rho = \lambda u_\rho - f$, we infer that

$$\|Au_\rho\|_\infty \le \lambda \|u_\rho\|_\infty + \|f\|_\infty \le 2\|f\|_\infty. \tag{1.9}$$

We now show that u_ρ converges uniformly on compact sets to the function u. For this, we fix $\sigma \le \sigma + 1 < \rho$. Given $1 < p < \infty$, Theorem C.1.1 leads to

$$\|u_\rho\|_{W^{2,p}(B_\sigma)} \le C_\sigma(\|u_\rho\|_{L^p(B_{\sigma+1})} + \|Au_\rho\|_{L^p(B_{\sigma+1})}), \tag{1.10}$$

for some constant $C_\sigma > 0$. Combining this with (1.8) and (1.9) we derive that there exists a positive constant $\tilde{C}_\sigma$ such that

$$\|u_\rho\|_{W^{2,p}(B_\sigma)} \leq \tilde{C}_\sigma \|f\|_\infty .$$

Hence, (u_ρ) is a bounded sequence in $W^{2,p}(B_\sigma)$ for any $1 < p < \infty$ and any fixed $\sigma > 0$. By the Sobolev embedding theorems for $p > d$, it follows that (u_ρ) is bounded in $C^1(\overline{B}_\sigma)$ too and the Ascoli-Arzelà theorem implies that u_ρ converges to u as $\rho \to \infty$ uniformly on compact subsets of $\mathbb{R}^d$. Using the fact that $Au_\rho = \lambda u_\rho - f$ on B_σ, also Au_ρ converges uniformly on compact subsets of $\mathbb{R}^d$.

Applying now (1.10) to the difference $u_{\rho_2} - u_{\rho_1}$, we get

$$\|u_{\rho_2} - u_{\rho_1}\|_{W^{2,p}(B_\sigma)} \leq C_\sigma(\|u_{\rho_2} - u_{\rho_1}\|_{L^p(B_\rho)} + \|Au_{\rho_2} - Au_{\rho_1}\|_{L^p(B_\rho)}),$$

for fixed $\sigma \leq \rho$. Therefore, $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and u_ρ converges to u as $\rho \to \infty$ strongly in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for any $1 < p < \infty$.

Finally, letting $\rho \to \infty$ in $Au_\rho = \lambda u_\rho - f$, we conclude that $u \in D_{\max}(A)$ and it solves the equation (1.2).

If now f is a general function belonging to $C_b(\mathbb{R}^d)$, we write $f = f^+ - f^-$ and, by (1.6), we have

$$u_\rho = R(\lambda, A_\rho)f = R(\lambda, A_\rho)(f^+) - R(\lambda, A_\rho)(f^-) =: u_{\rho,1} - u_{\rho,2}.$$

Using the above proof for $u_{\rho,1}$ and $u_{\rho,2}$, we may define the function u as before. Moreover, since (1.8) is satisfied also when f changes sign on $\mathbb{R}^d$, we obtain again inequality (1.7). Then, the result is valid even for general $f \in C_b(\mathbb{R}^d)$. $\square$

We point out that in general, given the datum $f \in C_b(\mathbb{R}^d)$, the function u we constructed in the previous theorem is not the unique solution to the elliptic equation (1.2) in $D_{\max}(A)$, namely the operator $\lambda I - A$ is not bijective from $D_{\max}(A)$ to $C_b(\mathbb{R}^d)$.

In the following result we define an operator $\hat{A} = (A, \hat{D})$ such that for every $\lambda > 0$, $\lambda I - A$ is bijective from $\hat{D}$ to $C_b(\mathbb{R}^d)$. The idea is to collect in $\hat{D}$ the solutions given by Theorem 1.1.2.

Theorem 1.1.3. *[48, Section 3] There is a family of bounded operators $(R(\lambda))_{\lambda>0}$ on $C_b(\mathbb{R}^d)$ such that for every $f \in C_b(\mathbb{R}^d)$ the solution of the equation (1.2) provided by Theorem 1.1.2 is given by*

$$u(x) = (R(\lambda)f)(x),$$

for any $x \in \mathbb{R}^d$. Moreover, there is a closed linear operator $\hat{A} = (A, \hat{D})$ in $C_b(\mathbb{R}^d)$ such that

$$R(\lambda, \hat{A}) = R(\lambda) \quad \text{and} \quad \hat{D} = R(\lambda)(C_b(\mathbb{R}^d)).$$

Proof. We define the family $(R(\lambda))_{\lambda>0}$ on $C_b(\mathbb{R}^d)$ by

$$R(\lambda)f = \lim_{\rho\to\infty} R(\lambda, A_\rho)f, \tag{1.11}$$

for any $f \in C_b(\mathbb{R}^d)$. We note that, since $u_\rho = R(\lambda, A_\rho)f$, $R(\lambda)f$ is exactly the solution $u \in D_{\max}(A)$ to the equation (1.2) constructed in Theorem 1.1.2 as the limit of the sequence (u_ρ). Moreover, by the proof of Theorem 1.1.2, we have that

$$\|\lambda R(\lambda)\| \le 1$$

and, if $f \ge 0$, the sequence $(R(\lambda, A_\rho)f)$ is nonnegative and increasing.

In view of the application of Proposition B.1.1, we prove that the family $(R(\lambda))_{\lambda>0}$ satisfies the resolvent identity

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\mu)R(\lambda). \tag{1.12}$$

Let $f \ge 0$. Clearly,

$$R(\lambda)R(\mu)f \ge \limsup_{\rho\to\infty} R(\lambda, A_\rho)R(\mu, A_\rho)f.$$

Furthermore, fixed ρ_1, we have

$$\liminf_{\rho\to\infty} R(\lambda, A_\rho)R(\mu, A_\rho)f \ge \liminf_{\rho\to\infty} R(\lambda, A_{\rho_1})R(\mu, A_\rho)f = R(\lambda, A_{\rho_1})R(\mu)f.$$

Letting $\rho_1 \to \infty$ yields $\liminf_{\rho\to\infty} R(\lambda, A_\rho)R(\mu, A_\rho)f \ge R(\lambda)R(\mu)f$. As a result, we derive that $\lim_{\rho\to\infty} R(\lambda, A_\rho)R(\mu, A_\rho)f = R(\lambda)R(\mu)f$. This is enough to deduce (1.12) because the family $(R(\lambda, A_\rho))_{\lambda>0}$ satisfies the resolvent identity. The general case when $f \in C_b(\mathbb{R}^d)$ follows as usual by writing $f = f^+ - f^-$. In addition, we note that the operators $R(\lambda)$ are injective. Indeed, if $R(\lambda)f = 0$, then $f = (\lambda I - A)R(\lambda)f = 0$ because $R(\lambda)$ is a solution of the equation (1.2).

We finally apply Proposition B.1.1 to infer that there exists a closed linear operator $\hat{A}$ whose resolvent is $R(\lambda)$. □

We show that $u = R(\lambda)f$ is the minimal positive solution if $f \ge 0$.

Proposition 1.1.4. *[48, Proposition 3.6] If $f \ge 0$, then $u = R(\lambda)f$ is the minimal element among the nonnegative solutions of (1.2) in $D_{\max}(A)$.*

Proof. Let $0 \le v \in D_{\max}(A)$ be a solution of (1.2). If u_ρ is defined as in (1.6), then $\lambda(v - u_\rho) - A(v - u_\rho) = 0$ in B_ρ and $v - u_\rho = v \ge 0$ on ∂B_ρ. According to Theorem C.2.2, we have that $v \ge u_\rho$ in B_ρ. If we now let $\rho \to \infty$ we deduce that $u \le v$. □

We now prove that the operator $\hat{A}$ is actually a restriction of $A_{\max}$.

Proposition 1.1.5. *[48, Proposition 3.5] The following statements hold.*

(a) $\hat{D} \subset D_{\max}(A)$ *and* $\hat{A}u = Au$ *for* $u \in \hat{D}$.

(b) $\hat{D} = D_{\max}(A)$ *holds if and only if* $\lambda I - A$ *is injective on* $D_{\max}(A)$ *for some (hence all) positive* λ.

(c) $D_{\max}(A) \cap C_0(\mathbb{R}^d) \subset \hat{D}$.

Proof. We recall that, by Theorem 1.1.3, $\hat{D}$ consists of the solutions to the equation (1.2) in $D_{\max}(A)$ given by Theorem 1.1.2, namely functions of the form $R(\lambda)f$ with $f \in C_b(\mathbb{R}^d)$. Then, statement (a) follows. Moreover, since $\lambda I - A$ is always bijective from $\hat{D}$ to $C_b(\mathbb{R}^d)$, it is injective on $D_{\max}(A)$ if and only if $\hat{D}$ and $D_{\max}(A)$ coincide. This proves (b).

We now turn to (c). Let $v \in D_{\max}(A) \cap C_0(\mathbb{R}^d)$ and consider $f = v - Av$ and $u = R(1, \hat{A})f$. Clearly, $f \in C_b(\mathbb{R}^d)$ and $u \in \hat{D}$. If we show that $v = u$, then the statement is proved. If $u_\rho = R(1, A_\rho)f$, then $(u_\rho - v) - A(u_\rho - v) = 0$ in B_ρ. As a result, since u_ρ vanishes on ∂B_ρ, then the maximum principle yields

$$\sup_{|x|\le\rho} |u_\rho(x) - v(x)| = \sup_{|x|=\rho} |u_\rho(x) - v(x)| = \sup_{|x|=\rho} |v(x)|.$$

Hence, letting $\rho \to \infty$, we get

$$|u(x) - v(x)| \le \lim_{|x|\to\infty} |v(x)| = 0,$$

for every $x \in \mathbb{R}^d$, where we used that $v \in C_0(\mathbb{R}^d)$. Thus, $u = v$. $\square$

1.2 The semigroup

Given the parabolic problem (1.1) associated to the operator A, namely

$$\begin{cases} \partial_t u(t,x) = Au(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

in this section we prove the existence of a classical solution. This will allow us to construct the related semigroup $T(\cdot)$.

Let us fix a ball B_ρ with $\rho > 0$. Proposition C.3.2 and Theorem C.3.3 show that, since A is uniformly elliptic on B_ρ, then for every $f \in C(\overline{B}_\rho)$ there is a unique solution to the problem (1.1) in B_ρ with Dirichlet boundary conditions on ∂B_ρ, that is the following problem

$$\begin{cases} \partial_t u_\rho(t,x) = Au_\rho(t,x), & t > 0,\ x \in B_\rho, \\ u_\rho(t,x) = 0, & t > 0,\ x \in \partial B_\rho, \\ u_\rho(0,x) = f(x), & x \in \overline{B}_\rho. \end{cases} \tag{1.13}$$

In other words, the realization A_ρ of the operator A with domain $D_\rho(A)$ defined in (1.5) generates an analytic semigroup $T_\rho(\cdot)$ in the space $C(\overline{B}_\rho)$ such that

$$u_\rho(t,x) = T_\rho(t)f(x),$$

is a solution of (1.13) for any $t > 0$ and $x \in B_\rho$.

We note that, since $D_\rho(A)$ is not dense in $C(\overline{B}_\rho)$, then $T_\rho(\cdot)$ is not strongly continuous. Indeed, the strong continuity at 0 fails. We now recall some properties of the semigroup $T_\rho(\cdot)$, which can be found for example in [39, Chapter 3] and [23, Chapter 3, Section 7].

Proposition 1.2.1. *The following statements hold true.*

(a) $T_\rho(t)f \to f$ *as* $t \to 0$ *uniformly in* $\overline{B}_\rho$ *if and only if* $f \in C_0(B_\rho)$.

(b) $T_\rho(t)f \to f$ *as* $t \to 0$ *uniformly in* $\overline{B}_\sigma$ *for every* $\sigma < \rho$, *hence pointwise in* B_ρ.

(c) $T_\rho(t)$ *is a bounded operator in* $L^p(B_\rho)$ *for every* $1 \le p < \infty$ *and* $t > 0$.

(d) $T_\rho(\cdot)$ *are integral operators, i.e. there exists a kernel* $p_\rho(t,x,y)$ *such that*

$$T_\rho(t)f(x) = \int_{B_\rho} p_\rho(t,x,y)f(y)\,dy,$$

for every $f \in C(\overline{B}_\rho)$. *Moreover, the kernel* p_ρ *is positive, the functions* $p_\rho(t,\cdot,\cdot)$ *and* $p_\rho(t,x,\cdot)$ *are measurable for any* $t > 0$, $x \in \mathbb{R}^d$, *and for every* $y \in B_\rho$, $0 < \varepsilon < \tau$ *we have* $p(\cdot,\cdot,y) \in C^{1+\zeta/2,2+\zeta}((\varepsilon,\tau)\times B_\rho)$ *and it satisfies* $\partial_t p_\rho = Ap_\rho$.

(e) $T_\rho(\cdot)$ *is positive, i.e.* $T_\rho(t)f \ge 0$ *if* $f \ge 0$.

(f) $T_\rho(t)$ *are contractions, i.e.* $\|T_\rho(t)f\|_\infty \le \|f\|_\infty$.

(g) $T_\rho(t)$ *preserves bounded pointwise convergence for every* $t > 0$, *i.e. if* $(f_n) \subset C(\overline{B}_\rho)$ *satisfies* $\|f_n\|_\infty \le C$ *for every* $n \in \mathbb{N}$ *and* $f_n \to f$ *pointwise, then* $T_\rho(t)f_n \to T_\rho(t)f$ *pointwise.*

(h) *For every* $f \in C(\overline{B}_\rho)$ *and* $0 < \varepsilon < \tau$ *the function* $u_\rho(t,x) = T_\rho(t)f(x)$ *belongs to* $C^{1+\zeta/2,2+\zeta}\ ((\varepsilon,\tau)\times B_\rho)$.

As a consequence of the weak maximum principle, we deduce that the semigroups $T_\rho(\cdot)$ are increasing in the sense of the following lemma.

Lemma 1.2.2. *[48, Lemma 4.1] Let* $f \in C_b(\mathbb{R}^d)$, $f \ge 0$ *and* $\rho < \rho_1 < \rho_2$. *Then*

$$0 \le T_{\rho_1}(t)f(x) \le T_{\rho_2}(t)f(x),$$

for every $t \ge 0$ *and* $x \in B_\rho$.

Proof. We start with proving the result in case of f vanishing on ∂B_{ρ_1}. We consider the function

$$w(t,x) = T_{\rho_2}(t)f(x) - T_{\rho_1}(t)f(x).$$

Since $f \in C_0(B_{\rho_1})$, then by Proposition 1.2.1(a) we obtain that w is continuous on $[0, \infty) \times \overline{B}_{\rho_1}$. Moreover, w solves the parabolic equation $\partial_t w = Aw$ and vanishes for $t = 0$. For $x \in \partial B_{\rho_1}$ we have that $T_{\rho_2}(t)f(x) \geq 0$ since $T_{\rho_2}(\cdot)$ is positive by Proposition 1.2.1(e) and $T_{\rho_1}(t)f(x) = 0$. Thus, $w \geq 0$ on $[0, \infty) \times \partial B_{\rho_1}$. Applying Proposition C.2.3 we get that $w(t, x) \geq 0$ in $[0, \infty) \times \overline{B}_{\rho_1}$.

We now turn to the general case. If $f \in C_b(\mathbb{R}^d)$ with $f \geq 0$, we approximate f in the $L^2(B_{\rho_2})$ norm with a nonnegative sequence of continuous functions (f_n) vanishing on ∂B_{ρ_1}. Then, for any $n \in \mathbb{N}$ we define

$$w_n(t, x) = T_{\rho_2}(t)f_n(x) - T_{\rho_1}(t)f_n(x),$$

for all $x \in B_{\rho_1}$. Since $T_{\rho_i}(\cdot)$ is bounded in $L^2(B_{\rho_i})$ for $i = 1, 2$ by Proposition 1.2.1(c) and $f_n \to f$ in $L^2(B_{\rho_2})$, then $w_n(t, \cdot) \to w(t, \cdot)$ in $L^2(B_{\rho_1})$ for all $t > 0$. Moreover, by what we proved above, $w_n \geq 0$ in $[0, \infty) \times \overline{B}_{\rho_1}$. It follows that $w \geq 0$ and, thus, $T_{\rho_1}(t)f(x) \leq T_{\rho_2}(t)f(x)$ for all $x \in B_{\rho_1}$.

Finally, by the positivity of $T_{\rho_1}(\cdot)$, we deduce that $T_{\rho_1}(t)f(x) \geq 0$. $\square$

In view of the previous lemma, we can define the semigroup $T(\cdot)$ associated with A in $\mathbb{R}^d$. We set

$$T(t)f(x) = \lim_{\rho \to \infty} T_\rho(t)f(x),$$

for all $0 \leq f \in C_b(\mathbb{R}^d)$ and $T(t)f = T(t)f^+ - T(t)f^-$ for general $f \in C_b(\mathbb{R}^d)$.

Proposition 1.2.3. *$T(\cdot)$ is a positive contraction semigroup in $C_b(\mathbb{R}^d)$.*

Proof. Clearly, $T(t)$ is a linear operator. Moreover, taking into account Proposition 1.2.1(e)-(f), the positivity and the contractivity of $T(\cdot)$ are inherited by that of $T_\rho(\cdot)$.

We now check the semigroup law. Let $0 \leq f \in C_b(\mathbb{R}^d)$. On one hand, we have

$$T(t+s)f(x) = \lim_{\rho \to \infty} T_\rho(t+s)f(x) = \lim_{\rho \to \infty} T_\rho(t)T_\rho(s)f(x) \leq T(t)T(s)f(x).$$

On the other hand, by the monoticity and the boundedness of the sequence $(T_\rho(t)f)$, for every $\rho_1 > 0$ we get

$$T(t+s)f(x) = \lim_{\rho \to \infty} T_\rho(t)T_\rho(s)f(x) \geq \lim_{\rho \to \infty} T_{\rho_1}(t)T_\rho(s)f(x) = T_{\rho_1}(t)T(s)f(x).$$

Letting $\rho_1 \to \infty$ we find that $T(t+s)f(x) \geq T(t)T(s)f(x)$, thus the semigroup law in case $f \geq 0$. For general f it suffices to write $f = f^+ - f^-$ and use the linearity of $T(t)$. $\square$

We now show that, for any $f \in C_b(\mathbb{R}^d)$, $T(t)f(x)$ is a solution to the parabolic equation $\partial_t u = Au$.

Theorem 1.2.4. *[48, Theorem 4.2] For $f \in C_b(\mathbb{R}^d)$, let $u(t,x) = T(t)f(x)$ for $t \geq 0$, $x \in \mathbb{R}^d$. Then u belongs to the space $C^{1+\zeta/2,2+\zeta}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ and satisfies the equation*

$$\partial_t u(t,x) = \sum_{i,j=1}^{d} q_{ij}(x) D_{ij} u(t,x) + \sum_{i=1}^{d} F_i(x) D_i u(t,x) - V(x) u(t,x).$$

Proof. Fix $\varepsilon, \tau, \sigma > 0$ such that $0 < \varepsilon < \tau$. By interior Schauder estimates (see Theorem C.1.3) there exists a constant C such that

$$\|u_\rho\|_{C^{1+\zeta/2,2+\zeta}([\varepsilon,\tau]\times \overline{B}_\sigma)} \leq C \|u_\rho\|_\infty ,$$

for any $\rho > \sigma$. Then, since $T_\rho(t)$ are contractions by Proposition 1.2.1(f), we have

$$\|u_\rho\|_{C^{1+\zeta/2,2+\zeta}([\varepsilon,\tau]\times \overline{B}_\sigma)} \leq C \|f\|_\infty .$$

It follows by Ascoli-Arzelà theorem that u_ρ converges to u uniformly in $[\varepsilon,\tau] \times \overline{B}_\sigma$. We now apply again interior Schauder estimates for $0 < \sigma_1 < \sigma$ and $\varepsilon < \varepsilon_1 < \tau_1 < \tau$ obtaining that there exists a constant C' such that

$$\|u_{\rho_2} - u_{\rho_1}\|_{C^{1+\zeta/2,2+\zeta}([\varepsilon_1,\tau_1]\times \overline{B}_{\sigma_1})} \leq C' \|u_{\rho_2} - u_{\rho_1}\|_{L^\infty([\varepsilon,\tau]\times \overline{B}_\sigma)} .$$

Combining this with the fact that (u_ρ) is a Cauchy sequence in $L^\infty([\varepsilon,\tau] \times \overline{B}_\sigma)$ we get that (u_ρ) is a Cauchy sequence in $C^{1+\zeta/2,2+\zeta}([\varepsilon_1,\tau_1] \times \overline{B}_{\sigma_1})$. Hence, $u_\rho \to u$ in $C^{1+\zeta/2,2+\zeta}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ and, thus, $u \in C^{1+\zeta/2,2+\zeta}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ and $\partial_t u = Au$. $\square$

In the following result we obtain an integral representation of the semigroup $T(\cdot)$ from that of $T_\rho(\cdot)$. Indeed, the integral kernel of $T(\cdot)$ is obtained as the limit of the kernels p_ρ that represent $T_\rho(\cdot)$.

Theorem 1.2.5. *[48, Theorem 4.4] The semigroup $T(\cdot)$ can be represented in the form*

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, dy, \tag{1.14}$$

for $f \in C_b(\mathbb{R}^d)$. Moreover, the integral kernel p enjoys the following properties:

(a) $p = p(t,x,y)$ is a positive function in $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and the functions $p(t,\cdot,\cdot)$ and $p(t,x,\cdot)$ are measurable for any $t > 0$, $x \in \mathbb{R}^d$;

(b) for almost every $y \in \mathbb{R}^d$ the function $p(\cdot,\cdot,y)$ belongs to $C^{1+\zeta/2,2+\zeta}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ and solves the equation $\partial_t p = Ap$.

Proof. Taking into account Proposition 1.2.1(d), we have that every semigroup $T_\rho(\cdot)$ is represented by the kernel p_ρ in B_ρ. Moreover, by Lemma 1.2.2, if $0 \leq f \in C_b(\mathbb{R}^d)$, then $T_\rho(t)f$ converge monotonically to $T(t)f$. Consequently, for every $0 \leq f \in C_b(\mathbb{R}^d)$ and $0 < \rho < \rho_1 < \rho_2$, we have

$$\int_{B_\rho} [p_{\rho_2}(t,x,y) - p_{\rho_1}(t,x,y)]\, f(y)\, dy \geq 0,$$

for any $t \geq 0$ and $x \in B_\rho$. Since the function $p_{\rho_2}(t,x,\cdot) - p_{\rho_1}(t,x,\cdot)$ is continuous in B_ρ for every $t > 0$ and $x \in B_\rho$, it follows that $p_{\rho_2}(t,x,y) \geq p_{\rho_1}(t,x,y)$ for all $t > 0$ and $x, y \in B_\rho$. As a result, also the kernels p_ρ increase with ρ.

We claim that the semigroup $T(\cdot)$ is represented by the kernel p defined as follows

$$p(t,x,y) = \lim_{\rho\to\infty} p_\rho(t,x,y).$$

First, we observe that p is finite almost everywhere. For that, it suffices to take $f = \mathbb{1}$ in Proposition 1.2.1 (f) to get

$$\int_{B_\rho} p_\rho(t,x,y)\, dy \leq 1,$$

for all $t > 0$, $x \in B_\rho$ and $\rho > 0$. If we now let $\rho \to \infty$, the monotone convergence leads to

$$\int_{\mathbb{R}^d} p(t,x,y)\, dy \leq 1,$$

for all $t > 0$ and $x \in \mathbb{R}^d$. This shows that $p(t,x,y)$ is finite for any $t > 0$, $x \in \mathbb{R}^d$ and almost any $y \in \mathbb{R}^d$. Moreover, by monotone convergence we have

$$T(t)f(x) = \lim_{\rho\to\infty} T_\rho(t)f(x) = \lim_{\rho\to\infty} \int_{B_\rho} p_\rho(t,x,y)f(y)\, dy = \int_{\mathbb{R}^d} p(t,x,y)f(y)\, dy,$$

for positive $f \in C_b(\mathbb{R}^d)$. For general f we put $f = f^+ - f^-$ in the previous expression and we obtain (1.14). In addition, each kernel p_ρ is positive and the functions $p_\rho(t,\cdot,\cdot)$ and $p_\rho(t,x,\cdot)$ are measurable for any $t > 0$, $x \in \mathbb{R}^d$. Thus, the limit p satisfies (a).

The rest of the proof is devoted to the regularity of p. We fix $y \in \mathbb{R}^d$ such that $p(t,x,y)$ is finite for any $t > 0$ and $x \in \mathbb{R}^d$. Then, we apply the parabolic Harnack inequality (see e.g. [35, Chapter VII]) for $0 < \varepsilon < \tau$, $t_1 > \tau$ and $\sigma > 1$, obtaining that there exists a constant $C > 0$ such that

$$\sup_{\varepsilon\leq t\leq\tau, x\in\overline{B}_\sigma} [p_{\rho_2}(t,x,y) - p_{\rho_1}(t,x,y)] \leq C \inf_{x\in\overline{B}_\sigma} [p_{\rho_2}(t_1,x,y) - p_{\rho_1}(t_1,x,y)].$$

Taking $\rho_1, \rho_2 \to \infty$ and considering that $p(t_1,x,y) < \infty$ for some $x \in B_\sigma$, we have that

$$\inf_{x\in\overline{B}_\sigma} [p_{\rho_2}(t_1,x,y) - p_{\rho_1}(t_1,x,y)] \to 0.$$

Therefore, $p_\rho(\cdot,\cdot,y)$ converges to $p(\cdot,\cdot,y)$ as $\rho \to \infty$ uniformly in $[\varepsilon,\tau] \times \overline{B}_\sigma$. Finally, fixed $\sigma_1 < \sigma$ and $\varepsilon < \varepsilon_1 < \tau_1 < \tau$, interior Schauder estimates (see Theorem C.1.3) yield

$$\begin{aligned}
&\|p_{\rho_2}(\cdot,\cdot,y) - p_{\rho_1}(\cdot,\cdot,y)\|_{C^{1+\zeta/2,2+\zeta}([\varepsilon_1,\tau_1]\times\overline{B}_{\sigma_1})} \\
&\leq C' \|p_{\rho_2}(\cdot,\cdot,y) - p_{\rho_1}(\cdot,\cdot,y)\|_{L^\infty([\varepsilon,\tau]\times\overline{B}_\sigma)},
\end{aligned}$$

for some constant $C' > 0$. Since the right hand side of the previous inequality converges to 0, then $(p_\rho(\cdot,\cdot,y))$ is a Cauchy sequence in $C^{1+\zeta/2,2+\zeta}([\varepsilon_1,\tau_1] \times \overline{B}_{\sigma_1})$. We conclude that $p(\cdot,\cdot,y)$ belongs to $C^{1+\zeta/2,2+\zeta}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ for almost all $y \in \mathbb{R}^d$ and $\partial_t p = Ap$. $\square$

Next, we investigate the continuity properties of the semigroup $T(\cdot)$, such as the continuity of the function $u(t,x) = T(t)f(x)$ up to $t = 0$. From that, it follows that, for any $f \in C_b(\mathbb{R}^d)$, $T(t)f(x)$ is actually a classical solution to the parabolic problem (1.1).

Proposition 1.2.6. *[48, Proposition 4.3, Theorem 4.5, Proposition 4.6] The following statements hold.*

(a) If $f \in C_0(\mathbb{R}^d)$, then $T(t)f \to f$ as $t \to 0$ uniformly on $\mathbb{R}^d$.

(b) If $f \in C_b(\mathbb{R}^d)$, then $T(t)f \to f$ as $t \to 0$ uniformly on compact subsets of $\mathbb{R}^d$.

(c) If (g_n) is a bounded sequence in $C_b(\mathbb{R}^d)$ and $g_n(x) \to g(x)$ for every $x \in \mathbb{R}^d$ with $g \in C_b(\mathbb{R}^d)$, then $T(t)g_n(x) \to T(t)g(x)$ as $n \to \infty$ in $C^{1,2}((0,\infty) \times \mathbb{R}^d)$.

Proof. We start with proving (a) if $f \in C^2(\mathbb{R}^d)$ with support contained in B_ρ. In such a case we have that $f \in D_\rho(A)$. Then, since A_ρ is sectorial by Theorem C.3.3, we apply Proposition B.3.4 to infer that

$$T_\rho(t)f(x) - f(x) = \int_0^t T_\rho(s)Af(x)\,ds,$$

for any $x \in \overline{B}_\rho$. Applying the dominated convergence theorem we get

$$T(t)f(x) - f(x) = \int_0^t T(s)Af(x)\,ds, \tag{1.15}$$

for any $x \in \overline{B}_\rho$, but also for $x \in \mathbb{R}^d$ by the arbitrarity of ρ. Hence, since $T(t)$ is a contraction by Proposition 1.2.3, we obtain

$$\|T(t)f - f\|_\infty \leq \int_0^t \|T(s)Af\|_\infty\,ds \leq t\,\|Af\|_\infty\,.$$

Letting $t \to 0$ we deduce that $T(t)f$ converges to f uniformly.

If now f is a generic function belonging to $C_0(\mathbb{R}^d)$, we approximate f with a sequence (f_n) of C^2-functions such that f_n has support contained in B_{σ_n}. Then, by the contractivity of $T(\cdot)$, we derive that

$$\begin{aligned}\|T(t)f - f\|_\infty &\leq \|T(t)f - T(t)f_n\|_\infty + \|T(t)f_n - f_n\|_\infty + \|f_n - f\|_\infty \\ &\leq 2\,\|f_n - f\|_\infty + \|T(t)f_n - f_n\|_\infty\,.\end{aligned}$$

Since $T(t)f_n$ converges to f_n uniformly as $t \to 0$ by what we proved above, taking the limsup as $t \to 0$ and then the limit as $n \to \infty$ in the previous inequality yields (a).

We now prove (b). We define

$$p(t,x,B) = \int_B p(t,x,y)\,dy,$$

for any measurable set $B \subset \mathbb{R}^d$. We preliminarily show that

$$p(t, x, \mathbb{R}^d \setminus B_{2\rho}) \to 0 \tag{1.16}$$

as $t \to 0$ uniformly on $\overline{B}_\rho$ for every $\rho > 0$. For that, let $f_1, f_2 \in C_0(\mathbb{R}^d)$ be such that $0 \le \chi_{B_\rho} \le f_1 \le \chi_{B_{2\rho}} \le f_2 \le 1$. Since $f_1 \le 1$ and vanishes in $\mathbb{R}^d \setminus B_{2\rho}$, we deduce by Theorem 1.2.5 that

$$T(t)f_1(x) = \int_{\mathbb{R}^d} p(t, x, y) f_1(y)\, dy \le \int_{B_{2\rho}} p(t, x, y)\, dy = p(t, x, B_{2\rho}),$$

for every $x \in \mathbb{R}^d$. Moreover, considering that $f_2 \ge 0$ and $f_2 = 1$ in $B_{2\rho}$, for any $x \in \mathbb{R}^d$ we get

$$p(t, x, B_{2\rho}) = \int_{B_{2\rho}} p(t, x, y) f_2(y)\, dy \le \int_{\mathbb{R}^d} p(t, x, y) f_2(y)\, dy = T(t)f_2(x).$$

Combining the both estimates, we find that

$$T(t)f_1(x) \le p(t, x, B_{2\rho}) \le T(t)f_2(x),$$

for any $x \in \mathbb{R}^d$. Therefore, given that $T(t)f_i \to 1$ as $t \to 0$ uniformly on $\overline{B}_\rho$ for $i = 1, 2$ by (a), we obtain that also $p(t, x, B_{2\rho}) \to 1$ uniformly on $\overline{B}_\rho$. Hence, we deduce (1.16) by the following chain of inequalities

$$0 \le p(t, x, \mathbb{R}^d \setminus B_{2\rho}) = p(t, x, \mathbb{R}^d) - p(t, x, B_{2\rho}) \le 1 - p(t, x, B_{2\rho}).$$

We are now ready to prove (b). Let $0 \le f \in C_b(\mathbb{R}^d)$ and consider a function $\eta \in C_0(\mathbb{R}^d)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B_{2\rho}$ and with support contained in $B_{3\rho}$. Using the positivity of $T(\cdot)$ by Proposition 1.2.3 and its integral representation given by Theorem 1.2.5, we have

$$\begin{aligned}
|T(t)f(x) - T(t)(\eta f)(x)| &= T(t)((1-\eta)f)(x) \\
&= \int_{\mathbb{R}^d} p(t, x, y)(1 - \eta(y)) f(y)\, dy \\
&\le \|f\|_\infty \int_{\mathbb{R}^d \setminus B_{2\rho}} p(t, x, y)\, dy \\
&= \|f\|_\infty\, p(t, x, \mathbb{R}^d \setminus B_{2\rho}).
\end{aligned}$$

Thus, according to (1.16), we have that the left hand side tends to 0 as $t \to 0$ uniformly on $\overline{B}_\rho$. Moreover, since $\eta f \in C_0(\mathbb{R}^d)$, then $\|T(t)(\eta f) - (\eta f)\|_\infty \to 0$ as $t \to 0$ by (a). In conclusion, since on B_ρ we have $T(t)f - f = T(t)f - T(t)(\eta f) + T(t)(\eta f) - \eta f$, we write

$$\|T(t)f - f\|_\infty \le \|T(t)f - T(t)(\eta f)\|_\infty + \|T(t)(\eta f) - (\eta f)\|_\infty .$$

It follows that $T(t)f$ converges to f as $t \to 0$ uniformly on $\overline{B}_\rho$. Considering $f = f^+ - f^-$, we get the result for general $f \in C_b(\mathbb{R}^d)$. This proves (b).

We now show that (c) holds. For that, fix $0 < \varepsilon < \tau$, $\sigma > 0$ and let (g_n) be a bounded sequence in $C_b(\mathbb{R}^d)$ such that $g_n(x) \to g(x) \in C_b(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$. We prove that $T(t)g_n(x) \to T(t)g(x)$ uniformly for $(t,x) \in [\varepsilon,\tau] \times \overline{B}_\sigma$.

Since Theorem 1.2.5 provides us with an integral representation for the semigroup $T(\cdot)$, it suffices to apply the dominated convergence theorem to infer that

$$T(t)g_n(x) \to T(t)g(x),$$

for any $x \in \mathbb{R}^d$. By the boundedness of the sequence (g_n), we have that $\sup_n \|g_n\|_\infty \le K$ for some constant $K > 0$. Combining this with the fact that $T(\cdot)$ is a contraction semigroup by Proposition 1.2.3, we deduce that

$$\sup_n \|T(t)g_n\|_\infty \le K,$$

for every $t \ge 0$. If we apply the interior Schauder estimates in Theorem C.1.2, we derive that the sequence $(T(\cdot)g_n)$ is bounded in $C^{1+\zeta/2,2+\zeta}([\varepsilon,\tau] \times \overline{B}_\sigma)$. The Ascoli-Arzelà theorem implies that there exists a subsequence $(T(\cdot)f_{n_k})$ converging uniformly in $[\varepsilon,\tau] \times \overline{B}_\sigma$ to a function $v \in C^{1+\zeta/2,2+\zeta}([\varepsilon,\tau] \times \overline{B}_\sigma)$. Since $T(\cdot)g_n$ converges pointwise to $T(\cdot)g$ in $(0,\infty) \times \mathbb{R}^d$, we obtain that $v = T(\cdot)g$ and $(T(\cdot)g_n)$ converges to $T(\cdot)g$ uniformly in $[\varepsilon,\tau] \times \overline{B}_\sigma$. We also get that $T(t)g_n(x) \to T(t)g(x)$ as $n \to \infty$ in $C^{1,2}((0,\infty) \times \mathbb{R}^d)$. □

In general, given $f \in C_b(\mathbb{R}^d)$, $u(t,x) = T(t)f(x)$ is not the unique classical solution to the problem (1.1) which is bounded in $[0,T] \times \mathbb{R}^d$ for any $T > 0$. However, if $f \ge 0$, u is the minimal positive solution in the sense of the next proposition.

Proposition 1.2.7. *For $f \ge 0$, let $u(t,x) = T(t)f(x)$ for $t \ge 0$, $x \in \mathbb{R}^d$. If v is another positive solution to the parabolic problem (1.1), then $v \ge u$.*

Proof. If we apply the weak maximum principle (Proposition C.2.3) to the function $v(t,x) - u_\rho(t,x)$ for any $t > 0$ and $x \in B_\rho$, then we get $v(t,x) \ge u_\rho(t,x)$. Taking $\rho \to \infty$ leads to $v \ge u$. □

Since $T(\cdot)$ selects the minimal from among all bounded positive solutions to the problem (1.1), we sometimes refer to it as the *minimal semigroup associated with A*.

1.3 The weak generator

Since the semigroup $T(\cdot)$ is not strongly continuous in $C_b(\mathbb{R}^d)$, it is not possible to consider the infinitesimal generator in the usual sense. However, we can take the Laplace transform of the semigroup $\int_0^\infty e^{-\lambda t} T(t)f(x)\,dt$ for $\lambda > 0$ and $x \in \mathbb{R}^d$. If $(\int_0^\infty e^{-\lambda t} T(t)f(x)\,dt)_{\lambda>0}$ is the resolvent family of a closed operator, then that operator is called the weak generator of $T(\cdot)$ and we write

$$R(\lambda, A)f(x) = \int_0^\infty e^{-\lambda t} T(t)f(x)\,dt,$$

for $\lambda > 0$ and $x \in \mathbb{R}^d$. In our situation the generator is $\hat{A} = (A, \hat{D})$ defined in Theorem 1.1.3.

Proposition 1.3.1. *[41, Proposition 5.1] The generator of $T(\cdot)$ is $\hat{A}$. In particular it coincides with $A_{\max}$ if and only if $\lambda I - A$ is injective on $D_{\max}(A)$.*

Proof. For $\lambda > 0$, $f \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have

$$\begin{aligned} R(\lambda, A)f(x) &= \int_0^\infty e^{-\lambda t} T(t)f(x)\,dt = \lim_{\rho\to\infty} \int_0^\infty e^{-\lambda t} T_\rho(t)f(x)\,dt \\ &= \lim_{\rho\to\infty} R(\lambda, A_\rho)f(x) = R(\lambda)f(x) = R(\lambda, \hat{A})f(x). \end{aligned}$$

This show that $R(\lambda, A)$ is exactly the resolvent family of $\hat{A}$ defined in (1.11). The last part of the statement follows from Proposition 1.1.5. $\square$

In [36, Proposition 2.3.1] it is proved that $\hat{A}$ is given by the following direct description

$$\begin{aligned} \hat{D} = \Big\{ & u \in C_b(\mathbb{R}^d) \colon \sup_{t>0} \left\| \frac{T(t)u - u}{t} \right\|_\infty < \infty, \ \exists g \in C_b(\mathbb{R}^d) \text{ such that} \\ & \lim_{t\to 0} \frac{T(t)u(x) - u(x)}{t} = g(x) \ \forall x \in \mathbb{R}^d \Big\}, \\ Au(x) = & \lim_{t\to 0} \frac{T(t)u(x) - u(x)}{t} \quad \text{for } u \in \hat{D}. \end{aligned}$$

Moreover, if $f \in \hat{D}$, then $T(t)f \in \hat{D}$ for every $t \geq 0$ and we have

$$\partial_t T(t)f(x) = AT(t)f(x) = T(t)Af(x), \tag{1.17}$$

for any $t > 0$ and $x \in \mathbb{R}^d$.

In the following result we point out that identity (1.17) is valid for C^2-functions with compact support.

Corollary 1.3.2. *If $f \in C_c^2(\mathbb{R}^d)$, then $\partial_t T(t)f(x) = T(t)Af(x)$ for any $t > 0$ and $x \in \mathbb{R}^d$.*

Proof. Let $f \in C_c^2(\mathbb{R}^d)$. Since $C_c^2(\mathbb{R}^d) \subset D_{\max}(A) \cap C_0(\mathbb{R}^d)$, we apply Proposition 1.1.5 to infer that $f \in \hat{D}$. Then the statement follows by (1.17). Alternatively, it is possible to directly compute the derivative of $T(t)f(x)$ by dividing by t equation (1.15) and then let $t \to 0$. $\square$

As a consequence of the previous result, we prove the following lemma that will be very useful in the next chapters.

Lemma 1.3.3. *[41, Lemma 2.1] Let $0 \leq a < b$ and $\varphi \in C_c^{1,2}(Q(a,b))$. Then*

$$\begin{aligned} &\int_{Q(a,b)} (\partial_t \varphi(t,y) + A\varphi(t,y)) p(t,x,y)\,dt\,dy \\ &\qquad = \int_{\mathbb{R}^d} (p(b,x,y)\varphi(b,y) - p(a,x,y)\varphi(a,y))\,dy. \end{aligned}$$

Proof. Let $\varphi \in C_c^{1,2}(Q(a,b))$. Then, applying Corollary 1.3.2, we have

$$\partial_t(T(t)\varphi(\cdot,t)) = T(t)\partial_t\varphi(\cdot,t) + T(t)A\varphi(\cdot,t).$$

The thesis follows by integrating the previous identity over $[a,b]$ and writing $T(t)$ in terms of the kernel p. $\square$

1.4 Lyapunov functions

In the previous sections we proved the existence of solutions to the problems (1.1) and (1.2) for any $f \in C_b(\mathbb{R}^d)$ and $\lambda > 0$. Now we show that the uniqueness of such solutions is ensured by the existence of a Lyapunov function. In the next chapter we will see some examples in case of operators in divergence form with polynomially or exponentially growing diffusion coefficients.

Definition 1.4.1. *We say that a function $Z : \mathbb{R}^d \to [0,\infty)$ is a* Lyapunov function for A *if $0 \le Z \in C^2(\mathbb{R}^d)$ for some $\zeta \in (0,1)$ such that $\lim_{|x|\to\infty} Z(x) = \infty$ and there is a constant $\lambda \ge 0$ such that*

$$AZ(x) \le \lambda Z(x) \tag{1.18}$$

for all $x \in \mathbb{R}^d$.

We sometimes say that Z is a Lyapunov function for A with respect to λ when we want to underline the constant $\lambda \ge 0$ which satifies inequality (1.18).

Remark 1.4.2. *If Z is a Lyapunov function for A, then $Z + C$ is also a Lyapunov function for A for any positive constant C. So, one can assume without loss of generality that a Lyapunov function Z for A satisfies $Z(x) \ge 1$ for all $x \in \mathbb{R}^d$.*

We need the following lemma which provides us with a local maximum principle for functions in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$.

Lemma 1.4.3. *[38, Proposition 4.2.1] Assume that $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, for any $p \in [1,\infty)$, and that $Au \in C(\mathbb{R}^d)$. If x_0 is a local maximum (resp. minimum) of u, then*

$$Au(x_0) + V(x_0)u(x_0) \le 0 \qquad (\textit{resp. } Au(x_0) + V(x_0)u(x_0) \ge 0)$$

We are now ready to prove the main theorem of this section.

Theorem 1.4.4. *Assume that A has a Lyapunov function Z. Then the following statements hold.*

(a) Fixed $T > 0$, if $u, v \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ are solutions of problem (1.1), then $u = v$.

(b) *The function* $u(t,x) = T(t)f(x)$ *is the unique classical solution of the Cauchy problem* (1.1) *which is bounded in each strip* $[0,T] \times \mathbb{R}^d$.

(c) *The operator* $\lambda I - A$ *is injective on* $D_{\max}(A)$ *for any* $\lambda > 0$.

(d) *The weak generator* $\hat{A} = (A, \hat{D})$ *coincides with* $A_{\max} = (A, D_{\max})$.

Proof. Statement (b) easily follows by (a) and, by Proposition 1.1.5, (c) implies (d). We now prove (a) and (c).

Let $u \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ be a solution of the parabolic problem (1.1) with $f \geq 0$. For $\varepsilon > 0$ we consider the function

$$v_\varepsilon(t,x) = e^{-\lambda t} u(t,x) + \varepsilon Z(x),$$

for any $(t,x) \in [0,T] \times \mathbb{R}^d$. Then, for every $R > 0$, v_ε has a minimum point $(t_0, x_0) \in [0,T] \times \overline{B}_R$. If $v_\varepsilon(t_0,x_0) < 0$, then $t_0 > 0$. Indeed, if $t_0 = 0$, then $v_\varepsilon(0,x_0) = f(x_0) + \varepsilon Z(x_0) \geq 0$, which is not possible. Therefore, $\partial_t v_\varepsilon(t_0,x_0) \leq 0$. On one hand, by Lemma 1.4.3, we deduce that

$$A v_\varepsilon(t_0,x_0) + V(x_0) v_\varepsilon(t_0,x_0) \geq 0. \tag{1.19}$$

On the other hand, since $AZ \leq \lambda Z$, we get

$$\partial_t v_\varepsilon - (A - \lambda I) v_\varepsilon = \varepsilon(\lambda Z - AZ) \geq 0. \tag{1.20}$$

Taking into account that $\partial_t v_\varepsilon(t_0,x_0) \leq 0$ and combining (1.19) with (1.20), we deduce that

$$\begin{aligned} 0 &\leq \partial_t v_\varepsilon(t_0,x_0) - (A - \lambda I) v_\varepsilon(t_0,x_0) \\ &\leq (A + V(x_0)) v_\varepsilon(t_0,x_0) - (A - \lambda I) v_\varepsilon(t_0,x_0) \\ &= (V(x_0) + \lambda) v_\varepsilon(t_0,x_0). \end{aligned}$$

Since $V(x_0) \geq 0$ and $\lambda > 0$, it follows that $v_\varepsilon(t_0,x_0) \geq 0$, so $v_\varepsilon \geq 0$. Letting $\varepsilon \to 0$ we obtain that $u \geq 0$.

We infer that this proves (a). Indeed, if we have $u, v \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ solutions of problem (1.1), then the difference $u - v$ solves the same problem with $f = 0$. Arguing as above yields $u \geq v$ and, taking $v - u$ instead, it leads to $u = v$.

For (c), let $v \in D_{\max}(A)$ with $\lambda v - Av = 0$, where $\lambda > 0$. By local regularity results for elliptic equations in bounded domains, we have that $v \in C^{2+\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, the function

$$u(t,x) = e^{\lambda t} v(x)$$

belongs to $C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ and satisfies problem (1.1) with $f = v$. Then, (b) implies that $u(t,x) = T(t)v(x)$. Since $T(\cdot)$ is a contraction by Proposition 1.2.3, we have $\|u(t,\cdot)\|_\infty \leq \|v\|_\infty$ for every $t > 0$. In addition, since

$$\sup_{x \in \mathbb{R}^d} |u(t,x)| = e^{\lambda t} \|v\|_\infty,$$

we conclude that $\|v\|_\infty = 0$. $\square$

1.5 Integrability of Lyapunov functions

In this section we deal with an operator A in divergence form, namely

$$A\varphi = \operatorname{div}(Q\nabla\varphi) + F\cdot\nabla\varphi - V\varphi,$$

where we assume that A satisfies Hypothesis 1.0.1 and the diffusion coefficients q_{ij} belong to $C^{1+\zeta}_{\rm loc}(\mathbb{R}^d)$ for all $i,j=1,\dots,d$.

We investigate the integrability of Lyapunov functions for A with respect to the measure $p(t,x,y)dy$. We first prove the following lemma.

Lemma 1.5.1. *Let Z be a Lyapunov function for A. Take $\vartheta\in C^\infty_c(\mathbb{R})$ with $\vartheta(t)=1$ for $|t|\le 1$, $\vartheta(t)=0$ for $|t|\ge 2$, $0\le\vartheta\le 1$ and set $\vartheta_m(x)=\vartheta(\left|\frac{x}{m}\right|)$, $F_m=\vartheta_m F$, $V_m=\vartheta_m V$ and*

$$q^{(m)}_{ij} = \vartheta_m q_{ij} + (1-\vartheta_m)\eta\delta_{ij},$$

where δ_{ij} is the Kronecker delta. Moreover, define $Q_m=(q^{(m)}_{ij})$ and

$$A_m = \operatorname{div}(Q_m\nabla) + F_m\cdot\nabla - V_m.$$

Consider the analytic semigroup $T_m(\cdot)$ generated by A_m in $C_b(\mathbb{R}^d)$. Then, for every $f\in C_b(\mathbb{R}^d)$ we have

$$T_m(\cdot)f(\cdot)\to T(\cdot)f(\cdot)$$

in $C^{1,2}((\varepsilon,T)\times B_R)$ as $m\to\infty$ for every $0<\varepsilon<T$ and $R>0$.

Proof. Let $f\in C_b(\mathbb{R}^d)$, $0<\varepsilon<T$ and $R>0$. By [38, Theorem 6.2.9] there exists a positive constant C depending on ε, T and R such that

$$\|T_m(\cdot)f\|_{C^{1+\zeta/2,2+\zeta}([\varepsilon,T]\times B_R)} \le C\,\|T_m(\cdot)f\|_{L^\infty([\varepsilon/2,T]\times B_{R+\varepsilon})} \le C\,\|f\|_{L^\infty(\mathbb{R}^d)}.$$

Then the Ascoli-Arzelà theorem infers that there is a sequence (m_k) such that $T_{m_k}(\cdot)f\to u$ in $C^{1,2}((\varepsilon,T)\times B_R)$ for some function $u\in C^{1+\zeta/2,2+\zeta}_{\rm loc}((0,+\infty)\times\mathbb{R}^d)$. Moreover, we have that $|u(t,x)|\le\|f\|_\infty$ and $\partial_t u - Au = 0$ in $(0,T)\times\mathbb{R}^d$ since $\partial_t T_{m_k}(\cdot)f - A_{m_k}T_{m_k}(\cdot)f = 0$ in $(0,T)\times B_R$ for $m_k>R$. From now on we write $T_m(\cdot)$ instead of $T_{m_k}(\cdot)$. Indeed, if we show the statement for the subsequence $(T_{m_k}(\cdot))$, then it is valid for the whole sequence $(T_m(\cdot))$. We set

$$u(0,x) = f(x),$$

for all $x\in\mathbb{R}^d$. If we prove that $u(t,x)$ is continuous up to $t=0$, then Theorem 1.4.4 implies that $u(t,x)=T(t)f(x)$ for any $t>0$ and $x\in\mathbb{R}^d$.

In the case $f\in C^{2+\zeta}_b(\mathbb{R}^d)$ the continuity of $u(t,x)$ for $t=0$ follows by applying the Schauder estimates for the operator A (see [38, Theorem 6.2.10])

$$\|T_m(\cdot)f\|_{C^{1+\zeta/2,2+\zeta}([0,T]\times B_R)} \le C\,\|f\|_{C^{2+\zeta}(\mathbb{R}^d)}.$$

Indeed, from that we have $T_m(\cdot)f \to T(\cdot)f$ as $m \to \infty$ in $C^{1,2}([0,T] \times B_R)$.

Let now assume $f \in C_c(\mathbb{R}^d)$. We approximate f with a sequence (f_k) of $C_c^{2+\zeta}$-functions such that $f_k \to f$ in $L^\infty(\mathbb{R}^d)$. For any $t > 0$ we have

$$\begin{aligned}\|T_m(t)f - f\|_{L^\infty(B_R)} &\le \|T_m(t)(f - f_k)\|_{L^\infty(B_R)} + \|T_m(t)f_k - f_k\|_{L^\infty(B_R)} \\ &\quad + \|f - f_k\|_{L^\infty(B_R)} \\ &\le \|T_m(t)f_k - f_k\|_{L^\infty(B_R)} + 2\,\|f - f_k\|_{L^\infty(\mathbb{R}^d)}\,.\end{aligned}$$

Since $f_k \in C_b^{2+\zeta}(\mathbb{R}^d)$, we deduce from the previous case that $T_m(\cdot)f_k \to T(\cdot)f_k$ as $m \to \infty$ in $C^{1,2}([0,T] \times B_R)$ for all $k \in \mathbb{N}$. Therefore, letting $m \to \infty$ in the previous inequality, we obtain that

$$\|u(t,\cdot) - f\|_{L^\infty(B_R)} \le \|T(t)f_k - f_k\|_{L^\infty(B_R)} + 2\,\|f - f_k\|_{L^\infty(\mathbb{R}^d)}\,,$$

for any $t > 0$. If we now let $t \to 0$ and $k \to \infty$ we find that $u(t,x) \to f(x)$ as $t \to 0$ for any $x \in B_R$. Thus, as a consequence of Theorem 1.4.4 we derive the statement.

We finally prove that $u(t,x)$ is continuous up to $t = 0$ for a general $f \in C_b(\mathbb{R}^d)$. We consider a function $\varphi \in C_c(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in $B_{R/2}$ and with support contained in B_R. For $t > 0$ we write

$$T_m(t)f = T_m(t)(\varphi f) + T_m(t)((1-\varphi)f).$$

We observe that the function $w = T_m(t)((1-\varphi)f) - \|f\|_\infty T_m(t)(1-\varphi)$ satisfies the equation $D_t w = A_m w$ and $w(0,\cdot) = (1-\varphi)f - (1-\varphi)\,\|f\|_\infty \le 0$. Thus, $w \le 0$. A similar inequality holds if we replace f with $-f$. Then we find that

$$|T_m(t)((1-\varphi)f)| \le \|f\|_\infty T_m(t)(1-\varphi).$$

Hence we get

$$\begin{aligned}|T_m(t)f - f| &\le |T_m(t)(\varphi f) - f| + |T_m(t)((1-\varphi)f)| \\ &\le |T_m(t)(\varphi f) - f| + \|f\|_\infty T_m(t)(1-\varphi) \\ &\le |T_m(t)(\varphi f) - f| + \|f\|_\infty (1 - T_m(t)\varphi).\end{aligned}$$

Since the functions φf and φ belong to $C_c(\mathbb{R}^d)$, we deduce that $T_m(t)(\varphi f) \to T(t)(\varphi f)$ and $T_m(t)\varphi \to T(t)\varphi$ as $m \to \infty$. Letting $m \to \infty$ in the previous inequality yields

$$|u(t,\cdot) - f| \le |T(t)(\varphi f) - f| + \|f\|_\infty (1 - T(t)\varphi).$$

Taking into account that $\varphi = 1$ in $B_{R/2}$ and considering the previous expression in $B_{R/2}$, we let $t \to 0$ and we proceed as above to gain the statement.

□

We now show the main result of this section.

Proposition 1.5.2. *Assume that A satisfies Hypothesis 1.0.1 with $q_{ij} \in C^{1+\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$. If Z is a Lyapunov function for A with respect to λ, then*

$$\int_{\mathbb{R}^d} p(t,x,y)Z(y)\,dy \leq e^{\lambda t}Z(x), \tag{1.21}$$

for every $x \in \mathbb{R}^d$ and $t \geq 0$.

Proof. Let $x \in \mathbb{R}^d$ and $t > 0$. We split the proof in several steps.

Step 1. First of all, we approximate Z with a $C^2_b(\mathbb{R}^d)$-function. Let $\alpha \geq 0$ and set

$$Z_\alpha := Z \wedge \alpha.$$

For every $0 < \varepsilon < 1$ we consider a function $\psi_\varepsilon \in C^\infty(\mathbb{R})$ such that $\psi_\varepsilon(t) = t$ for $t \leq \alpha$, $\psi_\varepsilon(t)$ is constant for $t \geq \alpha + \varepsilon$, $\psi'_\varepsilon \geq 0$ and $\psi''_\varepsilon \leq 0$. We approximate Z with the function $\psi_\varepsilon \circ Z \in C^2(\mathbb{R}^d)$. Indeed, we have that

$$\psi_\varepsilon \circ Z \to Z \ \text{ pointwise as } \varepsilon \to 0,\ \alpha \to +\infty.$$

Now we approximate A with the operator $A_m = \mathrm{div}(Q_m\nabla) + F_m \cdot \nabla - V_m$ and $T(\cdot)$ with the analytic semigroup $T_m(\cdot)$ as in Lemma 1.5.1. We observe that the domain of A_m is $D_{\max}(A_m)$.

Since $\lim_{|x|\to\infty} Z(x) = +\infty$, there exists $M_\alpha > 0$ such that $Z(x) > \alpha+1$ for $|x| > M_\alpha$. Since $\varepsilon < 1$, by definition of ψ_ε it follows that $(\psi_\varepsilon \circ Z)(x)$ is constant outside the compact set $\{Z \leq \alpha + 1\}$ and so $\psi_\varepsilon \circ Z$ is a bounded function. It implies that $A_m\psi_\varepsilon(Z(x))$ is bounded too. Hence, the function $\psi_\varepsilon \circ Z$ belongs to $D_{\max}(A_m)$.

Step 2. We now prove that

$$\partial_t T_m(t)\psi_\varepsilon(Z(x)) \leq T_m(t)(\psi'_\varepsilon(Z(x))A_m Z(x)). \tag{1.22}$$

Using (1.17), we write the left hand side of the previous inequality as follows

$$\partial_t T_m(t)\psi_\varepsilon(Z(x)) = A_m T_m(t)\psi_\varepsilon(Z(x)) = T_m(t)A_m\psi_\varepsilon(Z(x)) \tag{1.23}$$

Moreover, we have that

$$\begin{aligned} A_m\psi_\varepsilon(Z(x)) = &\sum_{i,j=1}^d D_i\left(q^{(m)}_{ij}(x)D_j\psi_\varepsilon(Z(x))\right) + \sum_{i=1}^d F_{m,i}(x)D_i\psi_\varepsilon(Z(x)) \\ &- V_m(x)\psi_\varepsilon(Z(x)) \\ =&\psi'_\varepsilon(Z(x))A_m Z(x) + V_m(x)[Z(x)\psi'_\varepsilon(Z(x)) - \psi_\varepsilon(Z(x))] \\ &+ \psi''_\varepsilon(Z(x))\sum_{i,j=1}^d q^{(m)}_{ij}(x)D_iZ(x)D_jZ(x). \end{aligned} \tag{1.24}$$

On one hand, since $\psi''_\varepsilon \leq 0$, it follows that $t\psi'_\varepsilon(t) \leq \psi_\varepsilon(t)$ for $t \geq 0$. Taking $t = Z(x)$ and given that $V_m \geq 0$, we obtain that

$$V_m(x)[Z(x)\psi'_\varepsilon(Z(x)) - \psi_\varepsilon(Z(x))] \leq 0. \tag{1.25}$$

On the other hand, by Hypothesis 1.0.1, Q is uniformly elliptic, so

$$\begin{aligned}\sum_{i,j=1}^d q_{ij}^{(m)}(x)D_iZ(x)D_jZ(x) =&\vartheta_m(x)\sum_{i,j=1}^d q_{ij}(x)D_iZ(x)D_jZ(x)\\ &+\eta(1-\vartheta_m(x))\,|\nabla Z(x)|^2\\ \geq&\eta\,|\nabla Z(x)|^2\geq 0,\end{aligned}$$

whereas $\psi_\varepsilon''(Z(x))\leq 0$. Then,

$$\psi_\varepsilon''(Z(x))\sum_{i,j=1}^d q_{ij}^{(m)}(x)D_iZ(x)D_jZ(x)\leq 0. \tag{1.26}$$

Combining (1.24) with (1.25) and (1.26) yields

$$A_m\psi_\varepsilon(Z(x))\leq\psi_\varepsilon'(Z(x))A_mZ(x).$$

Therefore, from (1.23) we gain inequality (1.22).

Step 3. Letting $m\to\infty$ in (1.22), we show that

$$\partial_t T(t)\psi_\varepsilon(Z(x))\leq T(t)(\psi_\varepsilon'(Z(x))AZ(x)). \tag{1.27}$$

Since $\psi_\varepsilon\circ Z$ is constant outside the compact set $\{Z\leq\alpha+1\}$ as observed in Step 1, in there we have that $\psi_\varepsilon'(Z(x))=0$. Hence, the right hand side of (1.27) makes sense and for m sufficiently large we infer that

$$T_m(t)(\psi_\varepsilon'(Z(x))A_mZ(x))=T_m(t)(\psi_\varepsilon'(Z(x))AZ(x)).$$

For m large we can then write (1.22) as follows

$$\partial_t T_m(t)\psi_\varepsilon(Z(x))\leq T_m(t)(\psi_\varepsilon'(Z(x))AZ(x)).$$

Letting $m\to\infty$ and using Lemma 1.5.1 since the functions $\psi_\varepsilon(Z(x))$ and $\psi_\varepsilon'(Z(x))AZ(x)$ belong to $C_b(\mathbb{R}^d)$ we obtain inequality (1.27).

Step 4. Letting $\varepsilon\to 0$ in (1.27), we now prove that

$$\partial_t T(t)Z_\alpha(x)\leq\int_{\{Z\leq\alpha\}}p(t,x,y)AZ(y)\,dy. \tag{1.28}$$

First, if we consider the sequence $(\psi_\varepsilon\circ Z)$ with respect to ε, we have that it is bounded, $\psi_\varepsilon\circ Z\in C_b^2(\mathbb{R}^d)$ and $\psi_\varepsilon\circ Z\to Z_\alpha$ pointwise as $\varepsilon\to 0$, with $Z_\alpha\in C_b(\mathbb{R}^d)$. Then, by Proposition 1.2.6(c), we deduce that $T(t)(\psi_\varepsilon\circ Z)\to T(t)Z_\alpha$ in $C^{1,2}((0,+\infty)\times\mathbb{R}^d)$. Consequently, if we look at the left hand side of (1.27), we have

$$\partial_t T(t)\psi_\varepsilon(Z(x))\to\partial_t T(t)Z_\alpha(x)\quad\text{as }\varepsilon\to 0.$$

Second, we apply the dominated convergence theorem in the right hand side of (1.27) because $\psi_\varepsilon'(t)\to\chi_{(-\infty,\alpha]}(t)$ as $\varepsilon\to 0$ and AZ is bounded on the compact set $\{Z\leq\alpha+1\}$. Then we obtain

$$\int_{\{Z\leq\alpha+1\}}p(t,x,y)\psi_\varepsilon'(Z(y))AZ(y)\,dy\to\int_{\{Z\leq\alpha\}}p(t,x,y)AZ(y)\,dy\quad\text{as }\varepsilon\to 0.$$

Then (1.28) follows.

Step 5. Finally, letting $\alpha \to \infty$ in (1.28), we get (1.21). Indeed, since Z is a Lyaponov function for A, (1.28) yields

$$\begin{aligned}\partial_t T(t) Z_\alpha(x) &\leq \int_{\{Z\leq\alpha\}} p(t,x,y) AZ(y)\,dy \leq \lambda \int_{\{Z\leq\alpha\}} p(t,x,y) Z(y)\,dy \\ &\leq \lambda \int_{\{Z\leq\alpha\}} p(t,x,y) Z(y)\,dy + \lambda \int_{\{Z>\alpha\}} p(t,x,y)\alpha\,dy \\ &\leq \lambda \int_{\mathbb{R}^d} p(t,x,y) Z_\alpha(y)\,dy = \lambda T(t) Z_\alpha(x).\end{aligned}$$

Then, by Gronwall's Lemma, it follows that

$$T(t)Z_\alpha(x) \leq e^{\int_0^t \lambda ds} T(0) Z_\alpha(x) = e^{\lambda t} Z_\alpha(x),$$

for all $x \in \mathbb{R}^d$ and $t > 0$. Letting $\alpha \to \infty$ we conclude that (1.21) holds true. □

Corollary 1.5.3. *Assume that A satisfies Hypothesis 1.0.1 with $q_{ij} \in C^{1+\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$ and that there exists a Lyapunov function Z for the operator A. Then, for fixed $T > 0$, the family $\{p(t,x,y)dy \mid t \in (0,T)\}$ is tight, namely for every $\varepsilon > 0$ there is a constant $R > 0$ such that $p(t,x,\mathbb{R}^d \setminus B_R) < \varepsilon$ for any $t \in (0,T)$.*

Proof. Let $R > 0$ be large enough. We have that

$$\int_{\mathbb{R}^d} p(t,x,y)Z(y)\,dy \geq \int_{\mathbb{R}^d\setminus B_R} p(t,x,y)Z(y)\,dy \geq \left(\inf_{\mathbb{R}^d\setminus B_R} Z\right) p(t,x,\mathbb{R}^d \setminus B_R).$$

Therefore, by Proposition 1.5.2 it follows that

$$p(t,x,\mathbb{R}^d \setminus B_R) \leq \frac{1}{\inf_{\mathbb{R}^d\setminus B_R} Z} \int_{\mathbb{R}^d} p(t,x,y)Z(y)\,dy \leq \frac{e^{\lambda T} Z(x)}{\inf_{\mathbb{R}^d\setminus B_R} Z} \to 0$$

as $R \to \infty$. □

Remark 1.5.4. *If $0 \leq Z \in C^2(\mathbb{R}^d)$ such that $\lim_{|x|\to\infty} Z(x) = +\infty$ and there is a constant $M \geq 0$ such that*

$$AZ(x) \leq M,$$

for all $x \in \mathbb{R}^d$, then Z is a Lyapunov function for A. Indeed, as in Remark 1.4.2, one can assume without loss of generality that $Z \geq 1$. So,

$$AZ(x) \leq M \leq MZ(x),$$

for all $x \in \mathbb{R}^d$. Hence, we find that Z is a Lyapunov function for A with respect to M.

1.6 Time dependent Lyapunov functions

As in the previous section, we deal with the operator A in divergence form defined by

$$A\varphi = \operatorname{div}(Q\nabla\varphi) + F \cdot \nabla\varphi - V\varphi.$$

In this section we keep the following assumptions.

Hypothesis 1.6.1. *We have* $Q = (q_{ij})_{i,j=1,\dots,d} \in C^{1+\zeta}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^{d\times d})$, $F = (F_j)_{j=1,\dots,d} \in C^{\zeta}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ *and* $0 \le V \in C^{\zeta}_{\rm loc}(\mathbb{R}^d)$ *for some* $\zeta \in (0,1)$. *Moreover,*

(a) the matrix Q is symmetric and uniformly elliptic, i.e. there is $\eta > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \ge \eta|\xi|^2 \quad \text{for all } x,\, \xi \in \mathbb{R}^d;$$

(b) there is a Lyapunov function Z for A.

We now introduce time dependent Lyapunov functions for $L := \partial_t + A$ as in [1, 30, 31, 41, 51].

Definition 1.6.2. *We say that a function $W : [0,T] \times \mathbb{R}^d \to [0,\infty)$ is a* time dependent Lyapunov function *for L if $W \in C^{1,2}((0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ such that $\lim_{|x|\to\infty} W(t,x) = \infty$ uniformly for t in compact subsets of $(0,T]$, $W \le Z$ and there is $0 \le h \in L^1((0,T))$ such that*

$$LW(t,x) \le h(t)W(t,x), \tag{1.29}$$

for all $(t,x) \in (0,T) \times \mathbb{R}^d$ and some $T > 0$.

To emphasize the dependence on Z and h, we also say that W is a time dependent Lyapunov function for L with respect to Z and h.

The first part of this section is devoted to showing that time dependent Lyapunov functions are integrable with respect to the measure $p(t,x,y)dy$ for any $(t,x) \in (0,T) \times \mathbb{R}^d$. For that, for any $(t,x) \in (0,T) \times \mathbb{R}^d$ we define

$$\xi_W(t,x) := \int_{\mathbb{R}^d} p(t,x,y)W(t,y)\,dy. \tag{1.30}$$

Proposition 1.6.3. *Assume that A satisfies Hypothesis 1.6.1. If W is a time dependent Lyapunov function for L with respect to Z and h, then*

$$\xi_W(t,x) \le e^{\int_0^t h(s)\,ds} W(0,x),$$

for any $(t,x) \in [0,T] \times \mathbb{R}^d$.

Proof. Let $x \in \mathbb{R}^d$ and $t \in [0,T]$. The proof is similar to the one of Proposition 1.5.2. We let $\alpha \geq 0$ and set

$$W_\alpha := W \wedge \alpha,$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. We approximate W with the $C_b^{1,2}(\mathbb{R}^d)$-function $\psi_\varepsilon \circ W$, where $0 < \varepsilon < 1$ and the function ψ_ε is such that $\psi_\varepsilon \in C^\infty(\mathbb{R})$, $\psi_\varepsilon(t) = t$ for $t \leq \alpha$, $\psi_\varepsilon(t)$ is constant for $t \geq \alpha + \varepsilon$, $\psi_\varepsilon' \geq 0$ and $\psi_\varepsilon'' \leq 0$. Similarly, as in Lemma 1.5.1, we approximate A with the operator $A_m = \operatorname{div}(Q_m \nabla) + F_m \cdot \nabla - V_m$ and $T(\cdot)$ with the analytic semigroup $T_m(\cdot)$. Furthermore, as observed in Step 1 of Proposition 1.5.2, the function $\psi_\varepsilon(W(t,\cdot))$ belongs to $D_{\max}(A_m)$ for any $t \in [\varepsilon, T]$. We note that by (1.17) we have

$$\begin{aligned}\partial_t T_m(t)\psi_\varepsilon(W(t,x)) &= A_m T_m(t)\psi_\varepsilon(W(t,x)) + T_m(t)\psi_\varepsilon'(W(t,x))\partial_t W(t,x)\\ &= T_m(t)[A_m\psi_\varepsilon(W(t,x)) + \psi_\varepsilon'(W(t,x))\partial_t W(t,x)],\end{aligned}$$

for all $t \in [\varepsilon, T]$ and $x \in \mathbb{R}^d$. By computing $A_m\psi_\varepsilon(W(t,y))$ and repeating the argument used in Step 2 of the proof of Proposition 1.5.2, we get

$$\partial_t T_m(t)\psi_\varepsilon(W(t,x)) \leq T_m(t)[\psi_\varepsilon'(W(t,x))(\partial_t + A_m)W(t,x)],$$

for all $t \in [\varepsilon, T]$. Moreover, as in Step 3, we may let $m \to \infty$ using Lemma 1.5.1 in order to obtain that

$$\begin{aligned}\partial_t T(t)\psi_\varepsilon(W(t,x)) &\leq T(t)(\psi_\varepsilon'(W(t,x))LW(t,x))\\ &= \int_{\{W \leq \alpha+1\}} p(t,x,y)\psi_\varepsilon'(W(t,y))LW(t,y)\,dy,\end{aligned}$$

for all $t \in [\varepsilon, T]$, where $L = \partial_t + A$. Then, since $\psi_\varepsilon(W(t,\cdot)) \to W_\alpha(t,\cdot)$ pointwise as $\varepsilon \to 0$, we let $\varepsilon \to 0$ in the previous inequality and we apply Theorem 1.2.6(c) to derive that

$$\partial_t T(t)W_\alpha(t,x) \leq \int_{\{W \leq \alpha\}} p(t,x,y)LW(t,y)\,dy,$$

for all $t \in [0,T]$. In addition, since W is a time dependent Lyapunov function for L, we use (1.29) and we find that

$$\begin{aligned}\partial_t T(t)W_\alpha(t,x) &\leq \int_{\{W \leq \alpha\}} p(t,x,y)LW(t,y)\,dy\\ &\leq h(t)\int_{\{W \leq \alpha\}} p(t,x,y)W(t,y)\,dy\\ &\leq h(t)\int_{\{W \leq \alpha\}} p(t,x,y)W(t,y)\,dy + h(t)\int_{\{W > \alpha\}} p(t,x,y)\alpha\,dy\\ &= h(t)\int_{\mathbb{R}^d} p(t,x,y)W_\alpha(t,y)\,dy = h(t)T(t)W_\alpha(t,x),\end{aligned}$$

for all $t \in [0,T]$. According to Gronwall's Lemma, for all $t \in [0,T]$ we get

$$T(t)W_\alpha(t,x) \leq e^{\int_0^t h(s)\,ds}W_\alpha(0,x).$$

The statement follows by letting $\alpha \to \infty$. $\square$

In the second part of this section we introduce a family of operators A_n with bounded diffusion coefficients approximating the operator A. It will be useful in the next chapters.

Assume that there exists a time dependent Lyapunov function W_1 with respect to Z for the operator L, where Z is the Lyapunov function given in Hypothesis 1.6.1(b). Let $\varphi \in C_c^\infty(\mathbb{R})$ be a function such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $(-1,1)$, $\varphi \equiv 0$ in $\mathbb{R} \setminus (-2,2)$, φ is decreasing in $(0,+\infty)$ and $|s\varphi'(s)| \le 2$ for all $s \in \mathbb{R}$. We set

$$\varphi_n(x) := \varphi(W_1(t_0,x)/n) \tag{1.31}$$

and

$$q_{ij}^{(n)}(x) := \varphi_n(x)q_{ij}(x) + (1-\varphi_n(x))\eta\delta_{ij}, \tag{1.32}$$

where $t_0 \in (0,T)$ and δ_{ij} is the Kronecker delta. We consider $Q_n := (q_{ij}^{(n)})$ and we approximate A with the family of operators A_n defined by

$$A_n = \mathrm{div}(Q_n\nabla) + F\cdot\nabla - V. \tag{1.33}$$

Lemma 1.6.4. *Assume that A satisfies Hypothesis 1.6.1 with the function Z in Hypothesis 1.6.1(b) being a Lyapunov function for both the operators A and $\eta\Delta + F\cdot\nabla - V$. Then, for every $n \in \mathbb{N}$, the diffusion coefficients $q_{ij}^{(n)}$ and their first order spatial derivatives are bounded on $\mathbb{R}^d$. Moreover the operator A_n satisfies Hypothesis 1.6.1 and if $\tilde{W}$ is a time dependent Lyapunov function for the operators L and $\partial_t + \eta\Delta + F\cdot\nabla - V$ with respect to Z and h such that $|\nabla\tilde{W}|$ is bounded on $(0,T)\times B_R$ for all $R > 0$, then $\tilde{W}$ is a time dependent Lyapunov function for $\partial_t + A_n$.*

Proof. Clearly, since $\lim_{|x|\to\infty} W_1(t_0,x) = +\infty$, the functions φ_n vanish outside a compact set. As a consequence, the coefficients $q_{ij}^{(n)}$ and their spatial derivatives $D_k q_{ij}^{(n)}$ are bounded on $\mathbb{R}^d$ for all $i,j,k = 1,\dots,d$. We now check Hypothesis 1.6.1 for A_n. First, we observe that Q_n is symmetric and, thanks to the uniformly ellipticity of Q, we get

$$\sum_{i,j=1}^d q_{ij}^{(n)}(x)\xi_i\xi_j = \varphi_n(x)\sum_{i,j=1}^d q_{ij}(x)\xi_i\xi_j + \eta(1-\varphi_n(x))\,|\xi|^2 \ge \eta|\xi|^2,$$

for any x, $\xi \in \mathbb{R}^d$.

It remains to prove that if Z is a Lyapunov function for the operators A and $\eta\Delta + F\cdot\nabla - V$, then Z is a Lyapunov function for A_n. As Remark 1.4.2 shows, without loss of generality we may assume that the infimum of Z is positive. Let $x \in \mathbb{R}^d$. Then

$$\begin{aligned}
A_nZ(x) =& \mathrm{div}(Q_n\nabla Z(x)) + F(x)\cdot\nabla Z(x) - V(x)Z(x)\\
=&\varphi_n(x)\mathrm{div}(Q\nabla Z(x)) + Q\nabla\varphi_n(x)\cdot\nabla Z(x) - \eta\nabla\varphi_n(x)\cdot\nabla Z(x)\\
&+\eta(1-\varphi_n(x))\Delta Z(x) + F(x)\cdot\nabla Z(x) - V(x)Z(x)\\
=&\varphi_n(x)AZ(x) + (1-\varphi_n(x))(\eta\Delta Z(x) + F(x)\cdot\nabla Z(x) - V(x)Z(x))\\
&+Q\nabla\varphi_n(x)\cdot\nabla Z(x) - \eta\nabla\varphi_n(x)\cdot\nabla Z(x).
\end{aligned}$$

For the first and the second term in the right hand side we use that $AZ(x) \leq \lambda Z(x)$ and $\eta\Delta Z(x) + F \cdot \nabla Z(x) - V(x)Z(x) \leq \lambda Z(x)$:

$$A_n Z(x) \leq \lambda Z(x) + Q\nabla\varphi_n(x) \cdot \nabla Z(x) - \eta\nabla\varphi_n(x) \cdot \nabla Z(x). \tag{1.34}$$

We can find a bound also for the last two terms since the functions φ_n vanish outside a compact set. As a result we find a constant $\lambda_n \geq 0$ such that

$$A_n Z(x) \leq \lambda_n Z(x),$$

for any $x \in \mathbb{R}^d$.

We now check that if $\tilde{W}$ is a time dependent Lyapunov function for the operators L and $\partial_t + \eta\Delta + F \cdot \nabla - V$ with respect to Z and h such that $|\nabla\tilde{W}|$ is bounded on $(0,T) \times B_R$ for all $R > 0$, then $\tilde{W}$ is a time dependent Lyapunov function for $\partial_t + A_n$. This can be seen by computing $\partial_t\tilde{W}(t,y) + A_n\tilde{W}(t,y)$ for $(t,y) \in (0,T) \times \mathbb{R}^d$:

$$\begin{aligned}
&\partial_t\tilde{W}(t,y) + A_n\tilde{W}(t,y) \\
&\quad= \partial_t\tilde{W}(t,y) + \operatorname{div}(Q_n\nabla\tilde{W}(t,y)) + F(y) \cdot \nabla\tilde{W}(t,y) - V(y)\tilde{W}(t,y) \\
&\quad= \varphi_n(y)L\tilde{W}(t,y) + (1-\varphi_n(y))[\partial_t\tilde{W}(t,y) + \eta\Delta\tilde{W}(t,y) + F(y) \cdot \nabla\tilde{W}(t,y) \\
&\qquad - V(y)\tilde{W}(t,y)] + Q\nabla\varphi_n(y) \cdot \nabla\tilde{W}(t,y) - \eta\nabla\varphi_n(y) \cdot \nabla\tilde{W}(t,y).
\end{aligned}$$

Since $\tilde{W}$ is a time dependent Lyapunov function for the operators L and $\partial_t + \eta\Delta + F \cdot \nabla - V$, the first two terms in the right hand side are bounded by $h(t)\tilde{W}(t,y)$:

$$\partial_t\tilde{W}(t,y) + A_n\tilde{W}(t,y) \leq h(t)\tilde{W}(t,y) + Q\nabla\varphi_n(y) \cdot \nabla\tilde{W}(t,y) - \eta\nabla\varphi_n(y) \cdot \nabla\tilde{W}(t,y),$$

where $h(t) \in L^1((0,T))$. Furthermore, the last terms are bounded by a nonnegative constant because φ_n vanishes outside a compact set and $|\nabla\tilde{W}|$ is bounded on $(0,T) \times B_R$ for all $R > 0$. Hence, there is a function $h_n(t) \in L^1((0,T))$ such that

$$\partial_t\tilde{W}(t,y) + A_n\tilde{W}(t,y) \leq h_n(t)\tilde{W}(t,y),$$

for all $(t,y) \in (0,T) \times \mathbb{R}^d$. Then $\tilde{W}$ is a time dependent Lyapunov function for $\partial_t + A_n$. $\square$

Remark 1.6.5. *Let Z be a Lyapunov function for the operators A and $\eta\Delta + F \cdot \nabla - V$ with respect to λ. If there exists a nonnegative function f such that*

$$\nabla Z(x) = f(x)\nabla W_1(t_0,x), \tag{1.35}$$

for all $x \in \mathbb{R}^d$, then, for any $n \in \mathbb{N}$, Z is a Lyapunov function for A_n with respect to the same λ. Indeed, by (1.34) we have

$$\begin{aligned}
A_n Z(x) \leq \lambda Z(x) + \frac{1}{n}\varphi'(W_1(t_0,x)/n)[&\langle Q\nabla W_1(t_0,x), \nabla Z(x)\rangle \\
&- \eta\langle\nabla W_1(t_0,x), \nabla Z(x)\rangle].
\end{aligned}$$

Making use of (1.35) and taking into account that Q is uniformly elliptic, $f \geq 0$ and φ is decreasing in $(0, +\infty)$, then we get

$$A_n Z(x) \leq \lambda Z(x) + \frac{1}{n}\varphi'(W_1(t_0,x)/n) f(x)[\langle Q\nabla W_1(t_0,x), \nabla W_1(t_0,x)\rangle - \eta|\nabla W_1(t_0,x)|^2] \leq \lambda Z(x),$$

for all $x \in \mathbb{R}^d$. Thus, we conclude that Z is a Lyapunov function for A_n with respect to λ for any $n \in \mathbb{N}$.

As a consequence of the previous lemma, for every $n \in \mathbb{N}$ the semigroup generated by A_n in $C_b(\mathbb{R}^d)$ is given by a kernel $p_n(t, x, y)$.

The following result similar to [31, Proposition 2.9].

Lemma 1.6.6. *Assume that A satisfies Hypothesis 1.6.1 with the function Z in Hypothesis 1.6.1(b) being a Lyapunov function for both the operators A and $\eta\Delta + F\cdot\nabla - V$. Let A_n be the operator defined by (1.33) and $p_n(t, x, y)$ be the integral kernel of the associated semigroup. Then, for $t > 0$ and $x \in \mathbb{R}^d$, we have that*

$$p_n(t, x, \cdot) \to p(t, x, \cdot)$$

locally uniformly in $\mathbb{R}^d$ as $n \to \infty$.

Proof. We consider the semigroup $T_n(\cdot)$ generated by A_n in $C_b(\mathbb{R}^d)$. By Lemma 1.6.4 we have that the function Z is a Lyapunov function for the operator A_n. Proceeding as in the proof of Lemma 1.5.1 it is possible to show that

$$\int_{\mathbb{R}^d} p_n(t, x, y) f(y)\, dy \to \int_{\mathbb{R}^d} p(t, x, y) f(y)\, dy,$$

for all $f \in C_c^{2+\zeta}(\mathbb{R}^d)$. Hence,

$$p_n(t, x, y)dy \to p(t, x, y)dy$$

weakly. On the other hand, from [8, Corollary 3.11] and Sobolev embedding, it follows that for any compact $K \subset \mathbb{R}^d$ there are a constant $C_1 > 0$ and $\gamma \in (0, 1)$ such that $\|p_n(t, x, \cdot)\|_{C^\gamma(K)} \leq C_1$ for all $n \in \mathbb{N}$. Thus, by compactness and a diagonal argument, up to a subsequence $p_n(t, x, \cdot)$ converges locally uniformly to a continuous function, that has to be $p(t, x, \cdot)$ by the weakly convergence proved above. $\square$

We now consider the function

$$\xi_{W,n}(t, x) := \int_{\mathbb{R}^d} p_n(t, x, y) W(t, y)\, dy, \tag{1.36}$$

for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$.

Lemma 1.6.7. *Assume that A satisfies Hypothesis 1.6.1 with the function Z in Hypothesis 1.6.1(b) being a Lyapunov function for both the operators A and $\eta\Delta + F \cdot \nabla - V$ and such that (1.35) holds. Let W be a time dependent Lyapunov function for the operators L and $\partial_t + \eta\Delta + F \cdot \nabla - V$ with respect to Z and h such that*

(a) $|\nabla W|$ is bounded on $(0, T) \times B_R$ for all $R > 0$;

(b) there are $c_0 > 0$ and $\sigma \in (0, 1)$ such that

$$W \le c_0 Z^{1-\sigma}.$$

Then

$$\xi_{W,n}(\cdot, x) \to \xi_W(\cdot, x)$$

uniformly in $(0, T)$ as $n \to \infty$, where the above functions are defined as in (1.30) and (1.36).

Proof. Let $t \in (0, T)$. Then

$$|\xi_{W,n}(t, x) - \xi_W(t, x)| \le \int_{\mathbb{R}^d} W(t, y)\, |p_n(t, x, y) - p(t, x, y)|\, dy.$$

We fix $R > 0$ and we split the above integral in the integral over B_R and the one over the complementary of B_R. Thus,

$$\begin{aligned}
&|\xi_{W,n}(t, x) - \xi_W(t, x)| \\
&\quad \le \int_{\mathbb{R}^d \setminus B_R} W(t, y)\, |p_n(t, x, y) - p(t, x, y)|\, dy \\
&\qquad + \int_{B_R} W(t, y)\, |p_n(t, x, y) - p(t, x, y)|\, dy \\
&\quad \le \int_{\mathbb{R}^d \setminus B_R} W(t, y) p(t, x, y)\, dy + \int_{\mathbb{R}^d \setminus B_R} W(t, y) p_n(t, x, y)\, dy \\
&\qquad + \int_{B_R} W(t, y)\, |p_n(t, x, y) - p(t, x, y)|\, dy.
\end{aligned} \tag{1.37}$$

On the other hand, Proposition 1.5.2 yields

$$\int_{\mathbb{R}^d} p(t, x, y) Z(y)\, dy \le e^{\lambda t} Z(x) \le e^{\lambda T} Z(x)$$

for all $x \in \mathbb{R}^d$, $t \in [0, T]$ and for some $\lambda \ge 0$. Hence, (b) and Hölder's inequality lead to

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus B_R} W(t, y) p(t, x, y)\, dy &\le c_0 \int_{\mathbb{R}^d \setminus B_R} Z^{1-\sigma}(y) p(t, x, y)\, dy \\
&\le c_0 \left(\int_{\mathbb{R}^d \setminus B_R} Z(y) p(t, x, y)\, dy \right)^{1-\sigma} p(t, x, \mathbb{R}^d \setminus B_R)^{\sigma} \\
&\le c_0 \left(e^{\lambda T} Z(x) \right)^{1-\sigma} p(t, x, \mathbb{R}^d \setminus B_R)^{\sigma}.
\end{aligned}$$

Since the family $\{p(t,x,y)dy \mid t \in (0,T)\}$ is tight by Corollary 1.5.3, the first term in the right hand side of (1.37) can be bounded by any given $\varepsilon > 0$ if R is large enough. We can argue similarly for the second term in the right hand side of (1.37) because Z is a Lyapunov function for A_n with respect to λ by Lemma 1.6.4 and Remark 1.6.5. Lastly, we look at the third term:

$$\begin{aligned}&\int_{B_R} W(t,y)\,|p_n(t,x,y) - p(t,x,y)|\,dy\\&\leq \|W\|_{L^\infty((0,T)\times B_R)}\,\|p_n(\cdot,x,\cdot) - p(\cdot,x,\cdot)\|_{L^\infty((0,T)\times B_R)}\,|B_R|\,,\end{aligned}$$

where $|B_R|$ denotes the Lebesgue measure of the ball B_R. Given $R > 0$, considering that $p_n(t,x,\cdot) \to p(t,x,\cdot)$ locally uniformly in $\mathbb{R}^d$ by Lemma 1.6.6, also the third term in the right hand side of (1.37) can be bounded by ε if n is large enough. To sum up, $\xi_{W,n}(\cdot,x) \to \xi_W(\cdot,x)$ uniformly on $(0,T)$ as $n \to \infty$. $\square$

Chapter 2

Schrödinger type operators with unbounded diffusion terms

In this chapter, we are concerned with Schrödinger type operators defined on smooth functions φ by

$$A\varphi = \operatorname{div}(Q\nabla\varphi) - V\varphi.$$

This operator has been studied in the paper [17]. We are interested in the situation when the diffusion coefficients Q and the potential V are unbounded functions.

In particular, we discuss the generation of a symmetric sub-Markovian and ultracontractive C_0-semigroup on $L^2(\mathbb{R}^d)$ which coincides on $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ with the minimal semigroup generated by a realization of A on $C_b(\mathbb{R}^d)$. Moreover, we look for pointwise upper bounds for the heat kernel of A and we apply the result in concrete examples, such as polynomial and exponential diffusion and potential coefficients.

Throughout, we make the following assumptions on Q and V.

Hypothesis 2.0.1. *We have* $Q = (q_{ij})_{i,j=1,\dots,d} \in C^{1+\zeta}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ *and* $0 \le V \in C^{\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$ *for some* $\zeta \in (0,1)$. *Moreover,*

(a) the matrix Q *is symmetric and uniformly elliptic, i.e. there is* $\eta > 0$ *such that*

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \ge \eta|\xi|^2 \quad \text{for all } x,\, \xi \in \mathbb{R}^d;$$

(b) there are $0 \le Z \in C^{2+\zeta}(\mathbb{R}^d)$ *and a constant* $M \ge 0$ *such that* $\lim_{|x|\to\infty} Z(x) = \infty$, $AZ(x) \le M$ *and* $\eta\Delta Z(x) - V(x)Z(x) \le M$ *for all* $x \in \mathbb{R}^d$.

It follows by Remark 1.5.4 that the function Z in Hypothesis 2.0.1(b) is a Lyapunov function for the operators A and $\eta\Delta - V$.

As shown in Chapter 1, a suitable realization of A generates a semigroup $T(\cdot) = (T(t))_{t\geq 0}$ on the space $C_b(\mathbb{R}^d)$ that is given through an integral kernel. Therefore, we can write

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy, \quad t>0,\, x\in\mathbb{R}^d,\, f\in C_b(\mathbb{R}^d),$$

where the kernel p is positive, $p(t,\cdot,\cdot)$ and $p(t,x,\cdot)$ are measurable for any $t>0$, $x\in\mathbb{R}^d$, and for a.e. fixed $y\in\mathbb{R}^d$, $p(\cdot,\cdot,y)\in C^{1+\zeta/2,2+\zeta}_{\rm loc}((0,\infty)\times\mathbb{R}^d)$. Moreover, since we assumed in Hypothesis 2.0.1(b) that there exists a Lyapunov function for A, then by Theorem 1.4.4 the domain of the weak generator is the maximal domain $D_{\max}(A)$ and $T(t)f$ is the unique classical solution of the corresponding Cauchy problem.

The chapter is organized as follows. In Section 2.1 we adapt the techniques in [4] to prove that $T(\cdot)$ can be extended to a symmetric sub-Markovian and ultracontractive C_0-semigroup on $L^2(\mathbb{R}^d)$. More precisely, given the maximal realization $A_{\max}$ in $L^2(\mathbb{R}^d)$

$$\begin{aligned} D(A_{\max}) &= \{u\in L^2(\mathbb{R}^d)\cap H^1_{\rm loc}(\mathbb{R}^d),\ Au\in L^2(\mathbb{R}^d)\},\\ A_{\max}u &= Au, \end{aligned}$$

we prove the uniqueness of the minimal realization in $L^2(\mathbb{R}^d)$, that is the operator $A_{\min}$ such that

(a) $A_{\min}\subset A_{\max}$;

(b) $A_{\min}$ generates a positive, symmetric C_0-semigroup $T_2(\cdot)$ on $L^2(\mathbb{R}^d)$;

(c) if $B\subset A_{\max}$ generates a positive C_0-semigroup $S(\cdot)$, then $T_2(t)\leq S(t)$ for all $t\geq 0$.

For that, we use an approximation argument. We consider balls B_ρ of increasing radius $\rho>0$ and we construct a sequence of semigroups $T^{(\rho)}(\cdot)$ on $L^2(B_\rho)$ via form methods. It turns out that $T^{(\rho)}(\cdot)$ are symmetric, sub-Markovian, contractive and strongly continuous. They increase to a semigroup $T_2(\cdot)$ which inherits all the above mentioned properties. Furthermore, $T_2(\cdot)$ is ultracontractive, its generator is $A_{\min}$ and it is consistent with $T(\cdot)$, namely it coincides with $T(\cdot)$ in the intersection $L^2(\mathbb{R}^d)\cap C_b(\mathbb{R}^d)$. Finally, classical results show that this semigroup extrapolates to a positive C_0-semigroup of contractions in all $L^p(\mathbb{R}^d)$, $p\in[1,\infty)$. Moreover, in the examples considered in Section 2.7, these semigroups are compact and the spectra of their corresponding generators are independent of p.

The second focus in this chapter lies in proving pointwise upper bounds for the kernel p. Recently, many papers addressed this question in case of polynomially growing coefficients (see for example [12, 13, 14, 15, 16, 21, 36, 37, 43, 44, 45, 46]). We adopt the technique of time dependent Lyapunov functions used in [1, 30, 31, 51] to our divergence form setting.

Because of their key role, in Section 2.2 we establish sufficient conditions under which certain exponential functions are time dependent Lyapunov functions in the case of polynomially and exponentially growing diffusion coefficients.

The strategy we use to find kernel estimates is based on an approximation argument. We approximate the diffusion coefficients q_{ij} with bounded ones $q_{ij}^{(n)}$ as at the beginning of Section 2.5 and we consider the family of the corresponding approximating operators

$$A_n = \mathrm{div}(Q_n \nabla) - V.$$

At this point one would be tempted to think that, since kernel estimates in case of bounded diffusion coefficients are available in works such as [1, 10, 34, 41], one could just apply such results for the approximating kernels p_n and then let $n \to \infty$. Unfortunately, it is not possible because the constant in the right hand side of the mentioned estimates depend on the diffusion coefficients, so it could explode as $n \to \infty$. Thus, we need a different approach to estimate the kernels p_n.

For this reason, in Section 2.3 we adapt [31, Theorem 3.7] in order to prove the key result that will allow us to overcome this problem, see Theorem 2.3.6. Moreover, we provide some global regularity results for the kernel p. Under suitable assumptions, they permit us to apply Theorem 2.3.6 in case of bounded diffusion coefficients to obtain in Section 2.4 the right estimate. Here the existence of time dependent Lyapunov functions plays a crucial role. With these ingredients at hand, in Section 2.5 we estimate the kernels p_n and we derive kernel estimates for the general operator A.

In the subsequent Section 2.6, we implement the previous results in concrete examples. We first deal with the operator with polynomial coefficients

$$A = \mathrm{div}((1 + |x|_*^m)\nabla) - |x|^s,$$

for $s > |m-2|$ and $m > 0$. Furthermore, since the method does not rely on the specific form of the coefficients, it is possible to consider even exponential functions, such as the operator

$$A = \mathrm{div}(e^{|x|^m}\nabla) - e^{|x|^s},$$

for $2 \le m < s$.

In the concluding Section 2.7, we present some consequences of our result for the spectrum and the eigenfunctions of the operator $A_{\min}$ from Section 2.1.

2.1 Generation of semigroups on $L^2(\mathbb{R}^d)$

In this section we show that, according to Definition B.4.1, a realization of A in $L^2(\mathbb{R}^d)$ generates a symmetric sub-Markovian and ultracontractive semigroup $T_2(\cdot)$ on $L^2(\mathbb{R}^d)$ which coincides with the semigroup $T(\cdot)$ on $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$.

We now take up our main line of study and consider the elliptic operator $\mathcal{A}$, defined by

$$\mathcal{A} : H^1_{\mathrm{loc}}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)', \quad \mathcal{A}\varphi = \operatorname{div}(Q\nabla\varphi) - V\varphi.$$

Its maximal realization $A_{\max}$ in $L^2(\mathbb{R}^d)$ is defined by

$$\begin{aligned} D(A_{\max}) &= \{u \in L^2(\mathbb{R}^d) \cap H^1_{\mathrm{loc}}(\mathbb{R}^d),\, \mathcal{A}u \in L^2(\mathbb{R}^d)\}, \\ A_{\max}u &= \mathcal{A}u. \end{aligned}$$

We adapt the proof of Theorem 1.1, Proposition 1.2 and Proposition 1.3 in [4] to our situation to show that there is also a minimal realization $A_{\min}$ of $\mathcal{A}$. The minimal realization of $\mathcal{A}$ in $L^2(\mathbb{R}^d)$ is the operator presented in the following theorem.

Theorem 2.1.1. *There exists a unique operator $A_{\min}$ on $L^2(\mathbb{R}^d)$ such that*

(a) $A_{\min} \subset A_{\max}$*;*

(b) $A_{\min}$ *generates a positive, symmetric C_0-semigroup $T_2(\cdot)$ on $L^2(\mathbb{R}^d)$;*

(c) if $B \subset A_{\max}$ generates a positive C_0-semigroup $S(\cdot)$, then $T_2(t) \le S(t)$ for all $t \ge 0$.

The operator $A_{\min}$ and the semigroup $T_2(\cdot)$ have the following additional properties:

(d) $D(A_{\min}) \subset H^1(\mathbb{R}^d)$ *and* $-\langle A_{\min}u, u\rangle \ge \eta \||\nabla u|\|_2^2$ *for all* $u \in D(A_{\min})$*;*

(e) $T_2(\cdot)$ *is sub-Markovian and ultracontractive;*

(f) the semigroup $T_2(\cdot)$ is consistent with $T(\cdot)$, i.e.

$$T_2(t)f = T(t)f, \quad t \ge 0,\ f \in L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d).$$

Proof. Step 1. We define approximate semigroups $T^{(\rho)}(\cdot)$ on $L^2(B_\rho)$.

To that end, we consider the sesquilinear form $\mathfrak{a}_\rho : H^1_0(B_\rho) \times H^1_0(B_\rho) \to \mathbb{C}$, defined by

$$\mathfrak{a}_\rho(u, v) = \int_{B_\rho} \sum_{i,j=1}^d q_{ij} D_i u D_j \bar{v}\, dx + \int_{B_\rho} V u\bar{v}\, dx.$$

This form is obviously symmetric. Using that Q and V are bounded on B_ρ, an easy application of Hölder's inequality yields

$$\begin{aligned} |\mathfrak{a}_\rho(u, v)| &\le \int_{B_\rho} |\langle Q\nabla u, \nabla\overline{v}\rangle|\, dx + \int_{B_\rho} V\,|u|\,|v|\, dx \\ &\le \|Q\|_{L^\infty(B_\rho;\mathbb{R}^{d\times d})} \int_{B_\rho} |\nabla u|\,|\nabla v|\, dx + \|V\|_{L^\infty(B_\rho)} \int_{B_\rho} |u|\,|v|\, dx \\ &\le \|Q\|_{L^\infty(B_\rho;\mathbb{R}^{d\times d})} \|\nabla u\|_{L^2(B_\rho)} \|\nabla v\|_{L^2(B_\rho)} \\ &\quad + \|V\|_{L^\infty(B_\rho)} \|u\|_{L^2(B_\rho)} \|v\|_{L^2(B_\rho)} \\ &\le \Big(\|Q\|_{L^\infty(B_\rho;\mathbb{R}^{d\times d})} + \|V\|_{L^\infty(B_\rho)}\Big) \|u\|_{H^1_0(B_\rho)} \|v\|_{H^1_0(B_\rho)}, \end{aligned}$$

for all $u, v \in H_0^1(B_\rho)$. Moreover, from the positivity of V, the uniform ellipticity of Q and Poincaré's inequality, it follows that

$$\mathfrak{a}_\rho(u,u) \geq \eta \|\nabla u\|^2_{L^2(B_\rho)} \geq \nu \|u\|^2_{H_0^1(B_\rho)},$$

for any $u \in H_0^1(B_\rho)$ and for some $\nu > 0$. This shows that the form $\mathfrak{a}_\rho$ satisfies (D.6) and (D.7). Thus, if we denote by $-A_\rho$ the associated operator on $L^2(B_\rho)$, Proposition D.1.2 and Remark D.1.3 imply that A_ρ generates a strongly continuous contraction semigroup $T^{(\rho)}(\cdot)$ on $L^2(B_\rho)$.

We now show that the semigroup $T^{(\rho)}(\cdot)$ is positive. In view of the first Beurling-Deny criterion on forms (see Theorem D.2.1) it suffices to prove that

(i) $(\mathrm{Re}\, u)^+ \in H_0^1(B_\rho)$,

(ii) $\mathfrak{a}_\rho(\mathrm{Re}\, u, \mathrm{Im}\, u) \in \mathbb{R}$,

(iii) $\mathfrak{a}_\rho((\mathrm{Re}\, u)^+, (\mathrm{Re}\, u)^-) \leq 0$,

for all $u \in H_0^1(B_\rho)$. Following the proof of [49, Proposition 4.4], we start by establishing (i) in $H^1(B_\rho)$, namely $(\mathrm{Re}\, u)^+ \in H^1(B_\rho)$ for any $u \in H^1(B_\rho)$. First, for $\varepsilon > 0$, we consider on $\mathbb{C}$ the function

$$f_\varepsilon(z) = \sqrt{|z|^2 + \varepsilon^2} - \varepsilon.$$

We note that $f_\varepsilon(u) \in H^1(B_\rho)$ for all $u \in H^1(B_\rho)$ because the partial derivatives $\frac{\partial}{\partial t} f_\varepsilon$ and $\frac{\partial}{\partial s} f_\varepsilon$ with respect to $t = \mathrm{Re} z$ and $s = \mathrm{Im} z$ are continuous and bounded on $\mathbb{C}$. Moreover, if we compute the j-th derivative, we get

$$\begin{aligned} D_j f_\varepsilon(u) &= \frac{\mathrm{Re}\, u}{\sqrt{|u|^2 + \varepsilon^2}} D_j(\mathrm{Re}\, u) + \frac{\mathrm{Im}\, u}{\sqrt{|u|^2 + \varepsilon^2}} D_j(\mathrm{Im}\, u) \\ &= \mathrm{Re}\left[D_j u \frac{\overline{u}}{\sqrt{|u|^2 + \varepsilon^2}} \right]. \end{aligned}$$

Then, for all $\varphi \in \mathcal{D}(B_\rho)$, we have

$$\int_{B_\rho} f_\varepsilon(u) D_j \varphi \, dx = -\int_{B_\rho} \mathrm{Re}\left[D_j u \frac{\overline{u}}{\sqrt{|u|^2 + \varepsilon^2}} \right] \varphi \, dx.$$

If we let $\varepsilon \to 0$ we obtain

$$\int_{B_\rho} |u| D_j \varphi \, dx = -\int_{B_\rho} \mathrm{Re}\left[\mathrm{sign}(\overline{u}) D_j u\right] \varphi \, dx.$$

Then $|u| \in H^1(B_\rho)$ and $D_j|u| = \mathrm{Re}\left[\mathrm{sign}(\overline{u}) D_j u\right]$. We can repeat the same argument for $\mathrm{Re}\, u$ so that we conclude that $|\mathrm{Re}\, u| \in H^1(B_\rho)$ and $D_j|\mathrm{Re}\, u| = \mathrm{sign}(\mathrm{Re}\, u) D_j(\mathrm{Re}\, u)$. Finally, using the fact that $(\mathrm{Re}\, u)^+ = \frac{1}{2}(|\mathrm{Re}\, u| + \mathrm{Re}\, u)$, we conclude that $(\mathrm{Re}\, u)^+ \in H^1(B_\rho)$ and

$$D_j (\mathrm{Re}\, u)^+ = D_j(\mathrm{Re}\, u) \chi_{\{\mathrm{Re}\, u > 0\}}.$$

Similarly, $(\operatorname{Re} u)^- \in H^1(B_\rho)$. We now prove (i). If we consider a function $f \in \mathcal{D}(B_\rho)$, then $f \in H^1(B_\rho)$, so by what we showed above we have that $(\operatorname{Re} f)^+ \in H^1(B_\rho)$. Moreover, $(\operatorname{Re} f)^+$ has compact support, thus it belongs to $H^1_0(B_\rho)$. If we take $u \in H^1_0(B_\rho)$, then we find a sequence $(f_n) \subset \mathcal{D}(B_\rho)$ such that

$$\lim_{n\to\infty} f_n = u$$

in $H^1(B_\rho)$. Since $(\operatorname{Re} f_n)^+ \to (\operatorname{Re} u)^+$ in $H^1(B_\rho)$ and $(\operatorname{Re} f_n)^+ \in H^1_0(B_\rho)$, we conclude that $(\operatorname{Re} u)^+ \in H^1_0(B_\rho)$. Furthermore, (ii) and (iii) hold true because $\mathfrak{a}_\rho$ has real-valued coefficients and $\mathfrak{a}_\rho((\operatorname{Re} u)^+, (\operatorname{Re} u)^-) = 0$. Making use of Theorem D.2.1 we see that the semigroup $T^{(\rho)}(\cdot)$ is positive.

Finally, we prove that $T^{(\rho)}(\cdot)$ is L^∞-contractive. By the second Beurling-Deny criterion for forms (see Theorem D.2.2), we need to show that

(i) $(1 \wedge |u|) \operatorname{sign} u \in H^1_0(B_\rho)$,

(ii) $\operatorname{Re} \mathfrak{a}_\rho((1 \wedge |u|) \operatorname{sign} u, (|u|-1)^+ \operatorname{sign} u) \geq 0$,

for all $u \in H^1_0(B_\rho)$.

We begin to state that $(1 \wedge |u|) \operatorname{sign} u \in H^1(B_\rho)$ for all $u \in H^1(B_\rho)$ as in [49, Proposition 4.11]. For $\varepsilon > 0$, we define on $\mathbb{R}$ the function

$$f_\varepsilon(t) = \begin{cases} \sqrt{(t-1)^2+\varepsilon^2} - \varepsilon & \text{if } t > 1, \\ 0 & \text{if } t \leq 1. \end{cases}$$

Since f_ε has bounded derivatives on $\mathbb{R}$, then $f_\varepsilon(u) \in H^1(B_\rho)$ for all real-valued $u \in H^1(B_\rho)$. Computing the j-th derivative of $f_\varepsilon(|u|)$ we obtain

$$D_j f_\varepsilon(|u|) = \frac{|u|-1}{\sqrt{(|u|-1)^2+\varepsilon^2}} D_j|u| \, \chi_{\{|u|>1\}}.$$

Thus, we get

$$\int_{B_\rho} f_\varepsilon(|u|) D_j\varphi \, dx = -\int_{B_\rho} \frac{|u|-1}{\sqrt{(|u|-1)^2+\varepsilon^2}} D_j|u| \, \chi_{\{|u|>1\}} \varphi \, dx,$$

for any $\varphi \in \mathcal{D}(B_\rho)$. Letting $\varepsilon \to 0$ we derive that $(|u|-1)^+ \in H^1(B_\rho)$ and

$$D_j(|u|-1)^+ = D_j|u| \, \chi_{\{|u|>1\}} = \operatorname{Re}(\operatorname{sign}(\overline{u}) D_j u) \chi_{\{|u|>1\}}.$$

It follows that $\frac{u}{\sqrt{|u|^2+\varepsilon}}(|u|-1)^+ \in H^1(B_\rho)$ and

$$\begin{aligned} D_j \left[\frac{u}{\sqrt{|u|^2+\varepsilon}} (|u|-1)^+ \right] &= \frac{(|u|-1)^+}{\sqrt{|u|^2+\varepsilon}} \left[D_j u - \frac{|u|}{|u|^2+\varepsilon} u D_j|u| \right] \\ &\quad + \frac{u}{\sqrt{|u|^2+\varepsilon}} D_j|u| \, \chi_{\{|u|>1\}}. \end{aligned}$$

Taking $\varepsilon \to 0$ yields $(|u|-1)^+ \operatorname{sign} u \in H^1(B_\rho)$ and

$$\begin{aligned}
&D_j((|u|-1)^+ \operatorname{sign} u) \\
&\quad = \frac{(|u|-1)^+}{|u|}(D_j u - \operatorname{sign} u\, D_j|u|) + \operatorname{sign} u\, D_j|u|\,\chi_{\{|u|>1\}} \\
&\quad = \left[D_j|u| + i(|u|-1)\frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_j u)}{|u|}\right] \operatorname{sign} u\,\chi_{\{|u|>1\}}.
\end{aligned} \tag{2.1}$$

We deduce that $(1 \wedge |u|) \operatorname{sign} u \in H^1(B_\rho)$ considering that $(1 \wedge |u|) \operatorname{sign} u = u - (|u|-1)^+ \operatorname{sign} u$. In addition,

$$\begin{aligned}
&D_j((1 \wedge |u|) \operatorname{sign} u) \\
&\quad = D_j u - \left[D_j|u| + i(|u|-1)\frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_j u)}{|u|}\right] \operatorname{sign} u\,\chi_{\{|u|>1\}} \\
&\quad = D_j u + i\frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_j u)}{|u|} \operatorname{sign} u\,\chi_{\{|u|>1\}} \\
&\qquad - [D_j|u| + i\operatorname{Im}(\operatorname{sign}(\overline{u})D_j u)] \operatorname{sign} u\,\chi_{\{|u|>1\}} \\
&\quad = i\frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_j u)}{|u|} \operatorname{sign} u\,\chi_{\{|u|>1\}} + D_j u - D_j u\,\chi_{\{|u|>1\}} \\
&\quad = i\frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_j u)}{|u|} \operatorname{sign} u\,\chi_{\{|u|>1\}} + D_j u\,\chi_{\{|u|\le 1\}}.
\end{aligned} \tag{2.2}$$

Since $(1 \wedge |u|) \operatorname{sign} u \in H^1_0(B_\rho)$ for all $u \in \mathcal{D}(B_\rho)$, arguing by density as above, we get (i) also for any $u \in H^1_0(B_\rho)$. We now show (ii). Taking into account (2.1) and (2.2) we find that

$$\begin{aligned}
&\operatorname{Re} \mathfrak{a}_\rho((1 \wedge |u|) \operatorname{sign} u, (|u|-1)^+ \operatorname{sign} u) \\
&= \operatorname{Re} \int_{B_\rho} \sum_{i,j=1}^d q_{ij} \left[i\frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_i u)}{|u|} \operatorname{sign} u\,\chi_{\{|u|>1\}} + D_i u\,\chi_{\{|u|\le 1\}}\right] \times \\
&\qquad \times \left[D_j|u| + i(|u|-1)\frac{\operatorname{Im}(\operatorname{sign} u\, D_j\overline{u})}{|u|}\right] \operatorname{sign}(\overline{u})\,\chi_{\{|u|>1\}}\,dx \\
&\quad + \operatorname{Re}\int_{B_\rho} V(1 \wedge |u|) \operatorname{sign} u\,(|u|-1)^+ \operatorname{sign} \overline{u}\,dx \\
&= \operatorname{Re}\left[i\int_{B_\rho} \sum_{i,j=1}^d q_{ij} \frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_i u)}{|u|} D_j|u|\,\chi_{\{|u|>1\}}\,dx\right] \\
&\quad - \operatorname{Re}\int_{B_\rho} \sum_{i,j=1}^d q_{ij} \frac{\operatorname{Im}(\operatorname{sign}(\overline{u})D_i u)\operatorname{Im}(\operatorname{sign} u\, D_j\overline{u})}{|u|^2}(|u|-1)\,\chi_{\{|u|>1\}}\,dx \\
&\quad + \operatorname{Re}\int_{B_\rho} V(1 \wedge |u|)(|u|-1)^+\,dx.
\end{aligned}$$

The first integral in the right hand side of the previous identity is real, hence the corresponding term is null. Moreover, $\operatorname{Im}(\operatorname{sign} u\, D_j\overline{u}) = -\operatorname{Im}(\operatorname{sign}(\overline{u})D_i u)$, thus the second term is nonnegative as the third one. We conclude that (ii) holds true. We now apply Theorem D.2.2 to infer that $T^{(\rho)}(\cdot)$ is L^∞-contractive. Combining this with the positivity proved above we conclude that $T^{(\rho)}(\cdot)$ is sub-Markovian.

Step 2. We prove that the semigroups $T^{(\rho)}(\cdot)$ are increasing to a semigroup $T_2(\cdot)$.

We now consider functions on B_ρ to be defined on all of $\mathbb{R}^d$, by extending them with 0 outside of B_ρ. Then, for any $0 < \rho_1 < \rho_2$, we identify $H^1_0(B_{\rho_1})$ with a subspace of $H^1_0(B_{\rho_2})$.

Given $0 < \rho_1 < \rho_2$, we have to show that

$$T^{(\rho_1)}(t) \le T^{(\rho_2)}(t),$$

for all $t \ge 0$ or, equivalently,

$$R(\lambda, A_{\rho_1})f \le R(\lambda, A_{\rho_2})f,$$

for all $0 \le f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$, where $R(\lambda, A_\rho) := (\lambda I - A_\rho)^{-1}$ is the resolvent operator of A_ρ for λ in the resolvent set $\rho(A_\rho)$ of A_ρ.

Let $0 \le f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$. We set $u_1 = R(\lambda, A_{\rho_1})f$ and $u_2 = R(\lambda, A_{\rho_2})f$. For $k = 1, 2$, since $(\lambda I - A_{\rho_k})u_k = f$, we have

$$\lambda \int_{B_{\rho_1}} u_k v\, dx + \int_{B_{\rho_1}} \sum_{i,j=1}^d q_{ij} D_j u_k D_i v\, dx + \int_{B_{\rho_1}} V u_k v\, dx = \int_{B_{\rho_1}} f v\, dx,$$

for all $v \in H^1_0(B_{\rho_1})$. Using the formula $g^+ = (|g| + g)/2$ with $g = (u_1 - u_2)^+$ we deduce that $(u_1 - u_2)^+ \in H^1_0(B_{\rho_1})$. Hence, we can take $v = (u_1 - u_2)^+$ in the previous identity. Then we derive that

$$\begin{aligned} \lambda \int_{B_{\rho_1}} u_k (u_1 - u_2)^+\, dx &+ \int_{B_{\rho_1}} \sum_{i,j=1}^d q_{ij} D_j u_k D_i (u_1 - u_2)^+\, dx \\ &+ \int_{B_{\rho_1}} V u_k (u_1 - u_2)^+\, dx = \int_{B_{\rho_1}} f (u_1 - u_2)^+\, dx. \end{aligned}$$

If we now subtract the two indentities, it follows that

$$\begin{aligned} \lambda \int_{B_{\rho_1}} (u_1 - u_2)(u_1 - u_2)^+\, dx &+ \int_{B_{\rho_1}} \sum_{i,j=1}^d q_{ij} D_j (u_1 - u_2) D_i (u_1 - u_2)^+\, dx \\ &+ \int_{B_{\rho_1}} V (u_1 - u_2)(u_1 - u_2)^+\, dx = 0. \end{aligned}$$

The uniform ellipticity of Q yields

$$\begin{aligned} \lambda \int_{B_{\rho_1}} [(u_1 - u_2)^+]^2\, dx &+ \eta \int_{B_{\rho_1}} \left|\nabla (u_1 - u_2)^+\right|^2\, dx \\ &+ \int_{B_{\rho_1}} V [(u_1 - u_2)^+]^2\, dx \le 0. \end{aligned}$$

Since $V \geq 0$, we obtain that $(u_1 - u_2)^+ = 0$. Thus, $u_1 \leq u_2$ on B_{ρ_1}. We conclude that $T^{(\rho)}(\cdot)$ are increasing.

As every semigroup $T^{(\rho)}(\cdot)$ is contractive, we may define

$$T_2(t)f := \sup_{n\in\mathbb{N}} T^{(n)}(t)f$$

for $0 \leq f \in L^2(\mathbb{R}^d)$ and then $T_2(t)f := T_2(t)f^+ - T_2(t)f^-$ for general $f \in L^2(\mathbb{R}^d)$. An easy computation shows that $T_2(\cdot)$ is a positive contraction semigroup. We prove that $T_2(\cdot)$ is strongly continuous. To that end, fix $0 \leq f \in \mathcal{D}(\mathbb{R}^d)$, and $\rho > 0$ such that $\operatorname{supp} f \subset B_\rho$. Let $t_n \downarrow 0$. Then

$$\begin{aligned}
&\limsup_{n\to\infty} \|T^{(\rho)}(t_n)f - T_2(t_n)f\|_2^2 \\
=&\limsup_{n\to\infty} \left[\|T^{(\rho)}(t_n)f\|_2^2 + \|T_2(t_n)f\|_2^2 - 2\langle T^{(\rho)}(t_n)f, T_2(t_n)f\rangle_2\right] \\
\leq&\limsup_{n\to\infty} \left[2\,\|f\|_2^2 - 2\langle T^{(\rho)}(t_n)f, T^{(\rho)}(t_n)f\rangle_2\right] = 2\|f\|_2^2 - 2\|f\|_2^2 = 0.
\end{aligned}$$

Here, in the third line we have used the contractivity of $T^{(\rho)}(\cdot)$ and $T_2(\cdot)$, that $0 \leq T^{(\rho)}(t_n)f \leq T_2(t_n)f$ and the strong continuity of $T^{(\rho)}(\cdot)$. Thus, $T_2(t_n)f \to f$ as $n \to \infty$. Splitting $f \in \mathcal{D}(\mathbb{R}^d)$ into positive and negative part, we see that this is true for general f. In view of the contractivity of $T_2(\cdot)$, a standard 3ε argument yields strong continuity of $T_2(\cdot)$.

As the form $\mathfrak{a}_\rho$ is symmetric, the semigroup $T^{(\rho)}(\cdot)$ consists of symmetric operators and thus, so does the limit semigroup $T_2(\cdot)$. Likewise, sub-Markovianity of $T_2(\cdot)$ is inherited by that of $T^{(\rho)}(\cdot)$.

Step 3. We identify the generator $A_{\min}$ of $T_2(\cdot)$.

Let us first note that $R(\lambda, A_\rho)f \to R(\lambda, A_{\min})f$ as $\rho \to \infty$ for every $\lambda > 0$. For $0 \leq f \in L^2(\mathbb{R}^d)$ this can be deduced from the construction of $T_2(\cdot)$ by taking Laplace transforms and using the monotone convergence as follows

$$R(\lambda, A_\rho)f = \int_0^\infty e^{-\lambda t} T_\rho(t)f\,dt \to \int_0^\infty e^{-\lambda t} T_2(t)f\,dt = R(\lambda, A_{\min})f.$$

Otherwise, for general $f \in L^2(\mathbb{R}^d)$, we write $f = f^+ - f^-$. Then, since $R(\lambda, A_\rho)f = R(\lambda, A_\rho)f^+ - R(\lambda, A_\rho)f^-$, the statement follows similarly by letting $\rho \to \infty$.

Now fix a sequence $\rho_n \uparrow \infty$ and $f \in L^2(\mathbb{R}^d)$. We put $u = R(1, A_{\min})f$ and $u_n = R(1, A_{\rho_n})f$. Then $u_n \to u$ and $A_{\rho_n}u_n = u_n - f \to u - f = A_{\min}u$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. By the coercivity of the form $\mathfrak{a}_{\rho_n}$, we have

$$\begin{aligned}
\eta \limsup_{n\to\infty} \int |\nabla u_n|^2\,dx \leq \limsup_{n\to\infty} \mathfrak{a}_{\rho_n}[u_n, u_n] &= \limsup_{n\to\infty} -\langle A_{\rho_n}u_n, u_n\rangle \\
&= -\langle A_{\min}u, u\rangle. \qquad (2.3)
\end{aligned}$$

It follows that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^d)$ and thus, by reflexivity of $H^1(\mathbb{R}^d)$, $u_n \to u$ weakly in $H^1(\mathbb{R}^d)$. The arbitrarity of $u \in D(A_{\min})$ implies that $D(A_{\min}) \subset H^1(\mathbb{R}^d)$. Moreover, using the weak lower semicontinuity of norms, we see that (2.3) implies $-\langle A_{\min}u, u\rangle \geq \eta\||\nabla u|\|_2^2$.

Now fix $v \in \mathcal{D}(\mathbb{R}^d)$. As u_n converges to u weakly in $H^1(\mathbb{R}^d)$, we see that

$$\langle \mathcal{A}u, v\rangle = \lim_{n\to\infty} \langle \mathcal{A}u_n, v\rangle = \lim_{n\to\infty} \langle A_{\rho_n} u_n, v\rangle = \langle A_{\min}u, v\rangle,$$

proving $A_{\min} \subset A_{\max}$. At this point, properties (a), (d) and (by definition of $A_{\min}$) (b) are proved.

Step 4. We establish the minimality property (c).

Let $B \subset A_{\max}$ be such that B generates a positive C_0-semigroup $S(\cdot)$ on $L^2(\mathbb{R}^d)$. To prove $T_2(t) \leq S(t)$ for all $t \geq 0$ it suffices to prove $R(\lambda, A_{\min}) \leq R(\lambda, B)$ for all $\lambda > 0$; this is an easy consequence of Euler's formula.

To see this, let us fix again a sequence $\rho_n \uparrow \infty$, $\lambda > 0$ and $0 \leq f \in L^2(\mathbb{R}^d)$. We put $u = R(\lambda, A_{\min})f$, $v = R(\lambda, B)f$ and $u_n = R(\lambda, A_{\rho_n})f$. As $B \subset A_{\max}$, we have $v \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\lambda \int_{B_{\rho_n}} (u_n - v)w\,dx + \int_{B_{\rho_n}} \sum_{i,j=1}^d q_{ij} D_j(u_n - v) D_i w\,dx + \int_{B_{\rho_n}} V(u_n - v)w\,dx = 0, \tag{2.4}$$

for all $w \in H^1_0(B_{\rho_n})$. As the semigroup $S(\cdot)$ is positive, $v \geq 0$ and thus $(u_n - v)^+ \leq u_n$. Consequently, $(u_n - v)^+ \in H^1_0(B_{\rho_n})$ and we may insert $w = (u_n - v)^+$ into (2.4). Taking the uniform ellipticity of Q into account, this yields

$$\lambda \int_{B_{\rho_n}} \big((u_n - v)^+\big)^2\,dx + \eta \int_{B_{\rho_n}} |\nabla(u_n - v)^+|^2\,dx + \int_{B_{\rho_n}} V\big((u_n - v)^+\big)^2\,dx \leq 0.$$

As $V \geq 0$, it follows that $(u_n - v)^+ = 0$ and thus $u_n \leq v$. Upon $n \to \infty$ we obtain $u \leq v$. Hence $R(\lambda, A_{\min})f \leq R(\lambda, B)f$ for $0 \leq f \in L^2(\mathbb{R}^d)$.

Step 5. We establish properties (e) and (f).

As we have already mentioned above, the semigroup $T_2(\cdot)$ is sub-Markovian and consists of symmetric operators. The latter implies that the generator $A_{\min}$ of $T_2(\cdot)$ is selfadjoint. In view of property (d), the ultracontractivity of the semigroup follows immediately from Proposition B.4.2.

As for consistency we note that the semigroup $T(\cdot)$ on $C_b(\mathbb{R}^d)$ constructed in Chapter 1 is obtained by the same approximation procedure as for $T_2(\cdot)$. But the semigroup solutions of the Cauchy–Dirichlet problem associated with $\mathcal{A}$ on $C_b(\overline{B_\rho})$ is consistent with the semigroup solution on $L^2(B_\rho)$ considered above. Thus, consistency of $T_2(\cdot)$ and $T(\cdot)$ follows. □

Remark 2.1.2. *(a) As the minimal realization $A_{\min}$ of the elliptic operator $\mathcal{A}$ generates a symmetric sub-Markovian C_0-semigroup $T_2(\cdot)$ on $L^2(\mathbb{R}^d)$, it follows from Theorem B.4.5, that $T_2(\cdot)$ extends to a positive C_0-semigroup* *of contractions $T_p(\cdot)$ on $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty)$. Moreover these semigroups are consistent, i.e.*

$$T_p(t)f = T_q(t)f, \qquad \text{for all } f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d),\ t \geq 0.$$

(b) *Since, by Theorem 2.1.1, $T_2(\cdot)$ is ultracontractive, and $T_2(\cdot)$ coincides with $T(\cdot)$ on $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$, it follows that $T_2(\cdot)$ is given through an integral kernel which coincides with the kernel p of the semigroup $T(\cdot)$.*

2.2 Time dependent Lyapunov functions

As in [1, 30, 31, 41, 51] we use time dependent Lyapunov functions introduced in Definition 1.6.2 to prove pointwise bounds of the kernel p. In particular, we will deal with time dependent Lyapunov functions W for the operators $L := \partial_t + A$ and $\partial_t + \eta\Delta - V$ with respect to Z and h, where Z is the Lyapunov function introduced in Hypothesis 2.0.1(b) and $0 \leq h \in L^1((0,T))$. Then, for fixed $T > 0$, they satisfy the following inequalities

$$LW(t,x) \leq h(t)W(t,x) \tag{2.5}$$

and

$$\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \leq h(t)W(t,x) \tag{2.6}$$

for all $(t,x) \in (0,T) \times \mathbb{R}^d$.

In this section we give conditions under which certain exponentials are time dependent Lyapunov functions for $L := \partial_t + A$ and $\partial_t + \eta\Delta - V$ also in the case of polynomially and exponentially growing diffusion coefficients.

2.2.1 Time dependent Lyapunov functions for polynomially growing diffusion

In the following result we assume that the diffusion coefficients grow polynomially. Here $x \mapsto |x|_*^\beta$ denotes any C^2-function which coincides with $x \mapsto |x|^\beta$ for $|x| \geq 1$. Moreover we take $T = 1$ in Definition 1.6.2.

Proposition 2.2.1. *Assume that there is a constant $c_q > 0$ such that*

$$\sum_{i,j=1}^d q_{ij}(x)\xi_i\xi_j \leq c_q(1+|x|^m)|\xi|^2 \tag{2.7}$$

holds for all ξ, $x \in \mathbb{R}^d$ and some $m > 0$. For $(t,x) \in [0,1] \times \mathbb{R}^d$, consider the function

$$W(t,x) = e^{\varepsilon t^\alpha |x|_*^\beta},$$

with $\beta > (2-m) \vee 0$, $\varepsilon > 0$ and $\alpha > \frac{\beta}{\beta+m-2}$. If

$$\limsup_{|x|\to\infty} |x|^{1-\beta-m}\left(G \cdot \frac{x}{|x|} - \frac{V}{\varepsilon\beta|x|^{\beta-1}}\right) < -\Lambda \tag{2.8}$$

is satisfied for $\Lambda > c_q\varepsilon\beta$ and

$$\lim_{|x|\to\infty} V(x)\,|x|^{2-2\beta-m} > c \tag{2.9}$$

holds true for some $c>0$, then W is a time dependent Lyapunov function for L and $\partial_t + \eta\Delta - V$ with respect to $Z(x) = e^{\varepsilon |x|_*^{\beta}}$ and $h(t) = C_1 t^{\alpha-\gamma(2\beta+m-2)}$ for some $\gamma > \frac{1}{\beta+m-2}$ and some constant $C_1>0$. Here $G_j := \sum_{i=1}^d D_i q_{ij}$ and Z is a Lyapunov function for A and $\eta\Delta - V$. Moreover,

$$\xi_W(t,x) \le e^{\int_0^1 h(s)\,ds} =: C_2$$

for all $(t,x)\in[0,1]\times\mathbb{R}^d$.

Proof. It is easy to see that $W \in C^{1,2}((0,1)\times\mathbb{R}^d)\cap C([0,1]\times\mathbb{R}^d)$, $W(t,x)\to\infty$ as $|x|\to\infty$ uniformly for t in compact subsets of $(0,1]$ and $W\le Z$. It remains to show that there is $0\le h\in L^1(0,1)$ such that (2.5) and (2.6) hold true.

In the following computations we assume that $|x|\ge 1$ so that $|x|_*^s = |x|^s$ for $s\ge 0$. Otherwise, if $|x|\le 1$, then by continuity the functions

$$(t,x)\to |W(t,x)^{-1}LW(t,x)|$$

and

$$(t,x)\to |W(t,x)^{-1}[\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x)]|$$

are bounded on $(0,1)\times B_1$. Thus, we possibly choose a larger constant C_1 to define the function $h(t)$.

Let $t\in(0,1)$ and $|x|\ge 1$. By straightforward computations we have

$$\begin{aligned}
D_jW(t,x) =&\varepsilon\beta t^{\alpha}\,|x|^{\beta-2}\,x_jW(t,x),\\
D_i(q_{ij}D_jW)(t,x) =&\varepsilon\beta t^{\alpha}\,|x|^{\beta-2}\,D_iq_{ij}(x)x_jW(t,x)\\
&+\varepsilon\beta(\beta-2)t^{\alpha}\,|x|^{\beta-4}\,q_{ij}(x)x_ix_jW(t,x)\\
&+\varepsilon\beta t^{\alpha}\,|x|^{\beta-2}\,q_{ij}(x)\delta_{ij}W(t,x)\\
&+\varepsilon^2\beta^2t^{2\alpha}\,|x|^{2\beta-4}\,q_{ij}(x)x_ix_jW(t,x).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
LW(t,x) =&\partial_tW(t,x) + AW(t,x)\\
=&\varepsilon\alpha t^{\alpha-1}\,|x|^{\beta}\,W(t,x) + \varepsilon\beta t^{\alpha}\,|x|^{\beta-2}\,W(t,x)\sum_{i,j=1}^d D_iq_{ij}(x)x_j\\
&+\varepsilon\beta(\beta-2)t^{\alpha}\,|x|^{\beta-4}\,W(t,x)\sum_{i,j=1}^d q_{ij}(x)x_ix_j\\
&+\varepsilon\beta t^{\alpha}\,|x|^{\beta-2}\,W(t,x)\sum_{i,j=1}^d q_{ij}(x)\delta_{ij}\\
&+\varepsilon^2\beta^2t^{2\alpha}\,|x|^{2\beta-4}\,W(t,x)\sum_{i,j=1}^d q_{ij}(x)x_ix_j - V(x)W(t,x).
\end{aligned}$$

We recall that $G_j := \sum_{i=1}^d D_i q_{ij}$ and we use the polynomially growth of the diffusion coefficients (2.7). We have

$$\begin{aligned} LW(t,x) \leq & \varepsilon\alpha t^{\alpha-1} |x|^\beta W(t,x) + \varepsilon\beta t^\alpha |x|^{\beta-2} W(t,x)G(x)\cdot x \\ & + c_q\varepsilon\beta(\beta-2)^+ t^\alpha |x|^{\beta-4}(1+|x|^m)|x|^2 W(t,x) \\ & + dc_q\varepsilon\beta t^\alpha |x|^{\beta-2}(1+|x|^m)W(t,x) \\ & + c_q\varepsilon^2\beta^2 t^{2\alpha}|x|^{2\beta-4}(1+|x|^m)|x|^2 W(t,x) - V(x)W(t,x). \end{aligned}$$

Since $(1+|x|^m) \leq 2|x|^m$ and $t^\alpha \leq 1$, we arrange the terms as follows.

$$\begin{aligned} LW(t,x) \leq \varepsilon\beta t^\alpha |x|^{2\beta+m-2} W(t,x) \Bigg[& \frac{\alpha}{\beta t}|x|^{2-\beta-m} + 2c_q((\beta-2)^+ + d)|x|^{-\beta} \\ & + c_q\varepsilon\beta t^\alpha + c_q\varepsilon\beta t^\alpha |x|^{-m} + |x|^{1-\beta-m}\left(G\cdot\frac{x}{|x|} - \frac{V}{\varepsilon\beta|x|^{\beta-1}}\right)\Bigg]. \end{aligned} \tag{2.10}$$

Let $\gamma > \frac{1}{\beta+m-2}$. We distinguish two cases.

Case 1: $|x| > \dfrac{1}{t^\gamma}$.

Since $t^\alpha \leq 1$ and using (2.10), we get

$$\begin{aligned} LW(t,x) \leq \varepsilon\beta t^\alpha |x|^{2\beta+m-2} W(t,x) \Bigg[& \frac{\alpha}{\beta}|x|^{\frac{1}{\gamma}+2-\beta-m} + 2c_q((\beta-2)^+ + d)|x|^{-\beta} \\ & + c_q\varepsilon\beta + c_q\varepsilon\beta |x|^{-m} + |x|^{1-\beta-m}\left(G\cdot\frac{x}{|x|} - \frac{V}{\varepsilon\beta|x|^{\beta-1}}\right)\Bigg]. \end{aligned} \tag{2.11}$$

We claim that, if we assume further that $|x|$ is large enough, then

$$LW(t,x) \leq 0,$$

for all $t \in (0,1)$. To see this, let $|x| > K$ for some $K > 1$. Combining (2.8) with (2.11) yields

$$\begin{aligned} LW(t,x) \leq \varepsilon\beta t^\alpha |x|^{2\beta+m-2} W(t,x) \Bigg[& \frac{\alpha}{\beta}|x|^{\frac{1}{\gamma}+2-\beta-m} + 2c_q((\beta-2)^+ + d)|x|^{-\beta} \\ & + c_q\varepsilon\beta + c_q\varepsilon\beta |x|^{-m} - \Lambda\Bigg]. \end{aligned} \tag{2.12}$$

Considering that $\gamma > \frac{1}{\beta+m-2}$, $\beta > 0$ and $m > 0$, we infer that

$$\begin{aligned} & \frac{\alpha}{\beta}|x|^{\frac{1}{\gamma}+2-\beta-m} + 2c_q((\beta-2)^+ + d)|x|^{-\beta} + c_q\varepsilon\beta + c_q\varepsilon\beta|x|^{-m} - \Lambda \\ & \leq \left(\frac{\alpha}{\beta} + 2c_q((\beta-2)^+ + d) + c_q\varepsilon\beta\right) K^{-l} + c_q\varepsilon\beta - \Lambda, \end{aligned}$$

where $l := \min(\frac{-1}{\gamma} - 2 + \beta + m, \beta, m) > 0$. Since $\Lambda > c_q \varepsilon \beta$, choosing

$$K \geq \left(\frac{\frac{\alpha}{\beta} + 2c_q((\beta-2)^+ + d) + c_q \varepsilon \beta}{\Lambda - c_q \varepsilon \beta} \right)^{\frac{1}{l}},$$

it follows that the quantity within square brackets in the right hand side of (2.12) is negative. Thus $LW(t,x) \leq 0$ for $|x| > \frac{1}{t^\gamma}$, $|x| > K$ and for all $t \in (0,1)$.

For the remaining values of x, $|x| \leq K$, we have that $LW(t,x) \leq C$ for a certain constant $C > 0$. Anyway, we conclude that

$$LW(t,x) \leq CW(t,x),$$

for all $t \in (0,1)$ and $|x| > \frac{1}{t^\gamma}$.

Case 2: $|x| \leq \dfrac{1}{t^\gamma}$.

We assume that $|x|$ is large enough. Otherwise, as in Case 1, $LW(t,x) \leq C \leq CW(t,x)$ for a certain constant C. We combine (2.8) and (2.10) to deduce that

$$\begin{aligned} LW(t,x) \leq & \big[\varepsilon \alpha t^{\alpha-1-\gamma\beta} + 2c_q \varepsilon \beta((\beta-2)^+ + d) t^{\alpha-\gamma(\beta+m-2)} \\ & + c_q \varepsilon^2 \beta^2 t^{2\alpha-\gamma(2\beta+m-2)} + c_q \varepsilon^2 \beta^2 t^{2\alpha-\gamma(2\beta-2)} \\ & - \varepsilon \beta t^\alpha |x|^{2\beta+m-2} \Lambda\big] W(t,x). \end{aligned}$$

We drop the term involving Λ because it is negative. Moreover, since $\gamma > 1/(\beta+m-2)$, we note that the leading term is $t^{\alpha-\gamma(2\beta+m-2)}$. Hence

$$LW(t,x) \leq h(t)W(t,x),$$

where

$$h(t) := C_1 t^{\alpha-\gamma(2\beta+m-2)}.$$

For the function $h(t)$ to be in the space $L^1((0,1))$, we set $\alpha > \frac{\beta}{\beta+m-2}$. In this way, choosing $\gamma < \frac{\alpha+1}{2\beta+m-2}$ so that $\alpha - \gamma(2\beta+m-2) > -1$, $h(t)$ is integrable in the interval $(0,1)$.

Summing up, considering a possibly larger constant C_1, we proved (2.5) for all $t \in (0,1)$ and $x \in \mathbb{R}^d$.

We now verify (2.6). An easy computation shows that

$$\Delta W(t,x) = \varepsilon \beta(\beta+d-2) t^\alpha |x|^{\beta-2} W(t,x) + \varepsilon^2 \beta^2 t^{2\alpha} |x|^{2\beta-2} W(t,x).$$

Thus, we get

$$\begin{aligned} & \partial_t W(t,x) + \eta \Delta W(t,x) - V(x)W(t,x) \\ & \quad = \varepsilon \alpha t^{\alpha-1} |x|^\beta W(t,x) + \eta \varepsilon \beta(\beta+d-2) t^\alpha |x|^{\beta-2} W(t,x) \\ & \qquad + \eta \varepsilon^2 \beta^2 t^{2\alpha} |x|^{2\beta-2} W(t,x) - V(x)W(t,x). \end{aligned} \tag{2.13}$$

As in the first part of the proof, we let $\gamma > \frac{1}{\beta+m-2}$ and we distinguish two cases.

Case 1: $|x| > \dfrac{1}{t^\gamma}$.

Since $t^\alpha \leq 1$, by (2.13) we obtain

$$\begin{aligned}
&\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x)\\
&\leq \varepsilon\beta t^\alpha |x|^{2\beta+m-2} W(t,x)\left[\frac{\alpha}{\beta}|x|^{\frac{1}{\gamma}+2-\beta-m} + \eta(\beta+d-2)|x|^{-\beta-m}\right.\\
&\qquad\qquad \left. + \eta\varepsilon\beta|x|^{-m} - \frac{1}{\varepsilon\beta}V(x)|x|^{2-2\beta-m}\right].
\end{aligned}$$

If $|x|$ large enough, then by (2.9) we have

$$\begin{aligned}
&\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x)\\
&\leq \varepsilon\beta t^\alpha |x|^{2\beta+m-2} W(t,x)\left[\frac{\alpha}{\beta}|x|^{\frac{1}{\gamma}+2-\beta-m} + \eta(\beta+d-2)|x|^{-\beta-m}\right.\\
&\qquad\qquad \left. + \eta\varepsilon\beta|x|^{-m} - \frac{c}{\varepsilon\beta}\right].
\end{aligned}$$

Arguing as in (2.12), we find that $\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x)$ is negative for $|x|$ large, whereas it is bounded for the remaining values of x. Therefore, we deduce that

$$\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \leq CW(t,x),$$

for all $t \in (0,1)$ and $|x| > \frac{1}{t^\gamma}$.

Case 2: $|x| \leq \dfrac{1}{t^\gamma}$.

Since $V \geq 0$, (2.13) leads to

$$\begin{aligned}
&\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x)\\
&\quad \leq \left[\varepsilon\alpha t^{\alpha-1-\gamma\beta} + \eta\varepsilon\beta((\beta-2)^+ + d)t^{\alpha-\gamma(\beta-2)} + \eta\varepsilon^2\beta^2 t^{2\alpha-\gamma(2\beta-2)}\right]W(t,x).
\end{aligned}$$

We can control the right hand side of the previous inequality with the function $h(t)W(t,x)$, obtaining that

$$\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \leq h(t)W(t,x),$$

where the constant C_1 in the function h has to be suitably adjusted. In both cases (2.6) holds true. We conclude that W is a time dependent Lyapunov function for L and $\partial_t + \eta\Delta - V$. In addition, we observe that, if we take $t = 1$ and we argue similarly, then it follows that Z is a Lyapunov function for A and $\eta\Delta - V$.

Moreover, by Proposition 1.6.3, we have

$$\xi_W(t,x) \leq e^{\int_0^t h(s)\,ds} W(0,x) \leq e^{\int_0^1 h(s)\,ds} =: C_2,$$

for all $(t,x) \in [0,1] \times \mathbb{R}^d$. □

Remark 2.2.2. *One can easily see that the same conclusion as in Proposition 2.2.1 remains valid if we replace the operator A with the more general operator $A_F := A + F \cdot \nabla$ with $F \in C^\zeta(\mathbb{R}^d, \mathbb{R}^d)$ for some $\zeta \in (0,1)$, and the condition (2.8) with*

$$\limsup_{|x|\to\infty} |x|^{1-\beta-m} \left((G+F) \cdot \frac{x}{|x|} - \frac{V}{\varepsilon\beta|x|^{\beta-1}} \right) < -\Lambda.$$

This generalizes Proposition 2.3 in [1].

2.2.2 Time dependent Lyapunov functions for exponentially growing diffusion

We now turn to the case of exponentially growing diffusion.

Proposition 2.2.3. *Assume that there is a constant $c_e > 0$ such that*

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \leq c_e e^{|x|^m} |\xi|^2 \tag{2.14}$$

holds for all ξ, $x \in \mathbb{R}^d$ and some $m \geq 2$. Consider the function

$$W(t,x) = \exp\left(\varepsilon t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}} \, d\tau \right)$$

for $(t,x) \in [0,1] \times \mathbb{R}^d$, with $\frac{m}{2} + 1 \leq \beta \leq m$, $\varepsilon > 0$ and $\alpha > \frac{2\beta+m-2}{2m}$. If

$$\limsup_{|x|\to\infty} |x|^{1-\beta-m} e^{-\frac{|x|^\beta}{2} - |x|^m} \left(G \cdot \frac{x}{|x|} - \frac{V}{\varepsilon e^{\frac{|x|^\beta}{2}}} \right) < -\Lambda \tag{2.15}$$

is satisfied for $\Lambda > 0$ and

$$\lim_{|x|\to\infty} V(x)\, |x|^{1-\beta-m}\, e^{-|x|^\beta - |x|^m} > c \tag{2.16}$$

holds true for some $c > 0$, then W is a time dependent Lyapunov function for L and $\partial_t + \eta\Delta - V$ with respect to $Z(x) = \exp\left(\varepsilon \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}} d\tau\right)$ and $h(t) = C_3 t^{\alpha-\gamma\left(\beta+\frac{3}{2}m-1\right)}$ for some $\gamma > \frac{1}{m}$ and some constant $C_3 > 0$. Here $G_j := \sum_{i=1}^d D_i q_{ij}$ and Z is a Lyapunov function for A and $\eta\Delta - V$. Moreover,

$$\xi_W(t,x) \leq e^{\int_0^1 h(s)\, ds} =: C_4$$

for all $(t,x) \in [0,1] \times \mathbb{R}^d$.

Proof. Throughout the proof we assume that $|x| \geq 1$ so that $|x|_*^s = |x|^s$ for $s \geq 0$. The estimates can be extended to $\mathbb{R}^d$ by possibly choosing larger constants.

Let $t \in (0,1)$ and $|x| \geq 1$. By direct computations we have

$$\begin{aligned}
D_j W(t,x) =& \varepsilon t^{\alpha} \frac{x_j}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x),\\
D_i(q_{ij} D_j W)(t,x) =& \varepsilon t^{\alpha} \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} D_i q_{ij}(x) x_j W(t,x)\\
&+ \frac{1}{2}\varepsilon \beta t^{\alpha} |x|^{\beta-3} e^{\frac{|x|^\beta}{2}} q_{ij}(x) x_i x_j W(t,x)\\
&+ \varepsilon t^{\alpha} \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} q_{ij}(x) \delta_{ij} W(t,x)\\
&- \varepsilon t^{\alpha} \frac{1}{|x|^3} e^{\frac{|x|^\beta}{2}} q_{ij}(x) x_i x_j W(t,x)\\
&+ \varepsilon^2 t^{2\alpha} \frac{1}{|x|^2} e^{|x|^\beta} q_{ij}(x) x_i x_j W(t,x).
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
LW(t,x) =& \partial_t W(t,x) + AW(t,x)\\
=& \varepsilon \alpha t^{\alpha-1} W(t,x) \int_0^{|x|} e^{\frac{\tau^\beta}{2}} d\tau + \varepsilon t^{\alpha} \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) \sum_{i,j=1}^d D_i q_{ij}(x) x_j\\
&+ \frac{1}{2}\varepsilon \beta t^{\alpha} |x|^{\beta-3} e^{\frac{|x|^\beta}{2}} W(t,x) \sum_{i,j=1}^d q_{ij}(x) x_i x_j\\
&+ \varepsilon t^{\alpha} \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) \sum_{i,j=1}^d q_{ij}(x) \delta_{ij}\\
&- \varepsilon t^{\alpha} \frac{1}{|x|^3} e^{\frac{|x|^\beta}{2}} W(t,x) \sum_{i,j=1}^d q_{ij}(x) x_i x_j\\
&+ \varepsilon^2 t^{2\alpha} \frac{1}{|x|^2} e^{|x|^\beta} W(t,x) \sum_{i,j=1}^d q_{ij}(x) x_i x_j - V(x) W(t,x).
\end{aligned}$$

First of all, we drop the negative term involving ε in the right hand side of the previous equality. Second, we use the exponentially growth of the diffusion coefficients (2.14) to obtain that

$$\begin{aligned}
LW(t,x) \leq& \varepsilon \alpha t^{\alpha-1} W(t,x) \int_0^{|x|} e^{\frac{\tau^\beta}{2}} d\tau + \varepsilon t^{\alpha} \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) G(x) \cdot x\\
&+ \frac{1}{2} c_e \varepsilon \beta t^{\alpha} |x|^{\beta-1} e^{\frac{|x|^\beta}{2} + |x|^m} W(t,x) + d c_e \varepsilon t^{\alpha} \frac{1}{|x|} e^{\frac{|x|^\beta}{2} + |x|^m} W(t,x)\\
&+ c_e \varepsilon^2 t^{2\alpha} e^{|x|^\beta + |x|^m} W(t,x) - V(x) W(t,x).
\end{aligned}$$

Since $t^\alpha \le 1$, we can write the previous inequality as follows:

$$\begin{aligned} LW(t,x) \le & \varepsilon t^\alpha |x|^{\beta+m-1} e^{|x|^\beta+|x|^m} W(t,x) \Bigg[\frac{\alpha}{t} |x|^{1-\beta-m} e^{-|x|^\beta-|x|^m} \int_0^{|x|} e^{\frac{\tau^\beta}{2}} d\tau \\ & + \frac{1}{2} c_e \beta |x|^{-m} e^{-\frac{|x|^\beta}{2}} + d c_e |x|^{-\beta-m} e^{-\frac{|x|^\beta}{2}} + c_e \varepsilon t^\alpha |x|^{1-\beta-m} \\ & + |x|^{1-\beta-m} e^{-\frac{|x|^\beta}{2}-|x|^m} \left(G\cdot \frac{x}{|x|} - \frac{V}{\varepsilon e^{\frac{|x|^\beta}{2}}} \right) \Bigg]. \end{aligned} \tag{2.17}$$

Let $\gamma > \frac{1}{m}$. We now distinguish two cases.

Case 1: $e^{|x|^m} \ge \dfrac{1}{t^{\gamma m}}$.

First, we observe that

$$\int_0^{|x|} e^{\frac{\tau^\beta}{2}} d\tau \le |x|\, e^{\frac{|x|^\beta}{2}}.$$

Then, since $t^\alpha \le 1$ and $e^{-\frac{|x|^\beta}{2}} \le 1$, by (2.17) we get

$$\begin{aligned} LW(t,x) \le & \varepsilon t^\alpha |x|^{\beta+m-1} e^{|x|^\beta+|x|^m} W(t,x) \Bigg[\alpha |x|^{2-\beta-m} e^{\left(\frac{1}{\gamma m}-1\right)|x|^m} \\ & + \frac{1}{2} c_e \beta |x|^{-m} + d c_e |x|^{-\beta-m} + c_e \varepsilon |x|^{1-\beta-m} \\ & + |x|^{1-\beta-m} e^{-\frac{|x|^\beta}{2}-|x|^m} \left(G\cdot \frac{x}{|x|} - \frac{V}{\varepsilon e^{\frac{|x|^\beta}{2}}} \right) \Bigg]. \end{aligned}$$

Moreover, $e^{\left(\frac{1}{\gamma m}-1\right)|x|^m} \le 1$ because $\gamma > \frac{1}{m}$. Thus, we derive that

$$\begin{aligned} LW(t,x) \le & \varepsilon t^\alpha |x|^{\beta+m-1} e^{|x|^\beta+|x|^m} W(t,x) \Bigg[\alpha |x|^{2-\beta-m} \\ & + \frac{1}{2} c_e \beta |x|^{-m} + d c_e |x|^{-\beta-m} + c_e \varepsilon |x|^{1-\beta-m} \\ & + |x|^{1-\beta-m} e^{-\frac{|x|^\beta}{2}-|x|^m} \left(G\cdot \frac{x}{|x|} - \frac{V}{\varepsilon e^{\frac{|x|^\beta}{2}}} \right) \Bigg]. \end{aligned}$$

If $|x|$ is large enough, say $|x| > K$ for some $K > 1$, then we apply (2.15) to deduce that

$$\begin{aligned} LW(t,x) \le & \varepsilon t^\alpha |x|^{\beta+m-1} e^{|x|^\beta+|x|^m} W(t,x) \Bigg[\alpha |x|^{2-\beta-m} + \frac{1}{2} c_e \beta |x|^{-m} \\ & + d c_e |x|^{-\beta-m} + c_e \varepsilon |x|^{1-\beta-m} - \Lambda \Bigg]. \end{aligned}$$

We now show that, for a suitable choice of K, the quantity within square brackets is negative. Since $\beta \geq \frac{m}{2}+1$ and $m \geq 2$, we have $\beta \geq 2$ and hence

$$\begin{aligned}&\alpha \left|x\right|^{2-\beta-m} + \frac{1}{2}c_e\beta \left|x\right|^{-m} + dc_e \left|x\right|^{-\beta-m} + c_e\varepsilon \left|x\right|^{1-\beta-m} - \Lambda \\ &\leq \left(\alpha + \frac{1}{2}c_e\beta + dc_e + c_e\varepsilon\right) K^{-m} - \Lambda.\end{aligned}$$

As a result, by taking

$$K \geq \left(\frac{\alpha + \frac{1}{2}c_e\beta + dc_e + c_e\varepsilon}{\Lambda}\right)^{\frac{1}{m}},$$

we finally get $LW(t,x) \leq 0$. For the remaining values of x, LW is bounded by a constant. In both cases we have

$$LW(t,x) \leq CW(t,x),$$

for all $t \in (0,1)$, $e^{|x|^m} \geq \frac{1}{t^{\gamma m}}$ and for some constant $C>0$.

Case 2: $e^{|x|^m} < \dfrac{1}{t^{\gamma m}}$.

Notice that $|x| < t^{-\gamma}$ and, since $\beta \leq m$, we have

$$e^{|x|^\beta} < \frac{1}{t^{\gamma m}} \quad \text{for } |x| \geq 1.$$

Then, if $|x|$ is large enough, using $\beta > 1$, and combining (2.15) and (2.17), we obtain that

$$\begin{aligned}LW(t,x) \leq \Bigg[&\varepsilon\alpha t^{\alpha-1-\gamma\left(\frac{m}{2}+1\right)} + \frac{1}{2}c_e\varepsilon\beta t^{\alpha-\gamma\left(\beta+\frac{3}{2}m-1\right)} + dc_e\varepsilon t^{\alpha-\frac{3}{2}\gamma m} + c_e\varepsilon^2 t^{2\alpha-2\gamma m} \\ &- \Lambda\varepsilon t^\alpha \left|x\right|^{\beta+m-1} e^{|x|^\beta+|x|^m}\Bigg] W(t,x).\end{aligned}$$

Dropping the last negative term, we find

$$\begin{aligned}LW(t,x) \leq \Bigg[&\varepsilon\alpha t^{\alpha-1-\gamma\left(\frac{m}{2}+1\right)} + \frac{1}{2}c_e\varepsilon\beta t^{\alpha-\gamma\left(\beta+\frac{3}{2}m-1\right)} \\ &+ dc_e\varepsilon t^{\alpha-\frac{3}{2}\gamma m} + c_e\varepsilon^2 t^{2\alpha-2\gamma m}\Bigg] W(t,x).\end{aligned}$$

Since $\gamma > \frac{1}{m}$ and $\beta \geq \frac{m}{2}+1$, the leading term is $t^{\alpha-\gamma\left(\beta+\frac{3}{2}m-1\right)}$. Therefore, we gain

$$LW(t,x) \leq Ct^{\alpha-\gamma\left(\beta+\frac{3}{2}m-1\right)} W(t,x),$$

for all $t \in (0,1)$, $e^{|x|^m} < \frac{1}{t^{\gamma m}}$ and for some constant $C>0$.

To sum up, there exists a constant $C_3 > 0$ such that

$$LW(t,x) \leq h(t)W(t,x),$$

for all $t \in (0,1)$ and $x \in \mathbb{R}^d$, where $h(t) = C_3 t^{\alpha-\gamma\left(\beta+\frac{3}{2}m-1\right)}$.

Moreover, we choose $\gamma < \frac{\alpha+1}{\beta+\frac{3}{2}m-1}$, which is possible since $\alpha > \frac{2\beta+m-2}{2m}$, so that $\alpha - \gamma\left(\beta + \frac{3}{2}m - 1\right) > -1$ and $h \in L^1((0,1))$. We conclude that condition (2.5) is satisfied.

To show (2.6) we compute

$$\begin{aligned}\Delta W(t,x) =& \frac{1}{2}\varepsilon\beta t^\alpha |x|^{\beta-1} e^{\frac{|x|^\beta}{2}} W(t,x) + d\varepsilon t^\alpha \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) \\ &- \varepsilon t^\alpha \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) + \varepsilon^2 t^{2\alpha} e^{|x|^\beta} W(t,x).\end{aligned}$$

Hence,

$$\begin{aligned}&\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \\ =&\varepsilon\alpha t^{\alpha-1} W(t,x) \int_0^{|x|} e^{\frac{\tau^\beta}{2}} d\tau + \frac{1}{2}\eta\varepsilon\beta t^\alpha |x|^{\beta-1} e^{\frac{|x|^\beta}{2}} W(t,x) \\ &+ d\eta\varepsilon t^\alpha \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) - \eta\varepsilon t^\alpha \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) \\ &+ \eta\varepsilon^2 t^{2\alpha} e^{|x|^\beta} W(t,x) - V(x)W(t,x) \\ \le&\varepsilon\alpha t^{\alpha-1} W(t,x) \int_0^{|x|} e^{\frac{\tau^\beta}{2}} d\tau + \frac{1}{2}\eta\varepsilon\beta t^\alpha |x|^{\beta-1} e^{\frac{|x|^\beta}{2}} W(t,x) \\ &+ d\eta\varepsilon t^\alpha \frac{1}{|x|} e^{\frac{|x|^\beta}{2}} W(t,x) + \eta\varepsilon^2 t^{2\alpha} e^{|x|^\beta} W(t,x) - V(x)W(t,x). \qquad (2.18)\end{aligned}$$

We use the same strategy as above. We let $\gamma > \frac{1}{m}$ and we consider two cases.

Case 1: $e^{|x|^m} \ge \dfrac{1}{t^{\gamma m}}$.

By (2.18) we obtain

$$\begin{aligned}&\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \\ \le&\varepsilon t^\alpha |x|^{\beta+m-1} e^{|x|^\beta+|x|^m} W(t,x) \Bigg[\alpha |x|^{2-\beta-m} e^{\left(\frac{1}{\gamma m}-1\right)|x|^m} + \frac{1}{2}\eta\beta |x|^{-m} \\ &+ d\eta |x|^{-\beta-m} + \eta\varepsilon |x|^{1-\beta-m} - \frac{1}{\varepsilon} V(x) |x|^{1-\beta-m} e^{-|x|^\beta-|x|^m}\Bigg].\end{aligned}$$

Using (2.16) and the fact that $\gamma > \frac{1}{m}$, we get

$$\begin{aligned}&\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \\ \le&\varepsilon t^\alpha |x|^{\beta+m-1} e^{|x|^\beta+|x|^m} W(t,x) \Big[\alpha |x|^{2-\beta-m} \\ &+ \frac{1}{2}\eta\beta |x|^{-m} + d\eta |x|^{-\beta-m} + \eta\varepsilon |x|^{1-\beta-m} - \frac{c}{\varepsilon}\Big].\end{aligned}$$

If $|x|$ is large enough, then the quantity within square brackets is negative. Otherwise, we can control it with a constant. In both cases, we deduce that

$$\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \le C,$$

for all $t \in (0,1)$, $e^{|x|^m} \geq \frac{1}{t^{\gamma m}}$ and for some constant $C > 0$.

Case 2: $e^{|x|^m} < \dfrac{1}{t^{\gamma m}}$.

Since $\beta \leq m$ and $V \geq 0$, (2.18) yields

$$\begin{aligned}
&\partial_t W(t,x) + \eta\Delta W(t,x) - V(x)W(t,x) \\
&\leq \Big[\varepsilon\alpha t^{\alpha-1-\gamma(\frac{m}{2}+1)} + \frac{1}{2}\eta\varepsilon\beta t^{\alpha-\gamma(\beta+\frac{m}{2}-1)} + d\eta\varepsilon t^{\alpha-\gamma\frac{m}{2}} + \eta\varepsilon^2 t^{2\alpha-\gamma m}\Big] W(t,x) \\
&\leq C t^{\alpha-\gamma(\beta+\frac{3}{2}m-1)} W(t,x),
\end{aligned}$$

for some constant C. Therefore, by possibly choosing a larger C_3, we gain (2.6). Then, W is a time dependent Lyapunov function for L and $\partial_t + \eta\Delta - V$. Moreover, taking $t = 1$ and arguing similarly, we obtain that Z is a Lyapunov function for A and $\eta\Delta - V$. Finally, the last assertion follows from Proposition 1.6.3. □

2.3 Preliminary results for bounded diffusion coefficients

Throughout this section we assume that the coefficients q_{ij} and their spatial derivatives $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for all $i, j, h = 1, \ldots, d$. Under this assumption, we establish global boundedness and Sobolev regularity of the kernel p. The results presented here are the main ingredients that in the next section will allow us to obtain an upper bound for the kernel p in case of bounded diffusion coefficients such that the constant in the right hand side of the estimate does not depend on the diffusion itself.

2.3.1 Global regularity results

We fix $T > 0$ and consider p as a function of $(t, y) \in (0, T) \times \mathbb{R}^d$ for fixed $x \in \mathbb{R}^d$. Moreover, we fix $0 < a_0 < a < b < b_0 \leq T$ and we set

$$\Gamma_2(k, x, a_0, b_0) = \left(\int_{Q(a_0,b_0)} V^k(y) p(t,x,y)\, dt\, dy \right)^{\frac{1}{k}}. \tag{2.19}$$

We now look for the values of s for which the transition kernel p belongs to the space $\mathcal{H}^{s,1}(Q(a,b))$ presented in Definition A.4.3. For that, we adapt [41, Lemma 3.1] for operators with potential term.

Lemma 2.3.1. *Assume that q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i, j, h = 1, \ldots, d$. If $\Gamma_2(k, x, a_0, b_0) < \infty$ for some $k > 1$ and $p \in L^r(Q(a_0, b_0))$ for some $1 < r \leq \infty$, then $p \in \mathcal{H}^{s,1}(Q(a,b))$ for $s = rk/(r + k - 1)$ if $r < \infty$, $s = k$ if $r = \infty$.*

Proof. Throughout the proof we consider a generic constant c depending on k, x, a_0, a, b, b_0 and the coefficients q_{ij}. Let $\vartheta \in C^\infty(\mathbb{R})$ be such that

- $\vartheta(t) = 1$ for $a \le t \le b$,
- $\vartheta(t) = 0$ for $t \le a_0$, $t \ge b_0$,
- $0 \le \vartheta \le 1$.

Moreover, let $\varphi \in C_c^{1,2}(Q(0,T))$. If we apply Lemma 1.3.3 to the function $\vartheta\varphi$ we get

$$\int_{Q(0,T)} q(\partial_t\varphi + A_1\varphi)\, dt\, dy = -\int_{Q(0,T)} (qG\cdot\nabla\varphi - qV\varphi + p\varphi\vartheta')\, dt\, dy, \quad (2.20)$$

where $q := \vartheta p$, $A_1 := \sum_{i,j=1}^d q_{ij}D_{ij}$ and $G_j := \sum_{i=1}^d D_i q_{ij}$. If $r < \infty$ and $s = rk/(r+k-1)$, then from Hölder's inequality with exponents k/s and $k/(k-s)$ we obtain that

$$\begin{aligned}\int_{Q(a_0,b_0)} V^s p^s\, dt\, dy &= \int_{Q(a_0,b_0)} V^s p^{\frac{s}{k}} p^{s\left(1-\frac{1}{k}\right)}\, dt\, dy \\ &\le \left(\int_{Q(a_0,b_0)} V^k p\, dt\, dy\right)^{\frac{s}{k}} \left(\int_{Q(a_0,b_0)} p^{\frac{s(k-1)}{k-s}}\, dt\, dy\right)^{1-s/k} \\ &= \left(\int_{Q(a_0,b_0)} V^k p\, dt\, dy\right)^{\frac{s}{k}} \left(\int_{Q(a_0,b_0)} p^r\, dt\, dy\right)^{\frac{k-s}{k}} \\ &= \Gamma_2(k,x,a_0,b_0)^s \, \|p\|_{L^r(Q(a_0,b_0))}^{\frac{s(k-1)}{k}}.\end{aligned}$$

If $r = \infty$ and $s = k$, we write

$$\int_{Q(a_0,b_0)} V^s p^s\, dt\, dy = \int_{Q(a_0,b_0)} V^k p^{k-1} p\, dt\, dy \le \Gamma_2(k,x,a_0,b_0)^k \, \|p\|_{L^\infty(Q(a_0,b_0))}^{k-1}.$$

In both cases it leads to

$$\|Vp\|_{L^s(Q(a_0,b_0))} \le c\, \|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}.$$

Therefore, applying Hölder's inequality with exponents s and s' such that $1/s + 1/s' = 1$, we have

$$\begin{aligned}\left|\int_{Q(0,T)} qV\varphi\, dt\, dy\right| &\le \|Vp\|_{L^s(Q(a_0,b_0))} \|\varphi\|_{L^{s'}(Q(0,T))} \\ &\le c\, \|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}} \|\varphi\|_{L^{s'}(Q(0,T))}. \quad (2.21)\end{aligned}$$

It is possible to repeat the same computations with $\mathbb{1}$ instead of V to prove that $p \in L^s(Q(a_0,b_0))$ and

$$\|p\|_{L^s(Q(a_0,b_0))} \le c\, \|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}. \quad (2.22)$$

Similarly, since $D_h q_{ij}$ are bounded on $\mathbb{R}^d$, it follows that

$$\|Gp\|_{L^s(Q(a_0,b_0))} \le c\, \|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}.$$

Consequently,

$$\left|\int_{Q(0,T)} qG\cdot\nabla\varphi\,dt\,dy\right| \le \|Gp\|_{L^s(Q(a_0,b_0))}\|\nabla\varphi\|_{L^{s'}(Q(0,T))}$$
$$\le c\,\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}\|\varphi\|_{W^{0,1}_{s'}(Q(0,T))}.$$

Putting everything together yields

$$\left|\int_{Q(0,T)} q(\partial_t\varphi + A_1\varphi)\,dt\,dy\right| \le c\,\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}\|\varphi\|_{W^{0,1}_{s'}(Q(0,T))}. \tag{2.23}$$

We now consider the difference quotient with respect to the variable y

$$\tau_{-h}\varphi(t,y) = |h|^{-1}(\varphi(t,y-he_{j_0}) - \varphi(t,y)),$$

for any $(t,y)\in Q(0,T)$, $0\ne h\in\mathbb{R}$ small enough and fixed $j_0\in\{1,\dots,d\}$. Substituting $\tau_{-h}\varphi$ instead of φ in (2.23) leads to

$$\left|\int_{Q(0,T)} q(\partial_t\tau_{-h}\varphi + A_1\tau_{-h}\varphi)\,dt\,dy\right| \le c\,\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}\|\tau_{-h}\varphi\|_{W^{0,1}_{s'}(Q(0,T))}. \tag{2.24}$$

By a change of variables, we find that

$$\begin{aligned}&\int_{Q(0,T)} qA_1\tau_{-h}\varphi\,dt\,dy\\ &\quad=\frac{1}{|h|}\int_{Q(0,T)} q(t,y+he_{j_0})\sum_{i,j=1}^d q_{ij}(y+he_{j_0})D_{ij}\varphi(t,y)\,dt\,dy\\ &\quad\quad-\frac{1}{|h|}\int_{Q(0,T)} q(t,y)\sum_{i,j=1}^d q_{ij}(y)D_{ij}\varphi(t,y)\,dt\,dy.\end{aligned}$$

Summing and subtracting $|h|^{-1}\int_{Q(0,T)} q(t,y+he_{j_0})\sum_{i,j=1}^d q_{ij}(y)D_{ij}\varphi(t,y)\,dt\,dy$ implies that

$$\begin{aligned}\int_{Q(0,T)} qA_1\tau_{-h}\varphi\,dt\,dy &= \int_{Q(0,T)} q(t,y+he_{j_0})\sum_{i,j=1}^d \tau_h q_{ij}(y)D_{ij}\varphi(t,y)\,dt\,dy\\ &\quad-\int_{Q(0,T)}\tau_h qA_1\varphi\,dt\,dy.\end{aligned} \tag{2.25}$$

Moreover, since there is ξ_y on the segment from y to $y+he_{j_0}$ such that $\tau_h q_{ij}(y) = D_{j_0}q_{ij}(\xi_y)$, applying Hölder's inequality and (2.22) we have

$$\begin{aligned}&\left|\int_{Q(0,T)} q(t,y+he_{j_0})\sum_{i,j=1}^d D_{j_0}q_{ij}(\xi_y)D_{ij}\varphi(t,y)\,dt\,dy\right|\\ &\le c\,\|p\|_{L^s(Q(a_0,b_0))}\|\varphi\|_{W^{1,2}_{s'}(Q(0,T))} \le c\,\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}\|\varphi\|_{W^{1,2}_{s'}(Q(0,T))},\end{aligned} \tag{2.26}$$

where we used the boundedness of the first order derivatives of the diffusion coefficients. Considering that $\|\tau_{-h}\varphi\|_{W^{0,1}_{s'}(Q(a_0,b_0))} \le \|\varphi\|_{W^{1,2}_{s'}(Q(a_0,b_0))}$ and combining (2.24) with (2.25) and (2.26) yields

$$\begin{aligned}
&\left|\int_{Q(0,T)} \tau_h q(\partial_t\varphi + A_1\varphi)\,dt\,dy\right| \le \left|\int_{Q(0,T)} q(\partial_t\tau_{-h}\varphi + A_1\tau_{-h}\varphi)\,dt\,dy\right| \\
&+ \left|\int_{Q(0,T)} q(t, y+he_{j_0}) \sum_{i,j=1}^{d} \tau_h q_{ij}(y) D_{ij}\varphi(t,y)\,dt\,dy\right| \\
&\le c\,\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\,\|\varphi\|_{W^{1,2}_{s'}(Q(0,T))}\,. \qquad (2.27)
\end{aligned}$$

Moreover, since $q \in L^s(Q(0,T))$ and $s = (s-1)s'$, we observe that $|\tau_h q|^{s-2}\tau_h q \in L^{s'}(Q(0,T))$ and we have

$$\left\||\tau_h q|^{s-2}\tau_h q\right\|_{L^{s'}(Q(0,T))} = \|\tau_h q\|^{s-1}_{L^s(Q(0,T))}\,. \qquad (2.28)$$

According to [33, Theorem 9.2.3] we choose $\varphi \in W^{1,2}_{s'}(Q(0,T))$ such that

$$\begin{cases} \partial_t\varphi + A_1\varphi = |\tau_h q|^{s-2}\tau_h q, & \text{in } Q(0,T), \\ \varphi(T,y) = 0, & y \in \mathbb{R}^d, \end{cases} \qquad (2.29)$$

and

$$\|\varphi\|_{W^{1,2}_{s'}(Q(0,T))} \le C\left\||\tau_h q|^{s-2}\tau_h q\right\|_{L^{s'}(Q(0,T))}\,. \qquad (2.30)$$

We note that we cannot insert directly such a φ in (2.27) because it does not have compact support with respect to the space variables. Thus, we approximate φ in $W^{1,2}_{s'}(Q(0,T))$ with a sequence $\varphi_n \in C^{1,2}_c(Q(0,T))$ defined by

$$\varphi_n(t,y) = \psi(y/n)\varphi(t,y),$$

where $\psi \in C^\infty_c(\mathbb{R}^d)$ is a fixed smooth function such that $\psi(y) = 1$ for $|y| \le 1$. Writing (2.27) for φ_n, letting $n \to \infty$ and using the dominated convergence theorem, we deduce that (2.27) is valid also for $\varphi \in W^{1,2}_{s'}(Q(0,T))$.

Therefore, since φ is the solution of the Cauchy problem (2.29), it follows that

$$\begin{aligned}
\int_{Q(0,T)} |\tau_h q|^s &= \int_{Q(0,T)} \tau_h q(\partial_t\varphi + A_1\varphi)\,dt\,dy \le c\,\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\,\|\varphi\|_{W^{1,2}_{s'}(Q(0,T))} \\
&\le c\,\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\left\||\tau_h q|^{s-2}\tau_h q\right\|_{L^{s'}(Q(0,T))} \\
&= c\,\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\,\|\tau_h q\|^{s-1}_{L^s(Q(0,T))}\,,
\end{aligned}$$

where we used (2.27), (2.30) and (2.28). In conclusion, we obtain that $\nabla q \in L^s(Q(0,T))$ and

$$\|\nabla q\|_{L^s(Q(0,T))} \le c\,\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\,, \qquad (2.31)$$

thus $p \in W^{0,1}_s(Q(a,b))$.

We are left to show that the distributional time derivative of p is in the space $(W^{0,1}_{s'}(Q(a,b)))'$. For that, we take $\varphi \in C^{1,2}_c(Q(0,T))$ and we apply again Lemma 1.3.3 to the function $\vartheta\varphi$. Then, integrating by parts, we get

$$\int_{Q(0,T)} q\partial_t\varphi \, dt\, dy = \int_{Q(0,T)} \langle Q\nabla\varphi, \nabla q\rangle \, dt\, dy + \int_{Q(0,T)} (qV\varphi - p\varphi\vartheta')\, dt\, dy.$$

If we take into account inequalities (2.21) and (2.22), then we have

$$\left|\int_{Q(0,T)} q\partial_t\varphi \, dt\, dy\right| \le \left|\int_{Q(0,T)} \langle Q\nabla\varphi, \nabla q\rangle \, dt\, dy\right| + c\, \|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\varphi\|_{L^{s'}(Q(0,T))}\,.$$

By the boundedness of the diffusion coefficients, Hölder's inequality and (2.31), we finally deduce that

$$\begin{aligned}\left|\int_{Q(0,T)} q\partial_t\varphi \, dt\, dy\right| \le & c\, \|\nabla q\|_{L^s(Q(0,T))} \|\varphi\|_{W^{0,1}_{s'}(Q(0,T))} \\ & + c\, \|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\varphi\|_{L^{s'}(Q(0,T))} \\ \le & c\, \|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))} \|\varphi\|_{W^{0,1}_{s'}(Q(0,T))}\,.\end{aligned}$$

We finally observe that the previous inequality can be extended to every $\varphi \in W^{0,1}_{s'}(Q(a,b))$. $\square$

Corollary 2.3.2. *Assume that q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i,j,h = 1,\ldots,d$. If $\Gamma_2(k,x,a_0,b_0) < \infty$ for some $k > 1$ and $p \in L^\infty(Q(a_0,b_0))$, then $p \in \mathcal{H}^{s,1}(Q(a,b))$ for all $s \in (1,k]$.*

Proof. Since $p \in L^1(Q(a_0,b_0)) \cap L^\infty(Q(a_0,b_0))$, by interpolation we have $p \in L^r(Q(a_0,b_0))$ for all $1 \le r \le \infty$. Then the statement follows from Lemma 2.3.1. $\square$

Remark 2.3.3. *Lemma 2.3.1 and thus Corollary 2.3.2 hold true for the more general operator $A_F := A + F\cdot\nabla$ with $F \in C^\zeta(\mathbb{R}^d,\mathbb{R}^d)$ for some $\zeta \in (0,1)$ if we further assume that $\Gamma_1(k,x,a_0,b_0) < \infty$, where*

$$\Gamma_1(k,x,a_0,b_0) = \left(\int_{Q(a_0,b_0)} |F(y)|^k p(t,x,y)\, dt\, dy\right)^{\frac{1}{k}}.$$

Indeed, inspecting the proof of Lemma 2.3.1, one realizes that it suffices to replace (2.20) by

$$\int_{Q(a_0,b_0)} q(\partial_t\varphi + A_1\varphi)\, dt\, dy = -\int_{Q(a_0,b_0)} [q(G+F)\cdot\nabla\varphi - qV\varphi + p\varphi\vartheta']\, dt\, dy.$$

Apart from that, the proof works the same.

2.3.2 Boundedness of weak solutions to parabolic problems

Here, we consider functions u which are, in some sense, weak solutions to an inhomogeneus parabolic equation $\partial_t u - Au = f$ and prove an estimate of their supremum norm. For that, we adapt the results in [31, Section 3.2].

Before stating the main theorem of this subsection, we show the following lemma.

Lemma 2.3.4. *Let $\ell > 0$ and $\vartheta\colon \mathbb{R}^d \to \mathbb{R}$ be a nonnegative, smooth and compactly supported function. Moreover, assume that one of the following situations applies:*

(a) $u \in \mathcal{H}^{p,1}(Q(a,b))$ for some $p > d+2$;

(b) $u \in \mathcal{H}^{p,1}(Q(a,b)) \cap C_b(\overline{Q(a,b)})$ for some $p \le d+2$.

Then, $\vartheta(u-\ell)_+ \in W^{0,1}_{p'}(Q(a,b))$ and

$$\int_{Q(a,b)} \vartheta(u-\ell)_+ \partial_t u \, dt\, dx = \frac{1}{2}\left[\int_{\mathbb{R}^d} \vartheta(u(b,\cdot)-\ell)_+^2\, dx - \int_{\mathbb{R}^d} \vartheta(u(a,\cdot)-\ell)_+^2\, dx\right].$$

Proof. We start by observing that $\vartheta(u-\ell)_+ \in W^{0,1}_{p'}(Q(a,b))$ because $\vartheta(u-\ell)_+ = (\vartheta(u-\ell))_+$, $\nabla(\vartheta(u-\ell))_+ = \chi_{\{u\ge\ell\}}\nabla(\vartheta(u-\ell))$ and $\vartheta(u-\ell) \in W^{0,1}_{p'}(Q(a,b))$.

If we are under condition (a), we apply Lemma A.4.4 to have a sequence $(u_n) \subset C_c^\infty(\mathbb{R}^{d+1})$ converging to u in the $\mathcal{H}^{p,1}$-norm. Moreover, since $\mathcal{H}^{p,1}(Q(a,b))$ is continuously embedded in $C_0(\overline{Q(a,b)})$ by Theorem A.4.5, then u_n converges to u uniformly in $\overline{Q(a,b)}$. Otherwise, under condition (b), the sequence u_n is provided by Lemma A.4.6. In both cases, we deduce that $\vartheta(u_n-\ell)_+$ tends to $\vartheta(u-\ell)_+$ in $W^{0,1}_{p'}(Q(a,b))$. As a result, since $\partial_t u \in (W^{0,1}_{p'}(Q(a,b)))'$, we have

$$\int_{Q(a,b)} \vartheta(u_n-\ell)_+\partial_t u\, dt\, dx \to \int_{Q(a,b)} \vartheta(u-\ell)_+\partial_t u\, dt\, dx,$$

as $n \to \infty$. We now write

$$\begin{aligned}
&\int_{Q(a,b)} \vartheta(u_n-\ell)_+\partial_t u\, dt\, dx \qquad (2.32)\\
&\quad= \int_{Q(a,b)} \vartheta(u_n-\ell)_+\partial_t u_n\, dt\, dx + \int_{Q(a,b)} \vartheta(u_n-\ell)_+(\partial_t u - \partial_t u_n)\, dt\, dx\\
&\quad=: I_1 + I_2.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{Q(a,b)} \vartheta\partial_t((u_n-\ell)_+^2)\, dt\, dx\\
&= \frac{1}{2}\int_{\mathbb{R}^d} \vartheta[(u_n(b,\cdot)-\ell)_+^2 - (u_n(a,\cdot)-\ell)_+^2]\, dx.
\end{aligned}$$

Since u_n converges to u uniformly in $\overline{Q(a,b)}$ and ϑ is compactly supported in $\mathbb{R}^d$, then

$$I_1 \to \frac{1}{2}\int_{\mathbb{R}^d} \vartheta[(u(b,\cdot)-\ell)_+^2 - (u(a,\cdot)-\ell)_+^2]\,dx, \tag{2.33}$$

as $n \to \infty$. Moreover, by the boundedness of the sequence $\vartheta(u_n - \ell)_+$ in $W^{0,1}_{p'}(Q(a,b))$ and the fact that $\partial_t u_n$ tends to $\partial_t u$ in $(W^{0,1}_{p'}(Q(a,b)))'$, it follows that $I_2 \to 0$ as $n \to \infty$. Combining this with (2.32) and (2.33) leads to the thesis. □

Moreover, we need an easy lemma stated below.

Lemma 2.3.5. *[26, Lemma 7.1] Let $\alpha > 0$ and let (x_n) be a sequence of real positive numbers such that*

$$x_{n+1} \le CB^n x_n^{1+\alpha} \tag{2.34}$$

with $C > 0$ and $B > 1$. If $x_0 \le C^{-\frac{1}{\alpha}}B^{-\frac{1}{\alpha^2}}$, then we have

$$x_n \le B^{-\frac{n}{\alpha}} x_0 \tag{2.35}$$

and hence in particular

$$\lim_{n\to\infty} x_n = 0.$$

Proof. We proceed by induction. Clearly, inequality (2.35) holds true for $n = 0$. If it is satisfied for n, then by (2.34) we have

$$x_{n+1} \le CB^n x_n^{1+\alpha} \le CB^n(B^{-\frac{n}{\alpha}}x_0)^{1+\alpha} = (CB^{\frac{1}{\alpha}}x_0^\alpha)B^{-\frac{n+1}{\alpha}}x_0 \le B^{-\frac{n+1}{\alpha}}x_0,$$

that is (2.35) for $n+1$. □

We are now ready to prove the main theorem of this subsection. It is a key result that will allow us to generalize the kernel estimates from bounded to unbounded diffusion coefficients.

Theorem 2.3.6. *Assume that q_{ij} is bounded on $\mathbb{R}^d$ for $i,j = 1,\dots,d$. Let $0 \le a_0 < b_0 \le T$, $k > d+2$ and let functions $f \in L^{\frac{k}{2}}(Q(a_0,b_0))$, $h = (h_i) \in L^k(Q(a_0,b_0);\mathbb{R}^d)$ and $u \in L^\infty(a_0,b_0;L^2(\mathbb{R}^d))$ be given such that $u(a_0) = 0$ and one of the following situations applies:*

(a) $u \in \mathcal{H}^{p,1}(Q(a_0,b_0))$ *for some* $p > d+2$;

(b) $u \in \mathcal{H}^{p,1}(Q(a_0,b_0)) \cap C_b(\overline{Q(a_0,b_0)})$ *for some* $p \le d+2$.

Moreover, assume that

$$\int_{Q(a_0,b_0)} [\langle Q\nabla u, \nabla\psi\rangle + \psi\partial_t u]\,dt\,dx = \int_{Q(a_0,b_0)} f\psi\,dt\,dx + \int_{Q(a_0,b_0)} \langle h, \nabla\psi\rangle\,dt\,dx, \tag{2.36}$$

for all $\psi \in C_c^\infty(\overline{Q(a_0,b_0)})$. Then u is bounded and there exists a constant $C > 0$, depending only on η, d and k (but not depending on $\|Q\|_\infty$) such that

$$\|u\|_\infty \le C(\|u\|_2 + \|f\|_{\frac{k}{2}} + \|h\|_k).$$

Proof. We initially assume that $\|u\|_2, \|f\|_{\frac{k}{2}}, \|h\|_k \leq 1$.

First, we observe that $u \in L^\infty(Q(a_0,b_0))$ not only under condition (b), but also under condition (a) since $\mathcal{H}^{p,1}(Q(a_0,b_0))$ is continuously embedded in $C_0(Q(a_0,b_0))$ for $p > d+2$ by Theorem A.4.5.

Second, we fix $\ell > 1$. Taking into account that $u \in L^\infty(Q(a_0,b_0))$ and $\nabla(u-\ell)_+ = \chi_{\{u\geq \ell\}}\nabla u$, we deduce that $(u-\ell)_+ \in W^{0,1}_r(Q(a_0,b_0))$ for any $r \in (1,\infty)$.

Next, we consider a standard sequence ϑ_n of cutoff functions (in x). Making use of a density argument one can see that (2.36) holds true even for functions $\psi \in W^{0,1}_r(Q(a_0,b_0))$ for any $r \in (1,\infty)$ such that there exists $R > 0$ with $\psi(t,x) = 0$ for all $t \in (a_0,b_0)$ and $|x| > R$. Hence, we may plug $\psi := \vartheta_n^2(u-\ell)_+$ in (2.36) obtaining that

$$
\begin{aligned}
&\int_{Q(a_0,b_0)} \vartheta_n^2 (u-\ell)_+ \partial_t u \, dt\, dx + \int_{Q(a_0,b_0)} \vartheta_n^2 \langle Q\nabla u, \nabla(u-\ell)_+\rangle \, dt\, dx \\
&+ 2\int_{Q(a_0,b_0)} \vartheta_n \langle Q\nabla u, \nabla\vartheta_n\rangle (u-\ell)_+ \, dt\, dx = \int_{Q(a_0,b_0)} f\vartheta_n^2 (u-\ell)_+ \, dt\, dx \\
&+ \int_{Q(a_0,b_0)} \vartheta_n^2 \langle h, \nabla(u-\ell)_+\rangle \, dt\, dx + 2\int_{Q(a_0,b_0)} \vartheta_n (u-\ell)_+ \langle h, \nabla\vartheta_n\rangle \, dt\, dx.
\end{aligned}
$$

Applying Lemma 2.3.4 and considering that $u(a_0) = 0$, it leads to

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d} \vartheta_n^2 (u(b_0,\cdot)-\ell)_+^2 \, dx + \int_{Q(a_0,b_0)} \vartheta_n^2 \langle Q\nabla(u-\ell)_+, \nabla(u-\ell)_+\rangle \, dt\, dx \\
&+ 2\int_{Q(a_0,b_0)} \vartheta_n \langle Q\nabla(u-\ell)_+, \nabla\vartheta_n\rangle (u-\ell)_+ \, dt\, dx = \int_{Q(a_0,b_0)} f\vartheta_n^2 (u-\ell)_+ \, dt\, dx \\
&+ \int_{Q(a_0,b_0)} \vartheta_n^2 \langle h, \nabla(u-\ell)_+\rangle \, dt\, dx + 2\int_{Q(a_0,b_0)} \vartheta_n (u-\ell)_+ \langle h, \nabla\vartheta_n\rangle \, dt\, dx.
\end{aligned}
$$

Therefore, we write

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d} \vartheta_n^2 (u(b_0,\cdot)-\ell)_+^2 \, dx + \int_{Q(a_0,b_0)} \vartheta_n^2 \langle Q\nabla(u-\ell)_+, \nabla(u-\ell)_+\rangle \, dt\, dx + I_1 \\
&= I_2 + I_3 + I_4,
\end{aligned} \tag{2.37}
$$

where

$$
\begin{aligned}
I_1 &= 2\int_{Q(a_0,b_0)} \vartheta_n \langle Q\nabla(u-\ell)_+, \nabla\vartheta_n\rangle (u-\ell)_+ \, dt\, dx, \\
I_2 &= \int_{Q(a_0,b_0)} f\vartheta_n^2 (u-\ell)_+ \, dt\, dx, \\
I_3 &= \int_{Q(a_0,b_0)} \vartheta_n^2 \langle h, \nabla(u-\ell)_+\rangle \, dt\, dx, \\
I_4 &= 2\int_{Q(a_0,b_0)} \vartheta_n (u-\ell)_+ \langle h, \nabla\vartheta_n\rangle \, dt\, dx.
\end{aligned}
$$

Since $|\langle Q\nabla(u-\ell)_+, \nabla\vartheta_n\rangle| \leq |Q^{\frac{1}{2}}\nabla(u-\ell)_+||Q^{\frac{1}{2}}\nabla\vartheta_n|$ by the Cauchy-Schwarz inequality, we may use Hölder's and Young's inequality to get

$$\begin{aligned}|I_1| \leq \frac{1}{2}\int_{Q(a_0,b_0)} &\vartheta_n^2 \langle Q\nabla(u-\ell)_+, \nabla(u-\ell)_+\rangle \, dt\, dx \\ &+ 2\int_{Q(a_0,b_0)} \langle Q\nabla\vartheta_n, \nabla\vartheta_n\rangle (u-\ell)_+^2 \, dt\, dx.\end{aligned}$$

Furthermore, since the diffusion coefficients are bounded on $\mathbb{R}^d$ and we can take ϑ_n such that $|\nabla\vartheta_n| \leq c/n$ for some positive constant c independent of n, we have

$$\begin{aligned}|I_1| \leq \frac{1}{2}\int_{Q(a_0,b_0)} &\vartheta_n^2 \langle Q\nabla(u-\ell)_+, \nabla(u-\ell)_+\rangle \, dt\, dx \\ &+ \frac{2c^2\,\|Q\|_\infty}{n^2}\int_{Q(a_0,b_0)} (u-\ell)_+^2 \, dt\, dx.\end{aligned} \tag{2.38}$$

We define $A_\ell(t) := \{u(t,\cdot) \geq \ell\}$, $A_\ell := \{u \geq \ell\}$ and we consider $|A_\ell(t)|$, the d-dimensional Lebesgue measure of $A_\ell(t)$, and $|A_\ell|$, the $d+1$-dimensional Lebesgue measure of A_ℓ. After that, we employ Hölder's inequality with exponents $\frac{k}{2}$, $2+\frac{4}{d}$ and s with $\frac{1}{s} = \frac{1}{2} - \frac{2}{k} + \frac{1}{d+2}$ to estimate $|I_2|$ as follows

$$|I_2| \leq \int_{A_\ell} |\vartheta_n f|(u-\ell)_+\vartheta_n \, dt\, dx \leq \|\vartheta_n f\|_{\frac{k}{2}} \, \|(u-\ell)_+\|_{2+\frac{4}{d}} \, |A_\ell|^{\frac{1}{2}-\frac{2}{k}+\frac{1}{d+2}}.$$

We now invoke Lemma A.4.8 with $p = q = 2+\frac{4}{d}$ in order to derive that

$$\|(u-\ell)_+\|_{2+\frac{4}{d}} \leq c_S(\|(u-\ell)_+\|_{\infty,2} + \|\nabla(u-\ell)_+\|_2). \tag{2.39}$$

Thus,

$$|I_2| \leq c_S(\|(u-\ell)_+\|_{\infty,2} + \|\nabla(u-\ell)_+\|_2)|A_\ell|^{\frac{1}{2}-\frac{2}{k}+\frac{1}{d+2}}, \tag{2.40}$$

where we used that $\|f\|_{\frac{k}{2}} \leq 1$. Similarly, applying Hölder's inequality with exponents $k, 2$ and s with $\frac{1}{s} = \frac{1}{2} - \frac{1}{k}$, since $\|h\|_k \leq 1$ and $|\nabla\vartheta_n| \leq c/n$, we find that

$$|I_3| \leq \|\nabla(u-\ell)_+\|_2 \, |A_\ell|^{\frac{1}{2}-\frac{1}{k}} \tag{2.41}$$

and

$$|I_4| \leq \frac{2c}{n} \, \|(u-\ell)_+\|_2 \, |A_\ell|^{\frac{1}{2}-\frac{1}{k}}. \tag{2.42}$$

We note that, according to the monotone convergence theorem, (2.40) and (2.41) imply that the integrals $\int_{Q(a_0,b_0)} f(u-\ell)_+ \, dt\, dx$ and $\int_{Q(a_0,b_0)} \langle h, \nabla(u-\ell)_+\rangle \, dt\, dx$ exist. Moreover, by (2.42), we have $I_4 \to 0$ as $n \to \infty$. Consequently, combining (2.37) with (2.38) and letting $n \to \infty$, it follows that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d} (u(b_0,\cdot)-\ell)_+^2 \, dx &+ \frac{1}{2}\int_{Q(a_0,b_0)} \langle Q\nabla(u-\ell)_+, \nabla(u-\ell)_+\rangle \, dt\, dx \\ &\leq \int_{Q(a_0,b_0)} |f|(u-\ell)_+ \, dt\, dx + \int_{Q(a_0,b_0)} |h||\nabla(u-\ell)_+| \, dt\, dx.\end{aligned} \tag{2.43}$$

Since $\langle Q\nabla(u-\ell)_+, \nabla(u-\ell)_+\rangle \geq \eta|\nabla(u-\ell)_+|^2$ by the uniform ellipticity of the matrix Q, one obtains from (2.40), (2.41) and (2.43) that

$$\frac{1}{2}\int_{\mathbb{R}^d}(u(b_0,\cdot)-\ell)_+^2\,dx+\frac{\eta}{2}\left\|\nabla(u-\ell)_+\right\|_2^2$$
$$\leq c_S(\|(u-\ell)_+\|_{\infty,2}+\|\nabla(u-\ell)_+\|_2)|A_\ell|^{\frac{1}{2}-\frac{2}{k}+\frac{1}{d+2}}+\left\|\nabla(u-\ell)_+\right\|_2|A_\ell|^{\frac{1}{2}-\frac{1}{k}}.$$

If we repeat the above proof for any $b_0'\in(a_0,b_0)$ and we take the supremum over such b_0' in the previous inequality, then we get

$$\begin{aligned}\min(1,\eta)(\|(u-\ell)_+\|_{\infty,2}^2+\|\nabla(u-\ell)_+\|_2^2)&\\ \leq 2c_S(\|(u-\ell)_+\|_{\infty,2}+\|\nabla(u-\ell)_+\|_2)|A_\ell|^{\frac{1}{2}-\frac{2}{k}+\frac{1}{d+2}}&\\ +2\left\|\nabla(u-\ell)_+\right\|_2|A_\ell|^{\frac{1}{2}-\frac{1}{k}}.&\end{aligned}\tag{2.44}$$

We observe that, since $\ell>1$ and $\|u\|_2\leq 1$, then

$$\begin{aligned}|A_\ell|&=\int_{\{u\geq\ell\}}dt\,dx<\int_{\{u\geq\ell\}}\ell^2dt\,dx\\ &\leq\int_{\{u\geq\ell\}}|u(t,x)|^2dt\,dx\leq\|u\|_2^2\leq 1.\end{aligned}\tag{2.45}$$

Therefore, $|A_\ell|\leq 1$. Considering also that $k>d+2$ implies $\frac{1}{2}-\frac{1}{k}<\frac{1}{2}-\frac{2}{k}+\frac{1}{d+2}$, by (2.44) we derive that

$$\|(u-\ell)_+\|_{\infty,2}+\|\nabla(u-\ell)_+\|_2\leq L|A_\ell|^{\frac{1}{2}-\frac{1}{k}},\tag{2.46}$$

for some constant L. As a result, taking $m>\ell$ yields

$$\begin{aligned}(m-\ell)^2|A_m|&=\int_{A_m}(m-\ell)^2\,dt\,dx\leq\int_{A_m}(u-\ell)^2\,dt\,dx\leq\int_{A_\ell}(u-\ell)^2\,dt\,dx\\ &\leq\left\|(u-\ell)_+^2\right\|_{1+\frac{2}{d}}|A_\ell|^{\frac{2}{d+2}}=\|(u-\ell)_+\|_{2+\frac{4}{d}}^2|A_\ell|^{\frac{2}{d+2}},\end{aligned}$$

where we used Hölder's inequality with exponents $1+\frac{2}{d}$ and $\frac{d+2}{2}$. Taking into account also (2.39) and (2.46), we find

$$\begin{aligned}(m-\ell)^2|A_m|&\leq c_S^2(\|(u-\ell)_+\|_{\infty,2}+\|\nabla(u-\ell)_+\|_2)^2|A_\ell|^{\frac{2}{d+2}}\\ &\leq L^2c_S^2|A_\ell|^{1-\frac{2}{k}+\frac{2}{d+2}}=:\nu_d|A_\ell|^{1-\frac{2}{k}+\frac{2}{d+2}}.\end{aligned}\tag{2.47}$$

Now let $\overline{\ell}\geq 1$ and consider $\ell_n=2\overline{\ell}-2^{-n}\overline{\ell}$, $y_n=|A_{\ell_n}|$ for $n\in\mathbb{N}$ and $\alpha=\frac{2}{d+2}-\frac{2}{k}>0$. If we write (2.47) with $m=\ell_{n+1}$ and $\ell=\ell_n$, we get

$$y_{n+1}\leq\frac{4\nu_d}{\overline{\ell}^2}4^ny_n^{1+\alpha}.$$

Easy computations show that, if we choose $\overline{\ell}=\max(1,2^{1+\frac{1}{\alpha}}\sqrt{\nu_d})$, then, since $|A_{\overline{\ell}}|\leq 1$ as in (2.45), we have

$$y_0=|A_{\overline{\ell}}|\leq 1\leq\left(\frac{4\nu_d}{\overline{\ell}^2}\right)^{-\frac{1}{\alpha}}4^{-\frac{1}{\alpha^2}}.$$

Consequently, we can apply Lemma 2.3.5 with $B = 4$ and $C = 4\nu_d/\overline{\ell}^2$ to infer that $|A_{\ell_n}| = y_n \to 0$ as $n \to \infty$. This implies that $|A_\ell| = 0$ for $\ell \geq 2\overline{\ell}$, i.e. $u \leq 2\overline{\ell} =: C$. Replacing u with $-u$, by linearity we obtain that also $-u \leq C$, hence $\|u\|_\infty \leq C$.

To end the proof, we remove the additional assumptions $\|u\|_2, \|f\|_{\frac{k}{2}}, \|h\|_k \leq 1$. We consider $M = \|u\|_2 + \|f\|_{\frac{k}{2}} + \|h\|_k$ and we repeat the above argument with $\tilde{u} = u/M$, $\tilde{f} = f/M$ and $\tilde{h} = h/M$, since they verify formula (2.36). Then, from $\|\tilde{u}\|_\infty \leq C$ we gain $\|u\|_\infty \leq CM$. □

2.4 Kernel estimates in case of bounded diffusion coefficients

In this section we establish pointwise upper bounds for the kernel p assuming that q_{ij} and $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for all $i, j, h = 1, \ldots, d$. In the next section they will be applied to a family of operators with bounded diffusion coefficients that approximates A. To this purpose we make the following assumptions.

Hypothesis 2.4.1. *Fix $T > 0$, $x \in \mathbb{R}^d$ and $0 < a_0 < a < b < b_0 < T$. Let us consider two time dependent Lyapunov functions W_1, W_2 for the operator $L := \partial_t + A$ with $W_1 \leq W_2$ and a weight function $1 \leq w \in C^{1,2}((0,T) \times \mathbb{R}^d)$ such that*

(a) the functions $w^{-2}\partial_t w$ and $w^{-2}\nabla w$ are bounded on $Q(a_0, b_0)$;

(b) there exist $k > d + 2$ and constants $c_1, \ldots, c_5$, possibly depending on the interval (a_0, b_0), with

(i) $w \leq c_1 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$, (ii) $|Q\nabla w| \leq c_2 w^{\frac{k-1}{k}} W_1^{\frac{1}{k}}$,

(iii) $|\mathrm{div}(Q\nabla w)| \leq c_3 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$, (iv) $|\partial_t w| \leq c_4 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$,

(v) $V^{\frac{1}{2}} \leq c_5 w^{-\frac{1}{k}} W_2^{\frac{1}{k}}$,

on $[a_0, b_0] \times \mathbb{R}^d$.

The following result can be deduced as in [30, Theorem 12.4] and [31, Theorem 4.2].

Theorem 2.4.2. *Assume Hypothesis 2.4.1, $k > d + 2$ and that q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i, j, h = 1, \ldots, d$. Then there is a constant $C > 0$ depending only on d, k and η such that*

$$w(t,y)p(t,x,y) \leq C\left[\left(c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b_0 - b)^{\frac{k}{2}}} + c_2^k + c_3^{\frac{k}{2}} + c_4^{\frac{k}{2}}\right)\int_{a_0}^{b_0} \xi_{W_1}(t,x)\,dt + c_5^k \int_{a_0}^{b_0} \xi_{W_2}(t,x)\,dt\right], \tag{2.48}$$

for all $(t, y) \in (a, b) \times \mathbb{R}^d$ *and any fixed* $x \in \mathbb{R}^d$.

Proof. Throughout the proof we consider p as a function of $(t, y) \in (0, T) \times \mathbb{R}^d$. We first prove the theorem under the assumption that w along with its first order partial derivatives are bounded. We split the proof in several steps.

Step 1. Let $a_0 < a_1 < a < b < b_1 < b_0$. We show that $p \in L^\infty(Q(a_1, b_1)) \cap \mathcal{H}^{s,1}(Q(a_1, b_1))$ for all $s \in (1, k/2)$.

The boundedness of p in $Q(a_1, b_1)$ follows from the fact that p is dominated by the kernel associated to the operator $A_0 := \operatorname{div}(Q\nabla)$ which satisfies Gaussian estimates, since the diffusion coefficients are assumed to be bounded.

Hence, the idea is to apply Corollary 2.3.2 to infer that $p \in \mathcal{H}^{s,1}(Q(a_1, b_1))$ for all $s \in (1, k/2)$. For that, it suffices to prove that the quantity $\Gamma_2(k/2, x, a_0, b_0)$ defined as in (2.19) is finite. On one hand, by Hypothesis 2.4.1, we have

$$
\begin{aligned}
\Gamma_2(k/2, x, a_0, b_0)^{\frac{k}{2}} &= \int_{Q(a_0,b_0)} V^{\frac{k}{2}}(y) p(t, x, y)\, dt\, dy \\
&\le c_5^k \int_{Q(a_0,b_0)} \frac{W_2(t, y)}{w(t, y)} p(t, x, y)\, dt\, dy \\
&\le c_5^k \int_{Q(a_0,b_0)} W_2(t, y) p(t, x, y)\, dt\, dy = c_5^k \int_{a_0}^{b_0} \xi_{W_2}(t, x)\, dt.
\end{aligned}
$$

On the other hand, Proposition 1.6.3 implies that the right hand side is finite. Then, we conclude that $\Gamma_2(k/2, x, a_0, b_0) < \infty$.

Step 2. Let $\vartheta \in C^\infty(\mathbb{R})$ be such that

- $\vartheta(t) = 1$ for $a \le t \le b$,
- $\vartheta(t) = 0$ for $t \le a_1$, $t \ge b_1$,
- $0 \le \vartheta \le 1$, $|\vartheta'| \le \frac{2}{b_1 - b}$.

We put $q := \vartheta^{\frac{k}{2}} p$ and we note that $wq \in L^\infty(Q(a_1, b_1)) \cap \mathcal{H}^{s,1}(Q(a_1, b_1))$ for all $s \in (1, k/2)$ because of Step 1 and since we are assuming that w and its derivatives are bounded. Moreover, given $\psi \in C_c^{1,2}(Q(a_1, b_1))$, we write

$$
\varphi(t, y) := \vartheta^{\frac{k}{2}}(t) w(t, y) \psi(t, y).
$$

Since $\varphi \in C_c^{1,2}(\overline{Q(a_1, b_1)})$, applying Lemma 1.3.3 we deduce that

$$
\int_{Q(a_1,b_1)} (\partial_t \varphi(t, y) + A\varphi(t, y)) p(t, x, y)\, dt\, dy = 0.
$$

After some computations, we derive from the previous identity that

$$
\begin{aligned}
\int_{Q(a_1,b_1)} wq(-\partial_t \psi - \operatorname{div}(Q\nabla\psi))\, dt\, dy &= \int_{Q(a_1,b_1)} 2q\langle Q\nabla w, \nabla\psi\rangle\, dt\, dy \\
&+ \int_{Q(a_1,b_1)} \left[q\partial_t w + q \operatorname{div}(Q\nabla w) - qVw + \frac{k}{2} pw\vartheta^{\frac{k-2}{2}} \vartheta' \right] \psi\, dt\, dy.
\end{aligned}
$$

Integrating by parts the left hand side, we have

$$\int_{Q(a_1,b_1)} \left[\langle Q\nabla(wq),\nabla\psi\rangle + \psi\partial_t(wq)\right] dt\,dy = \int_{Q(a_1,b_1)} 2q\langle Q\nabla w,\nabla\psi\rangle\,dt\,dy$$
$$+ \int_{Q(a_1,b_1)} \left[q\partial_t w + q\,\mathrm{div}(Q\nabla w) - qVw + \frac{k}{2}pw\vartheta^{\frac{k-2}{2}}\vartheta'\right]\psi\,dt\,dy.$$

We now apply Theorem 2.3.6 with

$$u := wq,$$
$$f := q\partial_t w + q\,\mathrm{div}(Q\nabla w) - qVw + \frac{k}{2}pw\vartheta^{\frac{k-2}{2}}\vartheta', \tag{2.49}$$
$$h := 2qQ\nabla w. \tag{2.50}$$

Then there is a constant $C > 0$ depending only on η, d and k (but not depending on $\|Q\|_\infty$) such that

$$\|u\|_\infty \le C(\|u\|_2 + \|f\|_{\frac{k}{2}} + \|h\|_k),$$

where for $p \in [1,\infty)$ we denote by $\|f\|_p$ the usual L^p-norm of the function $f\colon Q(a_1,b_1) \to \mathbb{R}$. In the following we consider C as a positive constant that can vary from line to line, but it will always depend only on η, d and k. Replacing the expressions of u, f and h in the previous inequality we obtain

$$\|wq\|_\infty \le C\Bigg(\|wq\|_2 + \|q\partial_t w\|_{\frac{k}{2}} + \|q\,\mathrm{div}(Q\nabla w)\|_{\frac{k}{2}} + \|qVw\|_{\frac{k}{2}}$$
$$+ \frac{k}{b_1 - b}\left\|pw\vartheta^{\frac{k-2}{2}}\right\|_{\frac{k}{2}} + \|qQ\nabla w\|_k\Bigg). \tag{2.51}$$

Step 3. We make use of Hypothesis 2.4.1 to estimate the terms in the right hand side of (2.51) in order to find an estimate for $\|wq\|_\infty$. We set

$$M_i := \int_{a_1}^{b_1} \xi_{W_i}(t,x)\,dt, \quad i = 1,2.$$

First of all, we apply Hypothesis 2.4.1 to estimate $\|wq\|_2$:

$$\|wq\|_2^2 = \int_{Q(a_1,b_1)} (wq)^2\,dt\,dy \le \|wq\|_\infty \int_{Q(a_1,b_1)} wq\,dt\,dy$$
$$\le c_1^{\frac{k}{2}}\|wq\|_\infty \int_{Q(a_1,b_1)} W_1 q\,dt\,dy \le c_1^{\frac{k}{2}}\|wq\|_\infty M_1.$$

Similarly, we estimate $\|q\partial_t w\|_{\frac{k}{2}}$ and $\|q\,\mathrm{div}(Q\nabla w)\|_{\frac{k}{2}}$ as follows

$$\|q\partial_t w\|_{\frac{k}{2}}^{\frac{k}{2}} = \int_{Q(a_1,b_1)} |\partial_t w|^{\frac{k}{2}} q^{\frac{k}{2}}\,dt\,dy \le c_4^{\frac{k}{2}} \int_{Q(a_1,b_1)} w^{\frac{k-2}{2}} W_1 q^{\frac{k}{2}}\,dt\,dy$$
$$\le c_4^{\frac{k}{2}}\|wq\|_\infty^{\frac{k-2}{2}} \int_{Q(a_1,b_1)} W_1 q\,dt\,dy \le c_4^{\frac{k}{2}}\|wq\|_\infty^{\frac{k-2}{2}} M_1,$$

$$\|q\,\mathrm{div}(Q\nabla w)\|_{\frac{k}{2}}^{\frac{k}{2}} = \int_{Q(a_1,b_1)} |\mathrm{div}(Q\nabla w)|^{\frac{k}{2}} q^{\frac{k}{2}}\,dt\,dy \le c_3^{\frac{k}{2}} \int_{Q(a_1,b_1)} w^{\frac{k-2}{2}} W_1 q^{\frac{k}{2}}\,dt\,dy$$
$$\le c_3^{\frac{k}{2}}\|wq\|_\infty^{\frac{k-2}{2}} M_1.$$

The same can be done for the rest of the terms in the right hand side of (2.51) applying Hypothesis 2.4.1. To sum up, we find

$$\begin{aligned}
&\|wq\|_2 \le c_1^{\frac{k}{4}} \|wq\|_\infty^{\frac{1}{2}} M_1^{\frac{1}{2}}, && \|q\partial_t w\|_{\frac{k}{2}} \le c_4 \|wq\|_\infty^{\frac{k-2}{k}} M_1^{\frac{2}{k}},\\
&\|q \operatorname{div}(Q\nabla w)\|_{\frac{k}{2}} \le c_3 \|wq\|_\infty^{\frac{k-2}{k}} M_1^{\frac{2}{k}}, && \|qVw\|_{\frac{k}{2}} \le c_5^2 \|wq\|_\infty^{\frac{k-2}{k}} M_2^{\frac{2}{k}},\\
&\frac{k}{b_1-b}\left\|pw\vartheta^{\frac{k-2}{2}}\right\|_{\frac{k}{2}} \le \frac{k}{b_1-b} c_1 \|wq\|_\infty^{\frac{k-2}{k}} M_1^{\frac{2}{k}}, && \|qQ\nabla w\|_k \le c_2 \|wq\|_\infty^{\frac{k-1}{k}} M_1^{\frac{1}{k}}.
\end{aligned}$$

Putting all together in (2.51), we gain the following inequality:

$$\begin{aligned}
\|wq\|_\infty \le & C c_1^{\frac{k}{4}} M_1^{\frac{1}{2}} \|wq\|_\infty^{\frac{1}{2}} + C c_2 M_1^{\frac{1}{k}} \|wq\|_\infty^{\frac{k-1}{k}}\\
&+ C\left[\left(\frac{c_1}{b_1-b} + c_3 + c_4\right) M_1^{\frac{2}{k}} + c_5^2 M_2^{\frac{2}{k}}\right] \|wq\|_\infty^{\frac{k-2}{k}}.
\end{aligned}$$

If we set

$$\begin{aligned}
X &:= \|wq\|_\infty^{\frac{1}{k}}, & \alpha &:= C c_1^{\frac{k}{4}} M_1^{\frac{1}{2}},\\
\beta &:= C c_2 M_1^{\frac{1}{k}}, & \gamma &:= C\left[\left(\frac{c_1}{b_1-b} + c_3 + c_4\right) M_1^{\frac{2}{k}} + c_5^2 M_2^{\frac{2}{k}}\right],
\end{aligned} \tag{2.52}$$

then we obtain

$$X^k \le \alpha X^{\frac{k}{2}} + \beta X^{k-1} + \gamma X^{k-2}.$$

If we apply Young's inequality $\alpha X^{\frac{k}{2}} \le \frac{1}{4} X^k + \alpha^2$ we get

$$X^k \le \frac{4}{3}\alpha^2 + \frac{4}{3}\beta X^{k-1} + \frac{4}{3}\gamma X^{k-2}. \tag{2.53}$$

We now prove that it leads to

$$X \le \frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + \left(\frac{4}{3}\alpha^2\right)^{\frac{1}{k}}. \tag{2.54}$$

We consider the function

$$\begin{aligned}
f(r) :=& r^k - \frac{4}{3}\beta r^{k-1} - \frac{4}{3}\gamma r^{k-2} - \frac{4}{3}\alpha^2 = r^{k-2}\left(r^2 - \frac{4}{3}\beta r - \frac{4}{3}\gamma\right) - \frac{4}{3}\alpha^2\\
=:& \, r^{k-2} g(r) - \frac{4}{3}\alpha^2.
\end{aligned}$$

First, we show that f is increasing in $\left(\frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + (\frac{4}{3}\alpha^2)^{\frac{1}{k}}, \infty\right)$. This can be seen by computing the first derivative:

$$f'(r) = (k-2) r^{k-3} g(r) + r^{k-2} g'(r).$$

Since the function g in positive and increasing in $\left(\frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + (\frac{4}{3}\alpha^2)^{\frac{1}{k}}, \infty\right)$, it follows that $f'(r) \ge 0$ in the given interval, so f is increasing.

Second, we have that

$$
\begin{aligned}
& f\left(\frac{4}{3}\beta+\sqrt{\frac{4}{3}\gamma}+\left(\frac{4}{3}\alpha^2\right)^{\frac{1}{k}}\right) \\
&= \left(\frac{4}{3}\beta+\sqrt{\frac{4}{3}\gamma}+\left(\frac{4}{3}\alpha^2\right)^{\frac{1}{k}}\right)^{k-2}\left[\left(\frac{4}{3}\beta+\sqrt{\frac{4}{3}\gamma}+\left(\frac{4}{3}\alpha^2\right)^{\frac{1}{k}}\right)^2\right. \\
&\quad \left. -\frac{4}{3}\beta\left(\frac{4}{3}\beta+\sqrt{\frac{4}{3}\gamma}+\left(\frac{4}{3}\alpha^2\right)^{\frac{1}{k}}\right)-\frac{4}{3}\gamma\right]-\frac{4}{3}\alpha^2 \\
&= \left(\frac{4}{3}\beta+\sqrt{\frac{4}{3}\gamma}+\left(\frac{4}{3}\alpha^2\right)^{\frac{1}{k}}\right)^{k-2}\left[\left(\frac{4}{3}\alpha^2\right)^{\frac{2}{k}}+\frac{8\sqrt{3}}{9}\sqrt{\gamma}\beta\right. \\
&\quad \left.+\frac{4\sqrt{3}}{3}\left(\frac{4}{3}\right)^{\frac{1}{k}}\alpha^{\frac{2}{k}}\left(\frac{\sqrt{3}}{3}\beta+\sqrt{\gamma}\right)\right]-\frac{4}{3}\alpha^2 \\
&> \left(\frac{4}{3}\alpha^2\right)^{\frac{k-2}{k}}\left(\frac{4}{3}\alpha^2\right)^{\frac{2}{k}}-\frac{4}{3}\alpha^2=0. \qquad (2.55)
\end{aligned}
$$

On one hand, from the previous observations we deduce that $f(r)>0$ if $r>\frac{4}{3}\beta+\sqrt{\frac{4}{3}\gamma}+(\frac{4}{3}\alpha^2)^{\frac{1}{k}}$. On the other hand, by (2.53), $f(X)\leq 0$. Thus, we conclude that (2.54) holds true. Consequently, there exists a positive constant K_1 such that

$$\|wq\|_\infty \leq K_1\left(\alpha^2+\beta^k+\gamma^{\frac{k}{2}}\right).$$

We get the desired estimate (2.48) by plugging in the previous inequality the definition of α,β,γ and letting $a_1\downarrow a_0$ and $b_1\uparrow b_0$.

Step 4. We now prove the theorem also if w is not necessary bounded. In such case we set

$$w_\varepsilon := \frac{w}{1+\varepsilon w}.$$

Since

$$D_i w_\varepsilon = (1+\varepsilon w)^{-2} D_i w$$

for all $i,j=1,\ldots,d$, then by Hypothesis 2.4.1(a) it follows that w_ε is bounded together with its first order partial derivatives. Moreover, making use of Hypothesis 2.4.1(b), we have

$$w_\varepsilon \leq w \leq c_1^{\frac{k}{2}} W_1,$$

$$|Q\nabla w_\varepsilon| = (1+\varepsilon w)^{-2}|Q\nabla w| \leq c_2(1+\varepsilon w)^{-2} w^{\frac{k-1}{k}} W_1^{\frac{1}{k}} \leq c_2 w_\varepsilon^{\frac{k-1}{k}} W_1^{\frac{1}{k}},$$

$$
\begin{aligned}
|\mathrm{div}(Q\nabla w_\varepsilon)| &\leq (1+\varepsilon w)^{-2}|\mathrm{div}(Q\nabla w)| + 2\varepsilon(1+\varepsilon w)^{-3}|Q\nabla w||\nabla w| \\
&\leq (2\eta^{-1}c_2^2+c_3) w_\varepsilon^{\frac{k-2}{k}} W_1^{\frac{2}{k}},
\end{aligned}
$$

$$|\partial_t w_\varepsilon| = (1+\varepsilon w)^{-2}|\partial_t w| \leq c_4 w_\varepsilon^{\frac{k-2}{k}} W_1^{\frac{2}{k}}.$$

Thus, w_ε satisfies Hypothesis 2.4.1 with the same constants c_1, c_2, c_4, c_5 and the constant $2\eta^{-1}c_2^2 + c_3$ instead of c_3. We repeat Steps 1-3 for w_ε and then, letting $\varepsilon \to 0$, we gain estimate (2.48). □

Notice that the assumption of bounded diffusion coefficients was crucial to apply Theorem 2.3.6. The fact that the constant C does not depend on $\|Q\|_\infty$ will allow us to extend this result to the general case.

2.5 Kernel estimates for general diffusion coefficients

In this section we bring all together in order to finally prove the second main result of this chapter: the pointwise upper bound of the kernel p.

In view of applying the results from the previous section, the first step is to approximate the operator A as in Chapter 1 with the family of operators A_n with bounded diffusion coefficients defined by

$$A_n = \mathrm{div}(Q_n\nabla) - V,$$

where the matrix $Q_n := (q_{ij}^{(n)})$ is defined by (1.32). Moreover, we take the function φ_n in (1.32) as in (1.31), where the function W_1 is the time dependent Lyapunov function from Hypothesis 2.4.1 and the constant $t_0 \in (0, T)$ will be chosen later on.

It follows by Lemma 1.6.4 that A_n satisfies Hypothesis 2.0.1. Then, for every $n \in \mathbb{N}$, the semigroup generated by A_n in $C_b(\mathbb{R}^d)$ is given by a kernel $p_n(t, x, y)$.

In order to show further properties about the operators A_n, we make the following assumptions.

Hypothesis 2.5.1. *Fix $T > 0$, $x \in \mathbb{R}^d$ and $0 < a_0 < a < b < b_0 < T$. Let us consider two time dependent Lyapunov functions W_1, W_2 for the operators $\partial_t + A$ and $\partial_t + \eta\Delta - V$ with $W_1 \le W_2$ and $|\nabla W_1|, |\nabla W_2|$ bounded on $(0, T) \times B_R$ for all $R > 0$ and a weight function $1 \le w \in C^{1,2}((0, T) \times \mathbb{R}^d)$ such that*

(a) *on $[a_0, b_0] \times \mathbb{R}^d$ we have*

$$|\Delta w| \le c_6 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}};$$

(b) *there is $t_0 \in (0, T)$ such that*

$$|Q\nabla W_1(t_0, \cdot)| \le c_7 W_1(t_0, \cdot) w^{-\frac{1}{k}} W_2^{\frac{1}{k}};$$

(c) *there are $c_0 > 0$ and $\sigma \in (0, 1)$ such that*

$$W_2 \le c_0 Z^{1-\sigma};$$

(d) there is a nonnegative function f such that

$$\nabla Z(x) = f(x)\nabla W_1(t_0, x),$$

for all $x \in \mathbb{R}^d$.

We observe that W_1 and W_2 are time dependent Lyapunov functions for $\partial_t + A_n$ by Lemma 1.6.4. We now prove that if the operator A satisfies Hypothesis 2.4.1, then the same is true for the operators A_n assuming further Hypothesis 2.5.1.

Lemma 2.5.2. *Assume that the operator A satisfies Hypotheses 2.4.1(b) and 2.5.1(a)-(b). Then the operator A_n satisfies Hypothesis 2.4.1(b) with the same constants c_1, c_4, c_5, with c_2 being replaced by $2c_2$ and with*

$$|\mathrm{div}(Q_n\nabla w)| \le (c_3 + \eta c_6)w^{\frac{k-2}{k}}W_1^{\frac{2}{k}} + 4\eta^{-1}c_2c_7w^{\frac{k-2}{k}}W_2^{\frac{2}{k}} \tag{2.56}$$

instead of (iii).

Proof. The constants c_1, c_4 and c_5 are the same because the corresponding inequalities do not depend on the diffusion coefficients. Let us note that Hypothesis 2.4.1(b)-(ii) implies that

$$|\nabla w| = \left|Q^{-1}Q\nabla w\right| \le \eta^{-1}c_2w^{\frac{k-1}{k}}W_1^{\frac{1}{k}}.$$

It follows that A_n satisfies Hypothesis 2.4.1(b) with $2c_2$ instead of c_2:

$$|Q_n\nabla w| = |\varphi_n Q\nabla w + (1-\varphi_n)\eta\nabla w| \le |Q\nabla w| + \eta\,|\nabla w| \le 2c_2w^{\frac{k-1}{k}}W_1^{\frac{1}{k}}.$$

In order to show the last estimate we observe that, for $(t, y) \in [a_0, b_0] \times \mathbb{R}^d$, we have

$$\begin{aligned}
&\mathrm{div}(Q_n\nabla w(t,y)) \\
&=\varphi_n(y)\mathrm{div}(Q\nabla w(t,y)) + \frac{\varphi'(W_1(t_0,y)/n)}{n}[Q\nabla W_1(t_0,y)\cdot\nabla w(t,y) \\
&\quad - \eta\nabla W_1(t_0,y)\cdot\nabla w(t,y)] + \eta(1-\varphi_n(y))\Delta w(t,y).
\end{aligned}$$

Moreover, since we took the function φ such that $|t\varphi'(t)| \le 2$ as in Section 1.6, we obtain that

$$\begin{aligned}
&\left|\frac{\varphi'(W_1(t_0,y)/n)}{n}[Q\nabla W_1(t_0,y)\cdot\nabla w(t,y) - \eta\nabla W_1(t_0,y)\cdot\nabla w(t,y)]\right| \\
&\le \frac{2}{W_1(t_0,y)}(|Q\nabla W_1(t_0,y)|\,|\nabla w(t,y)| + \eta\,|\nabla W_1(t_0,y)|\,|\nabla w(t,y)|).
\end{aligned}$$

We observe that $W_1(t_0, y) \ne 0$ because, by Hypothesis 2.4.1(b)-(i), we have that $1 \le w(t_0, y) \le c_1^{k/2}W_1(t_0, y)$. Now, applying Hypotheses 2.4.1(b) and 2.5.1(a)-(b) for the operator A, we gain inequality (2.56). □

As shown in the previous section, we can now get estimates for the kernels p_n.

Lemma 2.5.3. *Assume that the operator A satisfies Hypotheses 2.4.1 and 2.5.1. For $i = 1, 2$, we set*

$$\xi_{W_i,n}(t,x) := \int_{\mathbb{R}^d} p_n(t,x,y) W_i(t,y)\, dy.$$

Then for any $n \in \mathbb{N}$ there is a constant $C > 0$ depending only on d, k and η such that

$$\begin{aligned} w(t,y)p_n(t,x,y) \le C \Bigg[\Bigg(c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b_0-b)^{\frac{k}{2}}} + c_2^k + c_3^{\frac{k}{2}} + c_4^{\frac{k}{2}} + c_6^{\frac{k}{2}} \Bigg) \int_{a_0}^{b_0} \xi_{W_1,n}(t,x)\, dt \\ + (c_5^k + c_2^{\frac{k}{2}} c_7^{\frac{k}{2}}) \int_{a_0}^{b_0} \xi_{W_2,n}(t,x)\, dt \Bigg], \end{aligned} \tag{2.57}$$

for all $(t,y) \in (a,b) \times \mathbb{R}^d$ and fixed $x \in \mathbb{R}^d$.

Proof. Since the operators A_n have bounded diffusion coefficients and satisfy Hypotheses 2.0.1 and 2.4.1 by Lemmas 1.6.4 and 2.5.2, we can apply Theorem 2.4.2 to A_n. We note that we replaced inequality (iii) in Hypothesis 2.4.1(b) with (2.56), so in the proof of Theorem 2.4.2 (Step 3) we take

$$\gamma := C \Bigg[\Bigg(\frac{c_1}{b_1 - b} + c_3 + \eta c_6 + c_4 \Bigg) M_1^{\frac{2}{k}} + (c_5^2 + 4\eta^{-1} c_2 c_7) M_2^{\frac{2}{k}} \Bigg].$$

Then, (2.57) holds true. $\square$

We now prove our main result by letting $n \to \infty$ in (2.57) in order to obtain an upper bound for the transition kernel p even if the diffusion coefficients are unbounded.

Theorem 2.5.4. *Assume that the operator A satisfies Hypotheses 2.4.1 and 2.5.1. Then there is a constant $C > 0$ depending only on d, k and η such that*

$$\begin{aligned} w(t,y)p(t,x,y) \le C \Bigg[\Bigg(c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b_0-b)^{\frac{k}{2}}} + c_2^k + c_3^{\frac{k}{2}} + c_4^{\frac{k}{2}} + c_6^{\frac{k}{2}} \Bigg) \int_{a_0}^{b_0} \xi_{W_1}(t,x)\, dt \\ + (c_5^k + c_2^{\frac{k}{2}} c_7^{\frac{k}{2}}) \int_{a_0}^{b_0} \xi_{W_2}(t,x)\, dt \Bigg], \end{aligned} \tag{2.58}$$

for all $(t,y) \in (a,b) \times \mathbb{R}^d$ and fixed $x \in \mathbb{R}^d$.

Proof. Let $(t,y) \in (a,b) \times \mathbb{R}^d$ and $x \in \mathbb{R}^d$. First, by Lemma 1.6.6, we have that

$$p_n(t,x,y) \to p(t,x,y),$$

as $n \to \infty$ for all $x \in \mathbb{R}^d$. Second, thanks to Hypothesis 2.5.1(c), we apply Lemma 1.6.7 to infer that

$$\xi_{W_i,n}(\cdot, x) \to \xi_{W_i}(\cdot, x),$$

locally uniformly in $(0, T)$ as $n \to \infty$ for $i = 1, 2$. Then, inequality (2.58) follows by letting $n \to \infty$ in (2.57) and considering that the constant C in the right hand side of the latter inequality does not depend on the diffusion coefficients $q_{ij}^{(n)}$. □

2.6 Some applications

In this section we aim to show how the results of the previous sections work in some concrete examples. In particular, we apply Theorem 2.5.4 to obtain explicit kernel estimates in case of operators with polynomial or exponential diffusion coefficients and potential terms.

2.6.1 Kernel estimates in case of polynomial coefficients

We consider the operator

$$A = \mathrm{div}((1 + |x|_*^m)\nabla) - |x|^s,$$

with $s > |m - 2|$ and $m > 0$. Moreover we set

$$w(t, x) = e^{\varepsilon t^\alpha |x|_*^\beta} \text{ and } W_j(t, x) = e^{\varepsilon_j t^\alpha |x|_*^\beta},$$

where $j = 1, 2$, $\beta = \frac{s-m+2}{2}$, $0 < \varepsilon < \varepsilon_1 < \varepsilon_2 < \frac{1}{\beta}$ and $\alpha > \frac{\beta}{\beta+m-2}$.

Theorem 2.6.1. *Let p be the integral kernel associated with the operator A with $Q(x) = (1 + |x|_*^m)I$ and $V(x) = |x|^s$, where $s > |m - 2|$ and $m > 0$. Then*

$$p(t, x, y) \le Ct^{1-\frac{\alpha(2m\vee s)}{s-m+2}k} e^{-\frac{\varepsilon}{2}t^\alpha |x|_*^{\frac{s-m+2}{2}}} e^{-\frac{\varepsilon}{2}t^\alpha |y|_*^{\frac{s-m+2}{2}}},$$

for $k > d + 2$ and any $t \in (0, 1)$, $x, y \in \mathbb{R}^d$, where C is a positive constant.

Proof. Step 1. We apply Proposition 2.2.1 to verify that the operator A satisfies Hypothesis 2.0.1 with

$$Z(x) = e^{\varepsilon_2 |x|_*^\beta}$$

and that W_1 and W_2 are time dependent Lyapunov functions for the operators $L := \partial_t + A$ and $\partial_t + \eta\Delta - V$ with respect to Z. Clearly, (2.7) holds true with $c_q = 1$. Since $s > |m - 2|$, we have $\beta > (2 - m) \vee 0$. It remains to check (2.8) and (2.9). Let $|x| \ge 1$ and set $G_j = \sum_{i=1}^d D_i q_{ij} = m|x|^{m-2} x_j$. Then

$$\begin{aligned} |x|^{1-\beta-m}\left(G \cdot \frac{x}{|x|} - \frac{V}{\varepsilon_j \beta |x|^{\beta-1}}\right) &= |x|^{1-\beta-m}\left(m|x|^{m-1} - \frac{|x|^s}{\varepsilon_j \beta |x|^{\beta-1}}\right) \\ &= m|x|^{-\beta} - \frac{1}{\varepsilon_j \beta}. \end{aligned}$$

If $|x|$ is large enough, for example $|x| \geq K$ with

$$K > \left(\frac{m}{\frac{1}{\varepsilon_j \beta} - 1} \right)^{\frac{1}{\beta}},$$

then we get

$$|x|^{1-\beta-m} \left(G \cdot \frac{x}{|x|} - \frac{V}{\varepsilon_j \beta |x|^{\beta-1}} \right) = m|x|^{-\beta} - \frac{1}{\varepsilon_j \beta} \leq mK^{-\beta} - \frac{1}{\varepsilon_j \beta} < -1,$$

where we have used that $\varepsilon_j < \frac{1}{\beta}$. Hence, (2.8) is satisfied if we choose $\Lambda := 1$. Moreover, we have

$$\lim_{|x|\to\infty} V(x)\, |x|^{2-2\beta-m} = \lim_{|x|\to\infty} |x|^{2-2\beta-m+s} = 1.$$

Consequently, (2.9) holds true for any $c < 1$.

Step 2. We now show that A satisfies Hypothesis 2.4.1. Fix $T = 1$, $x \in \mathbb{R}^d$, $0 < a_0 < a < b < b_0 < T$ and $k > d + 2$. Hypothesis 2.4.1(a) obviously holds true. Let $(t, y) \in [a_0, b_0] \times \mathbb{R}^d$. We assume that $|y| \geq 1$; otherwise, in a neighborhood of the origin, all the quantities we are going to estimate are certainly bounded. First, since $\varepsilon < \varepsilon_1$, we have that

$$w \leq c_1 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}},$$

with $c_1 = 1$. Second, an easy computation shows that

$$\begin{aligned} \frac{|Q(y)\nabla w(t,y)|}{w(t,y)^{\frac{k-1}{k}} W_1(t,y)^{\frac{1}{k}}} &= \varepsilon\beta t^\alpha |y|^{\beta-1}(1+|y|^m) e^{-\frac{1}{k}(\varepsilon_1-\varepsilon)t^\alpha |y|^\beta} \\ &\leq 2\varepsilon\beta t^\alpha |y|^{\beta+m-1} e^{-\frac{1}{k}(\varepsilon_1-\varepsilon)t^\alpha |y|^\beta}. \end{aligned} \tag{2.59}$$

We make use of the following remark: since the function $t \mapsto t^p e^{-t}$ on $(0, \infty)$ attains its maximum at the point $t = p$, then for $\tau, \gamma, z > 0$ we have

$$z^\gamma e^{-\tau z^\beta} = \tau^{-\frac{\gamma}{\beta}} (\tau z^\beta)^{\frac{\gamma}{\beta}} e^{-\tau z^\beta} \leq \tau^{-\frac{\gamma}{\beta}} \left(\frac{\gamma}{\beta} \right)^{\frac{\gamma}{\beta}} e^{-\frac{\gamma}{\beta}} =: C(\gamma, \beta)\tau^{-\frac{\gamma}{\beta}}. \tag{2.60}$$

Applying (2.60) to the inequality (2.59) with $z = |y|$, $\tau = \frac{1}{k}(\varepsilon_1 - \varepsilon)t^\alpha$, $\beta = \beta$ and $\gamma = \beta + m - 1 > 0$ yields

$$\begin{aligned} \frac{|Q(y)\nabla w(t,y)|}{w(t,y)^{\frac{k-1}{k}} W_1(t,y)^{\frac{1}{k}}} &\leq 2C(\beta+m-1, \beta)\varepsilon\beta t^\alpha \left[\frac{1}{k}(\varepsilon_1 - \varepsilon)t^\alpha \right]^{-\frac{\beta+m-1}{\beta}} \\ &\leq \overline{c} t^{-\frac{\alpha(m-1)}{\beta}} \leq \overline{c} a_0^{-\frac{\alpha m}{\beta}}. \end{aligned}$$

Thus, we choose $c_2 = \overline{c} a_0^{-\frac{\alpha m}{\beta}}$, where $\overline{c}$ is a universal constant. Similarly,

$$\begin{aligned} \frac{|\mathrm{div}(Q(y)\nabla w(t,y))|}{w(t,y)^{\frac{k-2}{k}} W_1(t,y)^{\frac{2}{k}}} \leq& \frac{m|y|^{m-1}|\nabla w(t,y)| + (1+|y|^m)|\Delta w|}{w(t,y)^{\frac{k-2}{k}} W_1(t,y)^{\frac{2}{k}}} \\ \leq& \varepsilon\beta t^\alpha \big[m|y|^{\beta+m-2} + 2((\beta-2)^+ + d)|y|^{\beta+m-2} \\ &+ 2\varepsilon\beta t^\alpha |y|^{2\beta+m-2} \big] e^{-\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha |y|^\beta}. \end{aligned}$$

As a result, applying (2.60) to each term, we find that

$$\begin{aligned}&\frac{|\mathrm{div}(Q(y)\nabla w(t,y))|}{w(t,y)^{\frac{k-2}{k}}W_1(t,y)^{\frac{2}{k}}}\\&\le C(\beta,m)\varepsilon\beta t^\alpha\left\{[m+2((\beta-2)^++d)]\left[\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha\right]^{-\frac{\beta+m-2}{\beta}}\right.\\&\qquad\left.+2\varepsilon\beta t^\alpha\left[\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha\right]^{-\frac{2\beta+m-2}{\beta}}\right\}\le\overline{c}t^{-\frac{\alpha(m-2)}{\beta}}\le\overline{c}a_0^{-\frac{\alpha m}{\beta}}.\end{aligned}$$

Therefore, we pick $c_3=\overline{c}a_0^{-\frac{\alpha m}{\beta}}$. In the same way, we have

$$\begin{aligned}\frac{|\partial_t w(t,y)|}{w(t,y)^{\frac{k-2}{k}}W_1(t,y)^{\frac{2}{k}}}&=\varepsilon\alpha t^{\alpha-1}|y|^\beta e^{-\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha|y|^\beta}\\&\le C(\beta)\varepsilon\alpha t^{\alpha-1}\left[\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha\right]^{-1}\le\overline{c}a_0^{-1}.\end{aligned}$$

Then, we take $c_4=\overline{c}a_0^{-1}$. Finally,

$$\frac{V(y)^{\frac{1}{2}}}{w(t,y)^{-\frac{1}{k}}W_2(t,y)^{\frac{1}{k}}}=|y|^{\frac{s}{2}}e^{-\frac{1}{k}(\varepsilon_2-\varepsilon)t^\alpha|y|^\beta}\le C(s,\beta)\left[\frac{1}{k}(\varepsilon_2-\varepsilon)t^\alpha\right]^{\frac{-s}{2\beta}}\le\overline{c}a_0^{-\frac{\alpha s}{2\beta}},$$

so we set $c_5=\overline{c}a_0^{-\frac{\alpha s}{2\beta}}$.

Step 3. We check Hypothesis 2.5.1 assuming as above that $|y|\ge 1$. First, we have

$$\frac{|\Delta w(t,y)|}{w(t,y)^{\frac{k-2}{k}}W_1(t,y)^{\frac{2}{k}}}=\varepsilon\beta t^\alpha\left[(\beta-2+d)|y|^{\beta-2}+\varepsilon\beta t^\alpha|y|^{2\beta-2}\right]e^{-\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha|y|^\beta}.$$

Recalling that $|y|\ge 1$ and applying (2.60) yields

$$\begin{aligned}\frac{|\Delta w(t,y)|}{w(t,y)^{\frac{k-2}{k}}W_1(t,y)^{\frac{2}{k}}}&\le\varepsilon\beta t^\alpha\left[((\beta-2)^++d)|y|^\beta+\varepsilon\beta t^\alpha|y|^{2\beta}\right]e^{-\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha|y|^\beta}\\&\le C(\beta)\varepsilon\beta t^\alpha\left\{((\beta-2)^++d)\left[\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha\right]^{-1}\right.\\&\qquad\left.+\varepsilon\beta t^\alpha\left[\frac{2}{k}(\varepsilon_1-\varepsilon)t^\alpha\right]^{-2}\right\}\le\overline{c}.\end{aligned}$$

Thus, Hypothesis 2.5.1(a) is verified by taking $c_6=\overline{c}$. To choose the constant c_7, we let $t_0\in(0,t)$. Then, we get

$$\begin{aligned}\frac{|Q(y)\nabla W_1(t_0,y)|}{w(t,y)^{-1/k}W_1(t_0,y)W_2(t,y)^{1/k}}&=\frac{\varepsilon_1\beta t_0^\alpha|y|^{\beta-1}(1+|y|^m)W_1(t_0,y)}{w(t,y)^{-1/k}W_1(t_0,y)W_2(t,y)^{1/k}}\\&\le 2\varepsilon_1\beta t^\alpha|y|^{\beta+m-1}e^{-\frac{1}{k}(\varepsilon_2-\varepsilon)t^\alpha|y|^\beta}\\&\le 2C(\beta,m)\varepsilon_1\beta t^\alpha\left[\frac{1}{k}(\varepsilon_2-\varepsilon)t^\alpha\right]^{-\frac{\beta+m-1}{\beta}}\\&\le\overline{c}t^{-\frac{\alpha(m-1)}{\beta}}\le\overline{c}a_0^{-\frac{\alpha m}{\beta}}.\end{aligned}$$

Consequently, we set $c_7 = \overline{c} a_0^{-\frac{\alpha m}{\beta}}$ in Hypothesis 2.5.1(b). We observe that Hypothesis 2.5.1(c) is clearly satisfied. Finally, we have

$$\nabla Z(x) = \frac{\varepsilon_2}{\varepsilon_1 t_0^\alpha} e^{(\varepsilon_2 - \varepsilon_1 t_0^\alpha)|x|_*^\beta} \nabla W_1(t_0, x),$$

for all $x \in \mathbb{R}^d$, hence Hypothesis 2.5.1(d) holds.

To sum up, the constants $c_1, \dots, c_7$ are the following:

$$\begin{aligned} &c_1 = 1, && c_2 = c_3 = c_7 = \overline{c} a_0^{-\frac{\alpha m}{\beta}}, && c_4 = \overline{c} a_0^{-1}, \\ &c_5 = \overline{c} a_0^{-\frac{\alpha s}{2\beta}}, && c_6 = \overline{c}. \end{aligned}$$

Step 4. We are now ready to apply Theorem 2.5.4. Thus, there is a positive constant $C > 0$ depending only on d and k such that

$$\begin{aligned} w(t,y)p(t,x,y) \le C \Bigg[&\left(c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b_0 - b)^{\frac{k}{2}}} + c_2^k + c_3^{\frac{k}{2}} + c_4^{\frac{k}{2}} + c_6^{\frac{k}{2}} \right) \int_{a_0}^{b_0} \xi_{W_1}(t,x)\, dt \\ &+ (c_5^k + c_2^{\frac{k}{2}} c_7^{\frac{k}{2}}) \int_{a_0}^{b_0} \xi_{W_2}(t,x)\, dt \Bigg], \end{aligned} \tag{2.61}$$

for all $(t, y) \in (a, b) \times \mathbb{R}^d$ and fixed $x \in \mathbb{R}^d$. We set $a_0 = t/4, a = t/2, b = (t + 1)/2$ and $b_0 = (t + 3)/4$. Moreover, by Proposition 2.2.1, there are two constants H_1 and H_2 not depending on a_0 and b_0 such that $\xi_{W_j}(t,x) \le H_j$ for all $(t, x) \in [0, 1] \times \mathbb{R}^d$, so

$$\int_{a_0}^{b_0} \xi_{W_j}(t,x)\, dt \le H_j(b_0 - a_0) = \frac{3t}{4} H_j.$$

If we now replace in (2.61) the values of the constants $c_1, \dots, c_7$ determined in Step 3, we use the previous inequality and we consider C as a positive constant that can vary from line to line, we obtain

$$w(t,y)p(t,x,y) \le C \left[t^{1-\frac{\alpha m}{\beta}k} + t^{1-\frac{k}{2}} + t^{1-\frac{\alpha s}{2\beta}k} \right]. \tag{2.62}$$

We note that, since $\alpha > \frac{\beta}{\beta+m-2}$, $s > |m - 2|$ and $\beta = \frac{s-m+2}{2}$, it follows that

$$\frac{\alpha(m \vee \frac{s}{2})}{\beta} > \frac{m \vee \frac{s}{2}}{\beta + m - 2} > \frac{s}{2(\beta + m - 2)} = \frac{s}{s + m - 2} > \frac{1}{2}.$$

Hence,

$$t^{1-\frac{k}{2}} < t^{1-\frac{\alpha(m\vee\frac{s}{2})}{\beta}k}.$$

As a result, by (2.62), we find that

$$w(t,y)p(t,x,y) \le C t^{1-\frac{\alpha(m\vee\frac{s}{2})k}{\beta}} = C t^{1-\frac{\alpha(2m\vee s)k}{s-m+2}}.$$

Writing the expression of the weight function w we gain the following inequality:

$$p(t,x,y) \leq Ct^{1-\frac{\alpha(2m\vee s)}{s-m+2}k} e^{-\varepsilon t^\alpha |y|_*^{\frac{s-m+2}{2}}}, \tag{2.63}$$

for $k > d+2$ and for any $t \in (0,1)$, $x, y \in \mathbb{R}^d$.

Step 5. Since $A^* = A$, applying (2.63) to $p^*(t,y,x)$, we derive that

$$p^*(t,y,x) \leq Ct^{1-\frac{\alpha(2m\vee s)}{s-m+2}k} e^{-\varepsilon t^\alpha |x|_*^{\frac{s-m+2}{2}}},$$

for all $t \in (0,1)$ and $x, y \in \mathbb{R}^d$. Combining this with (2.63) and considering that $p^*(t,y,x) = p(t,x,y)$ yields

$$p(t,x,y) = p(t,x,y)^{1/2} p(t,x,y)^{1/2} \leq Ct^{1-\frac{\alpha(2m\vee s)}{s-m+2}k} e^{-\frac{\varepsilon}{2} t^\alpha |x|_*^{\frac{s-m+2}{2}}} e^{-\frac{\varepsilon}{2} t^\alpha |y|_*^{\frac{s-m+2}{2}}},$$

for $k > d+2$ and for any $t \in (0,1)$, $x, y \in \mathbb{R}^d$. □

2.6.2 Kernel estimates in case of exponential coefficients

Let A be the operator

$$A = \operatorname{div}(e^{|x|^m}\nabla) - e^{|x|^s},$$

with $2 \leq m < s$. Set

$$w(t,x) = \exp\left(\varepsilon t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau\right) \text{ and } W_j(t,x) = \exp\left(\varepsilon_j t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau\right),$$

where $j = 1, 2$, $\frac{m}{2} + 1 \leq \beta \leq m$, $0 < \varepsilon < \varepsilon_1 < \varepsilon_2$ and $\alpha > \frac{2\beta+m-2}{2m}$.

Theorem 2.6.2. *Let p be the integral kernel associated with the operator A with $Q(x) = e^{|x|^m} I$ and $V(x) = e^{|x|^s}$, where $2 \leq m < s$. Then*

$$\begin{aligned} p(t,x,y) \leq & Ct^{1-\frac{k}{2}} \exp(Ct^{-\alpha}) \exp\left(-\frac{\varepsilon}{2} t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau\right) \\ & \times \exp\left(-\frac{\varepsilon}{2} t^\alpha \int_0^{|y|_*} e^{\frac{\tau^\beta}{2}}\, d\tau\right), \end{aligned}$$

for $k > d+2$ and any $t \in (0,1)$, $x, y \in \mathbb{R}^d$, where C is a positive constant.

Proof. Step 1. We check conditions (2.14), (2.15) and (2.16) to apply Proposition 2.2.3 and show that W_1 and W_2 are time dependent Lyapunov functions for $L = \partial_t + A$ and $\partial_t + \eta\Delta - V$. It is clear that (2.14) holds true with $c_e = 1$. Moreover, since $s > m$, it follows that

$$\lim_{|x|\to\infty} V(x)\, |x|^{1-\beta-m}\, e^{-|x|^\beta - |x|^m} = \lim_{|x|\to\infty} |x|^{1-\beta-m}\, e^{|x|^s - |x|^\beta - |x|^m} = +\infty$$

and

$$\limsup_{|x|\to\infty} |x|^{1-\beta-m} e^{-\frac{|x|^\beta}{2}-|x|^m}\left(G\cdot\frac{x}{|x|}-\frac{V}{\varepsilon e^{\frac{|x|^\beta}{2}}}\right)$$
$$=\limsup_{|x|\to\infty}\left(m\,|x|^{-\beta}\,e^{-\frac{|x|^\beta}{2}}-\frac{1}{\varepsilon}|x|^{1-\beta-m}e^{|x|^s-|x|^\beta-|x|^m}\right)=-\infty.$$

Consequently, there exist constants $c,\Lambda>0$ such that (2.15) and (2.16) hold true. By Proposition 2.2.3 we conclude that W_1 and W_2 are time dependent Lyapunov functions for $L=\partial_t+A$ and $\partial_t+\eta\Delta-V$. In addition, we also note that Hypothesis 2.0.1 is verified with

$$Z(x)=\exp\left(\varepsilon_2\int_0^{|x|_*}e^{\frac{\tau^\beta}{2}}\,d\tau\right).$$

Step 2. We prove that A satisfies all the assumptions of Theorem 2.5.4. Fix $T=1$, $x\in\mathbb{R}^d$, $0<a_0<a<b<b_0<T$ and $k>d+2$. Let $(t,y)\in[a_0,b_0]\times\mathbb{R}^d$. If $|y|\le 1$, by continuity all the functions we are estimating are bounded by a constant. Thus, let $|y|\ge 1$. Since $\varepsilon<\varepsilon_1$, we have that $w\le W_1$. Hence, inequality

$$w\le c_1 w^{\frac{k-2}{k}}W_1^{\frac{2}{k}}$$

holds true with $c_1=1$. After, we observe that

$$\int_0^{|y|}e^{\frac{\tau^\beta}{2}}\,d\tau\ge\int_{|y|-1}^{|y|}e^{\frac{\tau^\beta}{2}}\,d\tau\ge e^{\frac{(|y|-1)^\beta}{2}},\tag{2.64}$$

which leads to

$$\begin{aligned}\frac{|Q(y)\nabla w(t,y)|}{w(t,y)^{\frac{k-1}{k}}W_1(t,y)^{\frac{1}{k}}}&=\varepsilon t^\alpha\exp\left(\frac{|y|^\beta}{2}+|y|^m-\frac{(\varepsilon_1-\varepsilon)}{k}t^\alpha\int_0^{|y|}e^{\frac{\tau^\beta}{2}}\,d\tau\right)\\&\le\varepsilon t^\alpha\exp\left(\frac{|y|^\beta}{2}+|y|^m-\frac{(\varepsilon_1-\varepsilon)}{k}t^\alpha e^{\frac{(|y|-1)^\beta}{2}}\right).\end{aligned}\tag{2.65}$$

We now consider the function

$$f(r):=\frac{r^\beta}{2}+r^m-\tilde{\varepsilon}t^\alpha e^{\frac{(r-1)^\beta}{2}},$$

where $r\ge 1$ and $\tilde{\varepsilon}:=(\varepsilon_1-\varepsilon)/k$. Considering that there exists a universal constant $\overline{c}>0$ (that can vary from line to line) depending on β and m such that

$$\frac{r^\beta}{2}+r^m\le\overline{c}e^{\frac{(r-1)^\beta}{4}},$$

for all $r\ge 1$, we get

$$f(r)\le\overline{c}e^{\frac{(r-1)^\beta}{4}}-\tilde{\varepsilon}t^\alpha e^{\frac{(r-1)^\beta}{2}}.$$

If we set $z = e^{\frac{(r-1)^\beta}{2}}$ and we compute the maximum of the function $h(z) = \overline{c}\sqrt{z} - \tilde{\varepsilon}t^\alpha z$, then we obtain that

$$f(r) \leq \frac{\overline{c}^2}{4\tilde{\varepsilon}} t^{-\alpha}.$$

As a result, by (2.65) we derive

$$\frac{|Q(y)\nabla w(t,y)|}{w(t,y)^{\frac{k-1}{k}} W_1(t,y)^{\frac{1}{k}}} \leq \varepsilon t^\alpha \exp\left(\frac{\overline{c}^2}{4\tilde{\varepsilon}} t^{-\alpha}\right) \leq \overline{c}\exp(\overline{c}a_0^{-\alpha}).$$

Then, we set $c_2 := \overline{c}\exp(\overline{c}a_0^{-\alpha})$. In a similar way, we have that

$$\begin{aligned}\frac{|\mathrm{div}(Q(y)\nabla w(t,y))|}{w(t,y)^{\frac{k-2}{k}} W_1(t,y)^{\frac{2}{k}}} \leq & \Bigg[(d-1)\varepsilon t^\alpha \frac{1}{|y|} e^{\frac{|y|^\beta}{2}+|y|^m} + m\varepsilon t^\alpha |y|^{m-1} e^{\frac{|y|^\beta}{2}+|y|^m} \\ & + \frac{\beta}{2}\varepsilon t^\alpha |y|^{\beta-1} e^{\frac{|y|^\beta}{2}+|y|^m} + \varepsilon^2 t^{2\alpha} e^{|y|^\beta+|y|^m}\Bigg] \\ & \times \exp\left(-\frac{2(\varepsilon_1-\varepsilon)}{k} t^\alpha \int_0^{|y|} e^{\frac{\tau^\beta}{2}}\, d\tau\right).\end{aligned}$$

Using again (2.64), we deduce

$$\begin{aligned}&\frac{|\mathrm{div}(Q(y)\nabla w(t,y))|}{w(t,y)^{\frac{k-2}{k}} W_1(t,y)^{\frac{2}{k}}} \\ \leq & (d-1)\varepsilon t^\alpha \exp\left(\frac{|y|^\beta}{2} + |y|^m - \frac{2(\varepsilon_1-\varepsilon)}{k} t^\alpha e^{\frac{(|y|-1)^\beta}{2}}\right) \\ & + m\varepsilon t^\alpha \exp\left(\log|y|^{m-1} + \frac{|y|^\beta}{2} + |y|^m - \frac{2(\varepsilon_1-\varepsilon)}{k} t^\alpha e^{\frac{(|y|-1)^\beta}{2}}\right) \\ & + \frac{\beta}{2}\varepsilon t^\alpha \exp\left(\log|y|^{\beta-1} + \frac{|y|^\beta}{2} + |y|^m - \frac{2(\varepsilon_1-\varepsilon)}{k} t^\alpha e^{\frac{(|y|-1)^\beta}{2}}\right) \\ & + \varepsilon^2 t^{2\alpha} \exp\left(|y|^\beta + |y|^m - \frac{2(\varepsilon_1-\varepsilon)}{k} t^\alpha e^{\frac{(|y|-1)^\beta}{2}}\right).\end{aligned}$$

Proceeding as above yields

$$\begin{aligned}\frac{|\mathrm{div}(Q(y)\nabla w(t,y))|}{w(t,y)^{\frac{k-2}{k}} W_1(t,y)^{\frac{2}{k}}} \leq & \left((d-1) + m + \frac{\beta}{2}\right)\varepsilon t^\alpha \exp\left(\frac{\overline{c}^2}{8\tilde{\varepsilon}} t^{-\alpha}\right) \\ & + \varepsilon^2 t^{2\alpha} \exp\left(\frac{\overline{c}^2}{8\tilde{\varepsilon}} t^{-\alpha}\right) \leq \overline{c}\exp(\overline{c}a_0^{-\alpha}).\end{aligned}$$

Thus, we choose $c_3 = \overline{c}\exp(\overline{c}a_0^{-\alpha})$. Concerning c_4, we have

$$\begin{aligned}\frac{|\partial_t w(t,y)|}{w(t,y)^{\frac{k-2}{k}} W_1(t,y)^{\frac{2}{k}}} = & \varepsilon\alpha t^{\alpha-1}\left(\int_0^{|y|} e^{\frac{\tau^\beta}{2}}\, d\tau\right) \exp\left(-\frac{2(\varepsilon_1-\varepsilon)}{k} t^\alpha \int_0^{|y|} e^{\frac{\tau^\beta}{2}}\, d\tau\right) \\ \leq & \overline{c}a_0^{-1}.\end{aligned}$$

We take $c_4 = \overline{c}a_0^{-1}$. Repeating the same procedure for the remaining estimates, we get $c_5 = c_6 = c_7 = c_2$. Finally, we have

$$\nabla Z(x) = \frac{\varepsilon_2}{\varepsilon_1 t_0^\alpha} \exp\left((\varepsilon_2 - \varepsilon_1 t_0^\alpha) \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau \right) \nabla W_1(t_0, x),$$

for all $x \in \mathbb{R}^d$, hence Hypothesis 2.5.1(d) holds.

Step 3. As in the proof of Theorem 2.6.1, we choose $a_0 = t/4, a = t/2, b = (t+1)/2$, $b_0 = (t+3)/4$ and we notice that, by Proposition 2.2.3, there are two constants H_1 and H_2 not depending on a_0 and b_0 such that

$$\int_{a_0}^{b_0} \xi_{W_j}(t,x)\, dt \le H_j(b_0 - a_0) = \frac{3t}{4} H_j.$$

Applying Theorem 2.5.4, we infer that there exists a positive constant $C > 0$ depending only on d, k and η such that (2.58) holds. From that, taking into account the values of the constants $c_1, \dots, c_7$ found in Step 2, keeping track only of powers of t and absorbing all other constants into the constant C, we get

$$w(t,y)p(t,x,y) \le C\left[t\exp(\overline{c}t^{-\alpha}) + t^{1-\frac{k}{2}} + t^{1+\frac{k}{2}} \right] \le Ct^{1-\frac{k}{2}} \exp(Ct^{-\alpha}).$$

Hence,

$$p(t,x,y) \le Ct^{1-\frac{k}{2}} \exp(Ct^{-\alpha}) \exp\left(-\varepsilon t^\alpha \int_0^{|y|_*} e^{\frac{\tau^\beta}{2}}\, d\tau \right), \tag{2.66}$$

for $k > d+2$ and for any $t \in (0,1)$, $x, y \in \mathbb{R}^d$, where C depends only on d, k, η, β and m.

Step 4. We conclude the proof by applying inequality (2.66) to $p^*(t,y,x)$. This is possible because $A^* = A$. Then we obtain

$$p^*(t,y,x) \le Ct^{1-\frac{k}{2}} \exp(Ct^{-\alpha}) \exp\left(-\varepsilon t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau \right),$$

for all $t \in (0,1)$ and $x, y \in \mathbb{R}^d$. As a consequence, since $p^*(t,y,x) = p(t,x,y)$, we get the desired inequality as follows:

$$\begin{aligned} p(t,x,y) = p(t,x,y)^{\frac{1}{2}} p^*(t,y,x)^{\frac{1}{2}} \le & Ct^{1-\frac{k}{2}} \exp(Ct^{-\alpha}) \exp\left(-\frac{\varepsilon}{2} t^\alpha \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\, d\tau \right) \\ & \times \exp\left(-\frac{\varepsilon}{2} t^\alpha \int_0^{|y|_*} e^{\frac{\tau^\beta}{2}}\, d\tau \right), \end{aligned}$$

for all $t \in (0,1)$ and $x, y \in \mathbb{R}^d$. $\square$

2.7 Spectral properties and eigenfunctions estimates

In this section we study some spectral properties of $A_{\min}$ with either polynomial or exponential coefficients. In particular we prove the following result.

Theorem 2.7.1. *If $Q(x) = (1 + |x|_*^m)I$ and $V(x) = |x|^s$ with $s > |m-2|$ and $m > 0$ or $Q(x) = e^{|x|^m}I$ and $V(x) = e^{|x|^s}$, where $2 \le m < s$, then $T_p(t)$ is compact for all $t > 0$ and $p \in (1,\infty)$. Moreover the spectrum of the generator of $T_p(\cdot)$ is independent of p for $p \in (1,\infty)$ and consists of a sequence of negative real eigenvalues which accumulates at $-\infty$.*

Proof. By [18, Theorem 1.6.3], it suffices to prove that $T_2(t)$ is compact for all $t > 0$. To this purpose let us assume that $Q(x) = (1 + |x|_*^m)I$ and $V(x) = |x|^s$ with $s > m-2$ and $m > 2$ or $Q(x) = e^{|x|^m}I$ and $V(x) = e^{|x|^s}$, where $2 \le m < s$. Applying [18, Corollary 1.6.7], one deduces that the L^2-realization A_0 of $\mathcal{A}_0 := \mathrm{div}(Q\nabla)$ has compact resolvent and thus the semigroup $S(t)$ generated by A_0 in $L^2(\mathbb{R}^d)$ is compact for all $t > 0$, cf. [19, Theorem 4.29]. Since $V \ge 0$ we have $0 \le T_2(t) \le S(t)$ for all $t \ge 0$. Applying the Aliprantis-Burkinshaw theorem [2, Theorem 5.15] we obtain the compactness of $T_2(t)$ for all $t > 0$.

Let us now show the compactness of $T_2(t)$ in the case where $Q(x) = (1 + |x|_*^m)I$ and $V(x) = |x|^s$ with $s > |m-2|$ and $0 < m \le 2$. The operator $A_{\min}$ can be considered as the sum of the operator $\widetilde{A}_2 u := (1 + |x|_*^m)\Delta u - |x|^s u$ and the operator $Bu := \nabla(1 + |x|_*^m) \cdot \nabla u$. From [37, Proposition 2.3] we know that B is a small perturbation of $\widetilde{A}_2$. Hence, $R(\lambda, A_{\min}) = R(\lambda, \widetilde{A}_2)(I - BR(\lambda, \widetilde{A}_2))^{-1}$ for all $\lambda \in \rho(\widetilde{A}_2)$. Moreover, by [37, Proposition 2.10], we know that $\widetilde{A}_2$ has compact resolvent and hence $A_{\min}$ has compact resolvent too. Since $T_2(\cdot)$ is an analytic semigroup, we deduce that $T_2(t)$ is compact for all $t > 0$. $\square$

Let us now estimate the eigenfunctions of $A_{\min}$. To this purpose let us note first that, by the semigroup law and the symmetry of $p(t,\cdot,\cdot)$ for any $t > 0$, we have

$$p(t+s,x,y) = \int_{\mathbb{R}^d} p(t,x,z)p(s,y,z)\,dz, \quad t,s > 0,\ x,y \in \mathbb{R}^d.$$

Thus,

$$p(t,x,x) = \int_{\mathbb{R}^d} p\left(\frac{t}{2},x,y\right)^2 dy, \quad t > 0,\ x \in \mathbb{R}^d.$$

So, if we denote by ψ an eigenfunction of $A_{\min}$ associated to the eigenvalue λ, then Hölder's inequality implies

$$\begin{aligned} e^{\lambda \frac{t}{2}}|\psi(x)| &= |T_2(t/2)\psi(x)| \\ &\leq \int_{\mathbb{R}^d} p\left(\frac{t}{2}, x, y\right)|\psi(y)|\,dy \\ &\leq \left(\int_{\mathbb{R}^d} p\left(\frac{t}{2}, x, y\right)^2 dy\right)^{\frac{1}{2}} \|\psi\|_2 \\ &= p(t, x, x)^{\frac{1}{2}}\|\psi\|_2 \end{aligned}$$

for any $t > 0$ and any $x \in \mathbb{R}^d$. Therefore, if we normalize ψ, i.e. $\|\psi\|_2 = 1$, then

$$|\psi(x)| \leq e^{-\lambda \frac{t}{2}} p(t, x, x)^{\frac{1}{2}}, \quad t > 0,\ x \in \mathbb{R}^d.$$

So, by Theorem 2.6.1 and Theorem 2.6.2 we deduce the following result.

Corollary 2.7.2. *Let ψ be any normalized eigenfunction of $A_{\min}$. Then,*

(a) in the case of polynomially growing coefficients, i.e., $Q(x) = (1 + |x|_*^m)I$ and $V(x) = |x|^s$, where $s > |m-2|$ and $m > 0$, we have

$$|\psi(x)| \leq c_1 e^{-c_2|x|_*^{\frac{s-m+2}{2}}},$$

for all $x \in \mathbb{R}^d$, for some constants $c_1, c_2 > 0$;

(b) in the case of exponentially growing coefficients, i.e., $Q(x) = e^{|x|^m} I$ and $V(x) = e^{|x|^s}$, where $2 \leq m < s$, we have

$$|\psi(x)| \leq c_1 \exp\left(-c_2 \int_0^{|x|_*} e^{\frac{\tau^\beta}{2}}\,d\tau\right),$$

for all $x \in \mathbb{R}^d$, for some constants $c_1, c_2 > 0$.

Chapter 3

Elliptic operators with unbounded diffusion, drift and potential terms

In Chapter 2 we treated Schrödinger type operators in divergence form. In this chapter, we are concerned with the more general elliptic operator defined on smooth functions φ by

$$A\varphi = \operatorname{div}(Q\nabla\varphi) + F \cdot \nabla\varphi - V\varphi,$$

where the diffusion coefficients Q, the drift F and the potential V are typically unbounded functions.

As studied in the paper [32], here we aim to prove global Sobolev regularity and pointwise upper bounds for the gradient of transition densities associated with A. Throughout, we make the following assumptions on Q, F and V.

Hypothesis 3.0.1. *We have $Q = (q_{ij})_{i,j=1,\dots,d} \in C^{1+\zeta}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d})$, $F = (F_j)_{j=1,\dots,d} \in C^{1+\zeta}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and $0 \le V \in C^{\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\zeta \in (0,1)$. Moreover,*

(a) The matrix Q is symmetric and uniformly elliptic, i.e. there is $\eta > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \ge \eta|\xi|^2 \quad \textit{for all } x,\, \xi \in \mathbb{R}^d;$$

(b) there are $0 \le Z \in C^{2+\zeta}(\mathbb{R}^d)$ and a constant $M \ge 0$ such that $\lim_{|x|\to\infty} Z(x) = \infty$, $AZ(x) \le M$ and $\eta\Delta Z(x) + F\cdot\nabla Z(x) - V(x)Z(x) \le M$ for all $x \in \mathbb{R}^d$.

We observe that, by Remark 1.5.4, Hypothesis 3.0.1(b) implies that Z is a Lyapunov function for A and $\eta\Delta + F\cdot\nabla - V$ as introduced in Chapter 1.

As in Chapter 2, the construction of the minimal semigroup $T(\cdot)$ in $C_b(\mathbb{R}^d)$ described in Chapter 1 applies for the more general elliptic operator A as well.

Then a suitable realization of the above operator A generates a (typically not strongly continuous) semigroup $T = (T(t))_{t\geq 0}$ on the space $C_b(\mathbb{R}^d)$ that is given through an integral kernel p, i.e.

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy, \quad t>0,\, x\in\mathbb{R}^d,\, f\in C_b(\mathbb{R}^d),$$

where the kernel p is positive, $p(t,\cdot,\cdot)$ and $p(t,x,\cdot)$ are measurable for any $t>0$, $x\in\mathbb{R}^d$, and for a.e. fixed $y\in\mathbb{R}^d$, $p(\cdot,\cdot,y)\in C^{1+\zeta/2,2+\zeta}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d)$.

In Section 3.1, the approach of Chapter 2 based on the existence of time dependent Lyapunov functions for $\partial_t + A$ and $\partial_t + \eta\Delta + F\cdot\nabla - V$ allows us to establish estimates for the kernel p. Such functions play an important role in the technique used in [41] in case of operators with bounded diffusion coefficients in order to obtain estimates not only for p, but also for ∇p, the gradient of p. In there, the key point is to prove the square integrability of the logarithmic gradient of p. From that, global regularity results follow.

The core of this chapter is to repeat the same steps to achieve bounds for the gradient of the transition kernel. For this purpose we make use of an approximation argument: we approximate the operator A with a family of operators A_n with bounded diffusion coefficients. As already underlined in Chapter 2, what is important for this procedure to work when dealing with the approximating kernels p_n is to ensure that the constant in the right hand side of the estimate of ∇p_n does not depend on the diffusion matrix. To this end, making use of Theorem 2.3.6, in Section 3.2 we establish a suitable estimate for ∇p in case of bounded diffusion coefficients. Then, in Section 3.3 we achieve the corresponding estimate for the general operator A.

Finally, in Section 3.4 we see our main result at work on the prototype operator

$$\operatorname{div}\big((1+|x|_*^m)\nabla u\big) - |x|^{p-1}x\cdot\nabla u - |x|^s,$$

for $p>(m-1)\vee 1$, $s>|m-2|$ and $m>0$, where $x\mapsto |x|_*$ is a C^2-function satisfying $|x|_* = |x|$ for $|x|\geq 1$.

3.1 Preliminaries

In this section we present the ingredients we will use in the next section to find pointwise upper bounds for the derivatives of the kernel in case of bounded diffusion coefficients. We will make use of time dependent Lyapunov functions W for the operators $L := \partial_t + A$ and $\partial_t + \eta\Delta + F\cdot\nabla - V$ with respect to Z and h introduced in Chapter 1, where Z is the Lyapunov function in Hypothesis 3.0.1(b) and $0\leq h\in L^1((0,T))$. According to Definition 1.6.2, for fixed $T>0$ and for all $(t,x)\in(0,T)\times\mathbb{R}^d$, they satisfy the following inequalities

$$LW(t,x)\leq h(t)W(t,x),$$

$$\partial_t W(t,x) + \eta\Delta W(t,x) + F(x)\cdot\nabla W(t,x) - V(x)W(t,x)\leq h(t)W(t,x).$$

3.1.1 Kernel estimates in case of bounded diffusion coefficients

As in Chapter 2, time dependent Lyapunov functions allow us to establish pointwise upper bounds for the kernel p.

Theorem 3.1.1. *Fix $T > 0$, $x \in \mathbb{R}^d$ and $0 < a_0 < a < b < b_0 < T$. Let us consider two time dependent Lyapunov functions W_1, W_2 for the operators $L = \partial_t + A$ with $1 \le W_1 \le W_2$ and a weight function $1 \le w \in C^{1,2}((0,T) \times \mathbb{R}^d)$ such that*

(a) the functions $w^{-2}\partial_t w$ and $w^{-2}\nabla w$ are bounded on $Q(a_0, b_0)$;

(b) there exist $k > d+2$ and constants $c_1, \dots, c_6 \ge 1$, possibly depending on the interval (a_0, b_0), with

(i) $w \le c_1 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$, *(ii) $|Q\nabla w| \le c_2 w^{\frac{k-1}{k}} W_1^{\frac{1}{k}}$,*

(iii) $|\mathrm{div}(Q\nabla w)| \le c_3 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$, *(iv) $|\partial_t w| \le c_4 w^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$,*

(v) $V^{\frac{1}{2}} \le c_5 w^{-\frac{1}{k}} W_2^{\frac{1}{k}}$, *(vi) $|F| \le c_6 w^{-\frac{1}{k}} W_2^{\frac{1}{k}}$,*

on $[a_0, b_0] \times \mathbb{R}^d$.

If q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i, j, h = 1, \dots, d$, then there is a constant $C > 0$ depending only on d, k and η such that

$$w(t,y)p(t,x,y) \le C\Bigg[\left(c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b_0-b)^{\frac{k}{2}}} + c_2^k + c_3^{\frac{k}{2}} + c_4^{\frac{k}{2}}\right)\int_{a_0}^{b_0} \xi_{W_1}(t,x)\,dt \\ + \left(c_5^k + c_6^k + c_2^{\frac{k}{2}} c_6^{\frac{k}{2}}\right)\int_{a_0}^{b_0} \xi_{W_2}(t,x)\,dt\Bigg],$$

for all $(t,y) \in (a,b) \times \mathbb{R}^d$ and any fixed $x \in \mathbb{R}^d$, where

$$\xi_{W_i}(t,x) := \int_{\mathbb{R}^d} p(t,x,y) W_i(t,y)\,dy,$$

for $i = 1, 2$.

Proof. The difference of Theorem 2.4.2 and the above theorem is the presence of the drift term in the operator A. For that, here we assumed further inequality (vi). We now inspect the proof of Theorem 2.4.2 and we highlight the changes to make in order to deal with it.

We first assume that w along with its first order spatial derivatives is bounded.

Taking into account Remark 2.3.3, in *Step 1* it is still possible to apply Corollary 2.3.2 to infer that $p \in \mathcal{H}^{s,1}(Q(a_1,b_1))$ for all $s \in (1,k/2)$. Indeed, by $w \geq 1$, (i), (vi) and $W_1 \leq W_2$, we have

$$\begin{aligned}
\Gamma_1(k/2,x,a_0,b_0)^{\frac{k}{2}} &= \int_{Q(a_0,b_0)} |F(y)|^{\frac{k}{2}} p(t,x,y)\,dt\,dy \\
&\leq \int_{Q(a_0,b_0)} w(t,y)|F(y)|^{\frac{k}{2}} p(t,x,y)\,dt\,dy \\
&\leq c_6^{\frac{k}{2}} \int_{Q(a_0,b_0)} w(t,y)^{\frac{1}{2}} W_2(t,y)^{\frac{1}{2}} p(t,x,y)\,dt\,dy \\
&\leq c_1^{\frac{k}{4}} c_6^{\frac{k}{2}} \int_{Q(a_0,b_0)} W_1(t,y)^{\frac{1}{2}} W_2(t,y)^{\frac{1}{2}} p(t,x,y)\,dt\,dy \\
&\leq c_1^{\frac{k}{4}} c_6^{\frac{k}{2}} \int_{a_0}^{b_0} \xi_{W_2}(t,x)\,dt.
\end{aligned}$$

Moreover, since time dependent Lyapunov functions are integrable with respect to $p(t,x,y)dy$ thanks to Proposition 1.6.3, we deduce that $\Gamma_1(k/2,x,a_0,b_0) < \infty$.

In *Step 2* the keypoint is to apply Theorem 2.3.6. For that, repeating the same computations, it turns out that we have to replace (2.49) and (2.50), respectively, by

$$\begin{aligned}
&f = q\partial_t w + q\,\mathrm{div}(Q\nabla w) - qVw + \frac{k}{2} pw\vartheta^{\frac{k-2}{2}}\vartheta' + qF\nabla w, \\
&h = 2qQ\nabla w + wqF.
\end{aligned}$$

As a consequence, in (2.51) the new terms $\|qF\nabla w\|_{\frac{k}{2}}$ and $\|wqF\|_k$ appear. Making use of inequalities (ii) and (vi), in *Step 3* we obtain

$$\begin{aligned}
\|qF\nabla w\|_{\frac{k}{2}} &\leq \eta^{-1} c_2 c_6 \|wq\|_\infty^{\frac{k-2}{k}} M_2^{\frac{2}{k}}, \\
\|wqF\|_k &\leq c_6 \|wq\|_\infty^{\frac{k-1}{k}} M_2^{\frac{1}{k}}.
\end{aligned}$$

Hence, we set

$$\begin{aligned}
&\beta = C\left(c_2 M_1^{\frac{1}{k}} + c_6 M_2^{\frac{1}{k}}\right), \\
&\gamma = C\left[\left(\frac{c_1}{b_1-b} + c_3 + c_4\right) M_1^{\frac{2}{k}} + (c_2c_6 + c_5^2) M_2^{\frac{2}{k}}\right],
\end{aligned}$$

instead of (2.52). The rest of the proof carries over verbatim. □

Remark 3.1.2. *If one assumes* $|Q\nabla w| \leq c_2 W_1^{\frac{1}{2k}}$, $|QD^2w| \leq c_3' W_1^{\frac{1}{k}}$ *and* $|\nabla Q| \leq c_7 w^{-\frac{1}{k}} W_1^{\frac{1}{2k}}$, *for some positive constants* c_2, c_3', c_7, *then, since* $w \geq 1$, *we have*

$$\begin{aligned}
|\mathrm{div}(Q\nabla w)| &\leq d\left(|\nabla Q|\,|\nabla w| + |QD^2w|\right) \\
&\leq d\left(c_2c_7\eta^{-1} w^{\frac{-1}{k}} W_1^{\frac{1}{k}} + c_3' W_1^{\frac{1}{k}}\right) \\
&\leq d\left(c_2c_7\eta^{-1} + c_3'\right) w^{\frac{k-2}{k}} W_1^{\frac{1}{k}}. \qquad (3.1)
\end{aligned}$$

So, since $1 \le W_1$*, the assumption (iii) of the above theorem is satisfied with*

$$c_3 = d\left(c_2 c_7 \eta^{-1} + c_3'\right).$$

For further purposes, we obtain from the above remark the following corollary.

Corollary 3.1.3. *Assume all the assumptions of Theorem 3.1.1 except (ii) and (iii). If* $|Q\nabla w| \le c_2 W_1^{\frac{1}{2k}}$, $|QD^2 w| \le c_3' W_1^{\frac{1}{k}}$ *and* $|\nabla Q| \le c_7 w^{-\frac{1}{k}} W_1^{\frac{1}{2k}}$ *hold for some positive constants* c_2, c_3', c_7*, then there is a constant* $C > 0$ *depending only on* d*,* k *and* η *such that*

$$w(t,y)p(t,x,y) \le C\left(A_1 \int_{a_0}^{b_0} \xi_{W_1}(t,x)\,dt + A_2 \int_{a_0}^{b_0} \xi_{W_2}(t,x)\,dt\right), \tag{3.2}$$

for all $(t,y) \in (a,b) \times \mathbb{R}^d$ *and any fixed* $x \in \mathbb{R}^d$*, with*

$$\begin{aligned} A_1 &= c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b_0 - b)^{\frac{k}{2}}} + c_2^k + \left[d\left(c_2 c_7 \eta^{-1} + c_3'\right)\right]^{\frac{k}{2}} + c_4^{\frac{k}{2}}, \\ A_2 &= c_5^k + c_6^k + c_2^{\frac{k}{2}} c_6^{\frac{k}{2}}. \end{aligned} \tag{3.3}$$

We aim to establish estimates for the derivatives of the kernel p. To this purpose we make the following assumptions.

Hypothesis 3.1.4. *Fix* $T > 0$, $x \in \mathbb{R}^d$ *and* $0 < a_0 < a < b < b_0 < T$*. Let us consider two time dependent Lyapunov functions* $1 \le W_1, W_2$ *with* $W_1 \le W_2$ *and a weight function* $1 \le w \in C^{1,3}((0,T) \times \mathbb{R}^d)$ *with* $\partial_t \nabla w \in C((0,T) \times \mathbb{R}^d)$ *such that for some* $\varepsilon \in (0,1)$ *and* $k > 2(d+2)$ *the following hold true:*

(a) $\displaystyle\int_{\mathbb{R}^d} \left(\frac{1}{w(t,y)}\right)^{1-\varepsilon} dy < \infty$ *for all fixed* $t \in [a,b]$ *and* $\displaystyle\int_{Q(a,b)} \left(\frac{1}{w(t,y)}\right)^{1-\varepsilon} dt\,dy < \infty$*;*

(b) the functions $w^{-2}\nabla w$, $w^{-2}\partial_t w$, $w^{-2}D^2 w$, $w^{-3}D_i w D_j w$, $w^{-2}\partial_t \nabla w$, $w^{-3}\partial_t w \nabla w$, $|\nabla w|^{-k-1} D^2 w$ *and* $|\nabla w|^{-k-1}\partial_t \nabla w$ *are bounded on* $Q(a_0, b_0)$*;*

(c) there exist constants $c_1, \ldots, c_{11} \ge 1$*, possibly depending on the interval* (a_0, b_0)*, such that*

(i) $w \le c_1 w^{\frac{k-2}{k}} W_1^{\frac{1}{k}}$, *(ii)* $|Q\nabla w| \le c_2 W_1^{\frac{1}{2k}}$,

(iii) $|QD^2 w| \le c_3 W_1^{\frac{1}{k}}$, *(iv)* $|\partial_t w| \le c_4 w^{\frac{k-2}{k}} W_1^{\frac{1}{2k}}$,

(v) $V^{\frac{1}{2}} \le c_5 w^{-\frac{1}{k}} W_2^{\frac{1}{2k}}$, *(vi)* $|F| \le c_6 w^{-\frac{1}{k}} W_2^{\frac{1}{2k}}$,

(vii) $|\nabla Q| \le c_7 w^{-\frac{1}{k}} W_1^{\frac{1}{2k}}$, *(viii)* $|\nabla F| \le c_8 w^{-\frac{1}{k}} W_2^{\frac{1}{k}}$,

(ix) $|\nabla V| \le c_9 w^{-\frac{2}{k}} W_2^{\frac{2}{k}}$, *(x)* $|D^3 w| \le c_{10} W_1^{\frac{3}{2k}}$,

(xi) $|\partial_t \nabla w| \le c_{11} W_1^{\frac{1}{k}}$,

on $[a_0, b_0] \times \mathbb{R}^d$.

From now on, we fix $0 < a_0 < a < a_1 < b_1 < b < b_0 < T$ with $T > 0$, $b - b_1 \geq a_1 - a \geq a - a_0$ and $x \in \mathbb{R}^d$. Moreover, we consider p as a function of $(t, y) \in (0, T) \times \mathbb{R}^d$.

3.1.2 Global regularity results for bounded diffusion coefficients

In this subsection we assume that the coefficients q_{ij} and their spatial derivatives $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for all $i, j, h = 1, \ldots, d$. In here we present some of the key results that in the next section will make our technique work.

Adapting [41, Theorem 5.1] to operators with potential term, we show that $p^{1/2}$ belongs to $W_2^{0,1}(Q(a,b))$.

Theorem 3.1.5. *Assume Hypothesis 3.1.4 and that q_{ij}, $D_k q_{ij}$ are bounded on $\mathbb{R}^d$ for $i, j, k = 1, \ldots, d$. Then the functions $p \log^2 p$ and $p \log p$ are integrable in $Q(a,b)$ and in $\mathbb{R}^d$ for all fixed $t \in [a,b]$ and*

$$\begin{aligned}\int_{Q(a,b)} \frac{|\nabla p(t,x,y)|^2}{p(t,x,y)}\,dt\,dy \leq & \frac{1}{\eta^2}\int_{Q(a,b)} \left(|F(y)|^2 + V^2(y)\right) p(t,x,y)\,dt\,dy \\ & + \int_{Q(a,b)} p(t,x,y)\log^2 p(t,x,y)\,dt\,dy \\ & - \frac{2}{\eta}\int_{\mathbb{R}^d} [p(t,x,y)\log p(t,x,y)]_{t=a}^{t=b} dy < \infty.\end{aligned}$$

In particular, $p^{\frac{1}{2}}$ belongs to $W_2^{0,1}(Q(a,b))$.

Proof. We first observe that, by Corollary 3.1.3 and Hypothesis 3.1.4(a), $p \log p$ is integrable in $\mathbb{R}^d$ for all fixed $t \in [a,b]$ and $p \log^2 p$ is integrable in $Q(a,b)$. Moreover, using Hypothesis 3.1.4 and Proposition 1.6.3, we have

$$\begin{aligned}\Gamma_1(k,x,a_0,b_0)^k &= \int_{Q(a_0,b_0)} |F(y)|^k p(t,x,y)\,dt\,dy \\ &\leq \int_{Q(a_0,b_0)} w(t,y)|F(y)|^k p(t,x,y)\,dt\,dy \\ &\leq c_6^k \int_{Q(a_0,b_0)} W_2(t,y)^{\frac{1}{2}} p(t,x,y)\,dt\,dy \\ &\leq c_6^k \int_{a_0}^{b_0} \xi_{W_2}(t,x)\,dt < \infty. \end{aligned} \tag{3.4}$$

Similarly, we get

$$\begin{aligned}
\Gamma_2(k,x,a_0,b_0)^k &= \int_{Q(a_0,b_0)} V^k(y)p(t,x,y)\,dt\,dy \\
&\le \int_{Q(a_0,b_0)} w^2(t,y)V^k(y)p(t,x,y)\,dt\,dy \\
&\le c_5^{2k}\int_{Q(a_0,b_0)} W_2(t,y)p(t,x,y)\,dt\,dy \\
&= c_5^{2k}\int_{a_0}^{b_0} \xi_{W_2}(t,x)\,dt < \infty.
\end{aligned}$$

Hence, Lemma 2.3.1 and Remark 2.3.3 imply that $p \in W_k^{0,1}(Q(a,b))$. As a consequence, since by Lemma 1.3.3 we have that for all $\varphi \in C_c^{1,2}(Q(a,b))$

$$\int_{Q(a,b)} (\partial_t\varphi(t,y) + A\varphi(t,y))p(t,x,y)\,dt\,dy = \int_{\mathbb{R}^d} [p(t,x,y)\varphi(t,y)]_{t=a}^{t=b}\,dy,$$

then integrating by parts we get

$$\begin{aligned}
\int_{Q(a,b)} p\partial_t\varphi\,dt\,dy &= \int_{Q(a,b)} [\langle Q\nabla\varphi, \nabla p\rangle - p\langle F, \nabla\varphi\rangle + V\varphi p]\,dt\,dy \\
&\quad + \int_{\mathbb{R}^d} [p(t,x,y)\varphi(t,y)]_{t=a}^{t=b}\,dy. \qquad (3.5)
\end{aligned}$$

By density, the previous identity holds if $\varphi \in W_2^{1,1}(Q(a,b))$ with compact support in y. We now consider $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that

- $\zeta(y) = 1$ for $|y| \le 1$,
- $\zeta(y) = 0$ for $|y| \ge 2$,
- $0 \le \zeta \le 1$.

We set $\zeta_n(y) = \zeta(y/n)$ for all $n \in \mathbb{N}$. Since $\zeta_n^2 \log p \in W_2^{1,1}(Q(a,b))$ by Proposition C.4.2, choosing $\varphi = \zeta_n^2 \log p$ in (3.5) yields

$$\begin{aligned}
\int_{Q(a,b)} \zeta_n^2\partial_t p\,dt\,dy = \int_{Q(a,b)} \Bigg(&\frac{\zeta_n^2}{p}\sum_{i,j=1}^d q_{ij}D_ipD_jp + 2\zeta_n\log p\sum_{i,j=1}^d q_{ij}D_ipD_j\zeta_n \\
&- \zeta_n^2\langle F, \nabla p\rangle - 2\zeta_n p\log p\langle F, \nabla\zeta_n\rangle + \zeta_n^2 Vp\log p\Bigg)\,dt\,dy \\
&+ \int_{\mathbb{R}^d} \left[p(t,x,y)\zeta_n^2(y)\log p(t,x,y)\right]_{t=a}^{t=b}\,dy.
\end{aligned}$$

Defining $q(u,v) := \sum_{i,j=1}^{d} q_{ij} u_i v_j$ for all $u, v \in \mathbb{R}^d$, we obtain

$$\begin{aligned}
&\int_{Q(a,b)} \zeta_n^2 \frac{q(\nabla p, \nabla p)}{p}\, dt\, dy \\
&\qquad = -2 \int_{Q(a,b)} \zeta_n \log p\, q(\nabla p, \nabla \zeta_n)\, dt\, dy + \int_{Q(a,b)} \zeta_n^2 \langle F, \nabla p\rangle\, dt\, dy \\
&\qquad\quad + 2 \int_{Q(a,b)} \zeta_n p \log p \langle F, \nabla \zeta_n\rangle\, dt\, dy - \int_{Q(a,b)} \zeta_n^2 V p \log p\, dt\, dy \\
&\qquad\quad + \int_{\mathbb{R}^d} \zeta_n^2 [p - p\log p]_{t=a}^{t=b}\, dy.
\end{aligned}$$

Then,

$$\int_{Q(a,b)} \zeta_n^2 \frac{q(\nabla p, \nabla p)}{p}\, dt\, dy = -2I_n + J_n + 2K_n - L_n + \int_{\mathbb{R}^d} \zeta_n^2 [p - p\log p]_{t=a}^{t=b}\, dy, \tag{3.6}$$

where

$$\begin{aligned}
I_n &= \int_{Q(a,b)} \zeta_n \log p\, q(\nabla p, \nabla \zeta_n)\, dt\, dy, \qquad & J_n &= \int_{Q(a,b)} \zeta_n^2 \langle F, \nabla p\rangle\, dt\, dy, \\
K_n &= \int_{Q(a,b)} \zeta_n p \log p \langle F, \nabla \zeta_n\rangle\, dt\, dy, \qquad & L_n &= \int_{Q(a,b)} \zeta_n^2 V p \log p\, dt\, dy.
\end{aligned}$$

In the following we estimate the previous quantities in order to show that $q(\nabla p, \nabla p)/p$ is integrable in $Q(a,b)$.
Applying the Cauchy-Schwarz inequality and Hölder's inequality, we deduce that

$$\begin{aligned}
|I_n| &\le \int_{Q(a,b)} \zeta_n |\log p|\, |q(\nabla p, \nabla \zeta_n)|\, dt\, dy \\
&\le \int_{Q(a,b)} \left(\zeta_n \frac{\sqrt{q(\nabla p, \nabla p)}}{\sqrt{p}}\right) \left(\sqrt{p} |\log p| \sqrt{q(\nabla \zeta_n, \nabla \zeta_n)}\right) dt\, dy \\
&\le \left(\int_{Q(a,b)} \zeta_n^2 \frac{q(\nabla p, \nabla p)}{p}\, dt\, dy\right)^{\frac{1}{2}} \left(\int_{Q(a,b)} p \log^2 p\, q(\nabla \zeta_n, \nabla \zeta_n)\, dt\, dy\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.7}$$

In addition, given that $|\nabla \zeta|$ is bounded by a constant M, we observe that

$$q(\nabla \zeta_n, \nabla \zeta_n) \le \|Q\|_\infty |\nabla \zeta_n|^2 \le \frac{M^2 \|Q\|_\infty}{n^2}.$$

Combining this with (3.7) and using Young's inequality, we derive that

$$|I_n| \le \varepsilon_1 \int_{Q(a,b)} \zeta_n^2 \frac{q(\nabla p, \nabla p)}{p}\, dt\, dy + \frac{c\|Q\|_\infty}{4\varepsilon_1 n^2} \int_{Q(a,b)} p \log^2 p\, dt\, dy. \tag{3.8}$$

In the same way, we have

$$
\begin{aligned}
|J_n| &\le \int_{Q(a,b)} \zeta_n |F|\,|\nabla p|\,dt\,dy \\
&\le \left(\int_{Q(a,b)} \zeta_n^2 \frac{|\nabla p|^2}{p}\,dt\,dy\right)^{\frac{1}{2}} \left(\int_{Q(a,b)} |F|^2 p\,dt\,dy\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.9}
$$

Since the matrix Q is uniformly elliptic by Hypothesis 3.0.1, by Young's inequality we get

$$
|J_n| \le \frac{\varepsilon_2}{\eta} \int_{Q(a,b)} \zeta_n^2 \frac{q(\nabla p, \nabla p)}{p}\,dt\,dy + \frac{1}{4\varepsilon_2} \int_{Q(a,b)} |F|^2 p\,dt\,dy. \tag{3.10}
$$

Moreover, by Hölder's and Young's inequality, we have

$$
\begin{aligned}
|K_n| &\le \int_{Q(a,b)} |F|p|\log p|\,|\nabla \zeta_n|\,dt\,dy \le \frac{M}{n} \int_{Q(a,b)} |F|p|\log p|\,dt\,dy \\
&\le \frac{M}{n} \left(\int_{Q(a,b)} |F|^2 p\,dt\,dy\right)^{\frac{1}{2}} \left(\int_{Q(a,b)} p\log^2 p\,dt\,dy\right)^{\frac{1}{2}} \\
&\le \frac{M}{n} \int_{Q(a,b)} |F|^2 p\,dt\,dy + \frac{M}{4n} \int_{Q(a,b)} p\log^2 p\,dt\,dy
\end{aligned} \tag{3.11}
$$

and

$$
\begin{aligned}
|L_n| &\le \int_{Q(a,b)} Vp|\log p|\,dt\,dy \le \left(\int_{Q(a,b)} V^2 p\,dt\,dy\right)^{\frac{1}{2}} \left(\int_{Q(a,b)} p\log^2 p\,dt\,dy\right)^{\frac{1}{2}} \\
&\le \frac{1}{4\varepsilon_3} \int_{Q(a,b)} V^2 p\,dt\,dy + \varepsilon_3 \int_{Q(a,b)} p\log^2 p\,dt\,dy.
\end{aligned} \tag{3.12}
$$

Using (3.8), (3.10), (3.11) and (3.12) in (3.6) yields

$$
\begin{aligned}
&\left(1 - 2\varepsilon_1 - \frac{\varepsilon_2}{\eta}\right) \int_{Q(a,b)} \zeta_n^2 \frac{q(\nabla p, \nabla p)}{p}\,dt\,dy \\
&\qquad \le \left(\frac{1}{4\varepsilon_2} + \frac{2M}{n}\right) \int_{Q(a,b)} |F|^2 p\,dt\,dy + \frac{1}{4\varepsilon_3} \int_{Q(a,b)} V^2 p\,dt\,dy \\
&\qquad\quad + \left(\frac{M}{2n} + \varepsilon_3 + \frac{c\,\|Q\|_\infty}{2\varepsilon_1 n^2}\right) \int_{Q(a,b)} p\log^2 p\,dt\,dy \\
&\qquad\quad + \int_{\mathbb{R}^d} \zeta_n^2 [p - p\log p]_{t=a}^{t=b}\,dy.
\end{aligned} \tag{3.13}
$$

We note that, taking into account Hypothesis 3.1.4 and Proposition 1.6.3 we deduce that

$$
\int_{Q(a,b)} |F|^2 p\,dt\,dy \le c_6^2 \int_{Q(a,b)} w^{-\frac{2}{k}} W_2^{\frac{1}{k}} p\,dt\,dy \le c_6^2 \int_a^b \xi_{W_2}(t,x)\,dt < \infty \tag{3.14}
$$

and

$$\int_{Q(a,b)} V^2 p\,dt\,dy \le c_5^4 \int_{Q(a,b)} w^{-\frac{4}{k}} W_2^{\frac{2}{k}} p\,dt\,dy \le c_5^4 \int_a^b \xi_{W_2}(t,x)\,dt < \infty.$$

Consequently, considering also that $p\log^2 p$ is integrable in $Q(a,b)$, letting $n \to \infty$ in (3.13) leads to

$$\int_{Q(a,b)} \frac{q(\nabla p, \nabla p)}{p}\,dt\,dy < \infty.$$

We now come back to (3.6). Choosing $\varepsilon_1 = 1/n$ in (3.8) we get

$$|I_n| \le \frac{1}{n}\int_{Q(a,b)} \zeta_n^2 \frac{q(\nabla p, \nabla p)}{p}\,dt\,dy + \frac{c\,\|Q\|_\infty}{4n}\int_{Q(a,b)} p\log^2 p\,dt\,dy.$$

Since both $q(\nabla p, \nabla p)/p$ and $p\log^2 p$ are integrable in $Q(a,b)$, we find that $I_n \to 0$ as $n \to \infty$. Similarly, by (3.11) and (3.14), $K_n \to 0$ as $n \to \infty$. Moreover, applying Young's inequality in (3.9), we have

$$|J_n| \le \varepsilon_4 \int_{Q(a,b)} \frac{|\nabla p|^2}{p}\,dt\,dy + \frac{1}{4\varepsilon_4}\int_{Q(a,b)} |F|^2 p\,dt\,dy.$$

Estimating J_n as above, L_n as in (3.12) and letting $n \to \infty$ in (3.6) we gain that

$$\begin{aligned}
\eta \int_{Q(a,b)} \frac{|\nabla p|^2}{p}\,dt\,dy &\le \int_{Q(a,b)} \frac{q(\nabla p, \nabla p)}{p}\,dt\,dy \\
\le \varepsilon_4 \int_{Q(a,b)} \frac{|\nabla p|^2}{p}\,dt\,dy &+ \frac{1}{4\varepsilon_4}\int_{Q(a,b)} |F|^2 p\,dt\,dy + \frac{1}{4\varepsilon_3}\int_{Q(a,b)} V^2 p\,dt\,dy \\
+ \varepsilon_3 \int_{Q(a,b)} p\log^2 p\,dt\,dy &+ \int_{\mathbb{R}^d} [p - p\log p]_{t=a}^{t=b}\,dy,
\end{aligned} \tag{3.15}$$

where in the first inequality we used the uniformly ellipticity of the matrix Q. We also observe that the function $t \mapsto (T(t)1)(x)$ is decreasing in $[0,+\infty)$ for any $x \in \mathbb{R}^d$ because the semigroup is contractive. Hence, we have

$$\int_{\mathbb{R}^d} p(b,x,y)\,dy \le \int_{\mathbb{R}^d} p(a,x,y)\,dy,$$

for all $x \in \mathbb{R}^d$. It follows that $\int_{\mathbb{R}^d} [p(t,x,y)]_{t=a}^{t=b}\,dy \le 0$. Combining this with (3.15) and setting $\varepsilon_3 = \varepsilon_4 = \eta/2$, then we derive the desired inequality. $\square$

As in [41, Lemma 5.1], we prove that ∇p is bounded.

Lemma 3.1.6. *Assume Hypothesis 3.1.4 and that q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i,j,h = 1,\dots,d$. Then $\nabla p \in L^s(Q(a_1,b_1))$ for all $1 \le s \le \infty$.*

Proof. Arguing as in the proof of Theorem 3.1.5 we know that $\nabla p \in L^k(Q(a,b))$. We now show that $\nabla p \in L^\infty(Q(a,b))$. For fixed $x \in \mathbb{R}^d$ we consider the function $q(t,y) := \vartheta^{k/2}(t)p(t,x,y)$, where $\vartheta \in C^\infty(\mathbb{R})$ be such that

- $\vartheta(t) = 1$ for $t \in [a_1, b_1]$,
- $\vartheta(t) = 0$ for $t \le a$, $t \ge b$,
- $0 \le \vartheta \le 1$.

We set

$$\begin{aligned}
&r_1 > 1 \text{ such that } \frac{1}{r_1} = \left(1 - \frac{2}{k}\right)\frac{1}{k} + \frac{2}{k},\\
&\alpha = \frac{k}{r_1},\\
&\beta > 1 \text{ such that } \frac{2}{\alpha} + \frac{1}{\beta} = 1.
\end{aligned}$$

Then, since $\sqrt{q} \in W_2^{0,1}(Q(a_0,b_0))$ by Theorem 3.1.5, Hölder's inequality with exponents $1/\alpha$, $1/\alpha$ and $1/\beta$ yields

$$\begin{aligned}
\int_{Q(a_0,b_0)} |F|^{r_1}|\nabla q|^{r_1}\,dt\,dy &= \int_{Q(a_0,b_0)} |F|^{r_1} q^{1/\alpha} q^{-1/\alpha} |\nabla q|^{2/\alpha} |\nabla q|^{r_1 - 2/\alpha}\,dt\,dy\\
&\le \left(\int_{Q(a_0,b_0)} \frac{|\nabla q|^2}{q}\,dt\,dy\right)^{\frac{1}{\alpha}} \left(\int_{Q(a_0,b_0)} |F|^{r_1\alpha} q\,dt\,dy\right)^{\frac{1}{\alpha}}\\
&\quad \times \left(\int_{Q(a_0,b_0)} |\nabla q|^{(r_1 - \frac{2}{\alpha})\beta}\,dt\,dy\right)^{\frac{1}{\beta}}\\
&\le \left(\int_{Q(a,b)} \frac{|\nabla p|^2}{p}\,dt\,dy\right)^{\frac{1}{\alpha}} \left(\int_{Q(a,b)} |F|^k q\,dt\,dy\right)^{\frac{1}{\alpha}}\\
&\quad \times \left(\int_{Q(a_0,b_0)} |\nabla q|^k\,dt\,dy\right)^{\frac{1}{\beta}}.
\end{aligned}$$

We observe that the right hand side of the previous inequality is finite because of Theorem 3.1.5, (3.4) and the fact that $\nabla p \in L^k(Q(a,b))$. Hence $F \cdot \nabla q$ belongs to $L^{r_1}(Q(a_0,b_0))$. Similarly, applying Hölder's inequality with exponents $2/\alpha$ and $1/\beta$ we get

$$\begin{aligned}
\int_{Q(a_0,b_0)} |\mathrm{div} F|^{r_1} q^{r_1}\,dt\,dy &= \int_{Q(a_0,b_0)} |\mathrm{div} F|^{r_1} q^{2/\alpha} q^{r_1 - 2/\alpha}\,dt\,dy\\
&\le \left(\int_{Q(a_0,b_0)} |\mathrm{div} F|^{r_1\alpha/2} q\,dt\,dy\right)^{\frac{2}{\alpha}} \left(\int_{Q(a_0,b_0)} q^{(r_1 - \frac{2}{\alpha})\beta}\,dt\,dy\right)^{\frac{1}{\beta}}\\
&= \left(\int_{Q(a_0,b_0)} |\mathrm{div} F|^{\frac{k}{2}} q\,dt\,dy\right)^{\frac{2}{\alpha}} \left(\int_{Q(a_0,b_0)} q^k\,dt\,dy\right)^{\frac{1}{\beta}}. \qquad (3.16)
\end{aligned}$$

By Hypothesis 3.1.4 and Proposition 1.6.3 we deduce that

$$\begin{aligned}\int_{Q(a_0,b_0)} |\mathrm{div}F|^{\frac{k}{2}} q\,dt\,dy &\le d^{\frac{k}{4}} \int_{Q(a_0,b_0)} |\nabla F|^{\frac{k}{2}} q\,dt\,dy \\ &\le d^{\frac{k}{4}} c_8^{\frac{k}{2}} \int_{Q(a_0,b_0)} w^{-\frac{1}{2}} W_2^{\frac{1}{2}} q\,dt\,dy \\ &\le d^{\frac{k}{4}} c_8^{\frac{k}{2}} \int_{a_0}^{b_0} \xi_{W_2}(t,x)\,dt < \infty. \end{aligned} \tag{3.17}$$

Thus from (3.16) we derive that $q\,\mathrm{div}F \in L^{r_1}(Q(a_0,b_0))$. Similar computations imply that $Vq \in L^{r_1}(Q(a_0,b_0))$. Since $F \in C^1(\mathbb{R}^d;\mathbb{R}^d)$, we now apply Proposition C.4.2(c) to infer that $p \in W^{1,2}_{r_1,\mathrm{loc}}(Q(a_0,b_0))$ and it satisfies the equation $\partial_t p - A^*_y p = 0$, where

$$A^* = \mathrm{div}(Q\nabla) - F\cdot\nabla - V - \mathrm{div}F$$

is the formal adjoint of A. As a consequence, q belongs to $W^{1,2}_{r_1,\mathrm{loc}}(Q(a_0,b_0)) \cap L^{r_1}(Q(a_0,b_0))$ and solves the parabolic problem

$$\begin{cases} \partial_t q - \mathrm{div}(Q\nabla q) = -F\cdot\nabla q - Vq - q\,\mathrm{div}F + p\partial_t(\vartheta^{\frac{k}{2}}), & \text{in } Q(a_0,b_0), \\ q(a_0,y) = 0, & y \in \mathbb{R}^d. \end{cases} \tag{3.18}$$

Since we proved that the right hand side belongs to $L^{r_1}(Q(a_0,b_0))$, then by [33, Theorem IV.9.1] it follows that $q \in W^{1,2}_{r_1}(Q(a_0,b_0))$.

If $r_1 < d+2$, then $\nabla q \in L^{s_1}(Q(a_0,b_0))$ for $1/s_1 = 1/r_1 - 1/(d+2)$ according to the Sobolev embedding theorem. In this case we iterate the procedure described above with

$$\begin{aligned} &\frac{1}{r_{n+1}} = \left(1-\frac{2}{k}\right)\frac{1}{s_n} + \frac{2}{k}, \\ &\frac{1}{s_n} = \frac{1}{r_n} - \frac{1}{d+2}, \\ &s_0 = k, \end{aligned}$$

for every $n \in \mathbb{N}$. If $r_n < d+2$ for every $n \in \mathbb{N}$, then $0 \le s_n \le s_{n+1}$. If we take $s = \lim_{n\to\infty} s_n$, we derive that

$$\frac{1}{s} = \left(1-\frac{2}{k}\right)\frac{1}{s} + \frac{2}{k} - \frac{1}{d+2} < 0,$$

where we used that $k > 2(d+2)$. As a result, there exists $n \in \mathbb{N}$ such that $r_n > d+2$, so $\nabla q \in L^\infty(Q(a_0,b_0))$ by the Sobolev embedding theorem. Otherwise, if $r_n = d+2$ for some $n \in \mathbb{N}$, then $s_n < \infty$ is arbitrary, thus $r_{n+1} > d+2$, taking s_n sufficiently large and since $k > 2(d+2)$.

To sum up, $\nabla p \in L^\infty(Q(a,b))$. If we now take into account Theorem 3.1.5, we finally prove that $\nabla p \in L^1(Q(a,b))$ observing that

$$\int_{Q(a_0,b_0)} |\nabla q|\,dt\,dy \le \left(\int_{Q(a_0,b_0)} \frac{|\nabla q|^2}{q}\,dt\,dy\right)^{1/2} \left(\int_{Q(a_0,b_0)} q\,dt\,dy\right)^{1/2}. \qquad \square$$

Moreover, a result similar to [41, Theorem 5.2] is valid.

Theorem 3.1.7. *Assume Hypothesis 3.1.4 and that q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i,j,h=1,\dots,d$. Then $p \in W^{1,2}_{k/2}(Q(a_1,b_1))$.*

Proof. Let q and ϑ be as in Lemma 3.1.6. Since $\nabla q \in L^\infty(Q(a_0,b_0))$ we have

$$\begin{aligned}\int_{Q(a_0,b_0)} |F|^{\frac{k}{2}}|\nabla q|^{\frac{k}{2}}\,dt\,dy &= \int_{Q(a_0,b_0)} |F|^{\frac{k}{2}}|\nabla q|^{\frac{k-2}{2}}\frac{|\nabla q|}{\sqrt{q}}\sqrt{q}\,dt\,dy\\ &\le \|\nabla q\|_\infty^{\frac{k-2}{2}}\left(\int_{Q(a_0,b_0)}\frac{|\nabla p|^2}{p}\,dt\,dy\right)^{\frac12}\left(\int_{Q(a_0,b_0)}|F|^k q\,dt\,dy\right)^{\frac12}.\end{aligned}$$

Considering Theorem 3.1.5 and (3.4), we obtain that $F\cdot\nabla q \in L^{\frac{k}{2}}(Q(a_0,b_0))$. In a similar way we have

$$\begin{aligned}\int_{Q(a_0,b_0)} |\mathrm{div}F|^{\frac{k}{2}}q^{\frac{k}{2}}\,dt\,dy &= \int_{Q(a_0,b_0)} |\mathrm{div}F|^{\frac{k}{2}}q^{\frac{k-2}{2}}q^{\frac12}q^{\frac12}\,dt\,dy\\ &\le \|q\|_\infty^{\frac{k-2}{2}}\left(\int_{Q(a_0,b_0)}|\mathrm{div}F|^k q\,dt\,dy\right)^{\frac12}\left(\int_{Q(a_0,b_0)} q\,dt\,dy\right)^{\frac12}.\end{aligned}$$

Inequality (3.17) implies that $q\,\mathrm{div}F \in L^{\frac{k}{2}}(Q(a_0,b_0))$. If we repeat the computation with V instead of $\mathrm{div}F$, we find that Vq belongs to $L^{\frac{k}{2}}(Q(a_0,b_0))$ as well. As in the proof of Lemma 3.1.6, q solves the parabolic problem (3.18). Since the right hand side belongs to $L^{\frac{k}{2}}(Q(a_0,b_0))$, we obtain that $q \in W^{1,2}_{k/2}(Q(a,b))$. Hence, $p \in W^{1,2}_{k/2}(Q(a_1,b_1))$. ☐

It is possible to prove even more regularity on ∇p, as the following result shows.

Theorem 3.1.8. *Assume Hypothesis 3.1.4 and that q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i,j,h=1,\dots,d$. Then $\nabla p \in \mathcal{H}^{\frac{k}{2},1}(Q(a_1,b_1))$.*

Proof. In view of Theorem 3.1.7, we are left to show that

$$\partial_t\nabla p(\cdot,x,\cdot) \in (W^{0,1}_{(k/2)'}(Q(a_1,b_1)))'.$$

Let q and ϑ be as in Lemma 3.1.6 and consider $\varphi \in C_c^{1,2}(Q(a,b))$. By Lemma 1.3.3, we have

$$\int_{Q(a,b)}(\partial_t\varphi(t,y)+A\varphi(t,y))p(t,x,y)\,dt\,dy = \int_{\mathbb{R}^d}[p(t,x,y)\varphi(t,y)]_{t=a}^{t=b}\,dy.$$

Substituting $\vartheta^{\frac{k}{2}}\varphi$ instead of φ in the previous equation, we get

$$\int_{Q(a,b)}\left(q\partial_t\varphi - \langle Q\nabla\varphi,\nabla q\rangle + \langle F,\nabla\varphi\rangle q - V\varphi q + p\varphi\partial_t\vartheta^{\frac{k}{2}}\right)dt\,dy = 0.$$

We replace again φ by the difference quotients with respect to the variable y

$$\tau_{-h}\varphi(t,y) = \frac{\varphi(t,y-he_j)-\varphi(t,y)}{|h|},$$

for $(t,y) \in Q(a,b)$, $0 \neq h \in \mathbb{R}$ and we obtain

$$\begin{aligned}
&\int_{Q(a,b)} q\partial_t(\tau_{-h}\varphi)\,dt\,dy - \int_{Q(a,b)} \langle Q\nabla(\tau_{-h}\varphi), \nabla q\rangle\,dt\,dy \\
&+ \int_{Q(a,b)} \langle F, \nabla(\tau_{-h}\varphi)\rangle q\,dt\,dy - \int_{Q(a,b)} Vq(\tau_{-h}\varphi)\,dt\,dy \\
&+ \int_{Q(a,b)} p(\tau_{-h}\varphi)\partial_t\vartheta^{\frac{k}{2}}\,dt\,dy = I_1 - I_2 + I_3 - I_4 + I_5 = 0,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{Q(a,b)} q\partial_t(\tau_{-h}\varphi)\,dt\,dy, & I_2 &= \int_{Q(a,b)} \langle Q\nabla(\tau_{-h}\varphi), \nabla q\rangle\,dt\,dy, \\
I_3 &= \int_{Q(a,b)} \langle F, \nabla(\tau_{-h}\varphi)\rangle q\,dt\,dy, & I_4 &= \int_{Q(a,b)} Vq(\tau_{-h}\varphi)\,dt\,dy, \\
I_5 &= \int_{Q(a,b)} p(\tau_{-h}\varphi)\partial_t\vartheta^{\frac{k}{2}}\,dt\,dy.
\end{aligned}$$

By a change of variables we have

$$I_1 = \int_{Q(a,b)} (\tau_h q)\partial_t\varphi\,dt\,dy$$

and

$$\begin{aligned}
I_2 = \frac{1}{|h|}\int_{Q(a,b)} \big(&\langle Q(y+he_j)\nabla\varphi(t,y), \nabla q(t,y+he_j)\rangle \\
&- \langle Q(y)\nabla\varphi(t,y), \nabla q(t,y)\rangle\big)\,dt\,dy.
\end{aligned}$$

Summing and subtracting $|h|^{-1}\int_{Q(a,b)}\langle Q(y+he_j)\nabla\varphi(t,y), \nabla q(t,y)\rangle\,dt\,dy$ in the previous expression yields

$$\begin{aligned}
I_2 = \int_{Q(a,b)} \big(&\langle Q(y+he_j)\nabla\varphi(t,y), \nabla\tau_h q(t,y)\rangle \\
&+ \langle \tau_h Q(y)\nabla\varphi(t,y), \nabla q(t,y)\rangle\big)\,dt\,dy.
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
I_3 &= \int_{Q(a,b)} \big(\tau_h q(t,y)\langle F(y+he_j), \nabla\varphi(t,y)\rangle \\
&\qquad + q(t,y)\langle \tau_h F(t,y), \nabla\varphi(t,y)\rangle\big)\,dt\,dy, \\
I_4 &= \int_{Q(a,b)} \left(\tau_h V(y)q(t,y) + V(y+he_j)\tau_h q(t,y)\right)\varphi(t,y)\,dt\,dy, \\
I_5 &= \int_{Q(a,b)} (\tau_h p)\varphi\partial_t\vartheta^{\frac{k}{2}}\,dt\,dy.
\end{aligned}$$

Since $q_{ij} \in C_b^1(\mathbb{R}^d)$, applying the Cauchy-Schwarz inequality and Hölder's inequality we deduce that

$$|I_2| \leq c\left(\|\nabla \tau_h q\|_{L^{k/2}(Q(a,b))} + \|\nabla q\|_{L^{k/2}(Q(a,b))}\right) \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))}.$$

Moreover,

$$\begin{aligned}
|I_3| \leq & \left(\int_{Q(a,b)} |\tau_h q(t,y)|^{\frac{k}{2}} |F(y+he_j)|^{\frac{k}{2}}\, dt\, dy\right)^{\frac{2}{k}} \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))} \\
& + \left(\int_{Q(a,b)} q^{\frac{k}{2}} |\tau_h F|^{\frac{k}{2}}\, dt\, dy\right)^{\frac{2}{k}} \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))} \\
& \leq \|\tau_h q\|_\infty^{\frac{k-2}{k}} \left(\int_{Q(a,b)} \frac{|\tau_h p(t,y)|^2}{p}\, dt\, dy\right)^{\frac{1}{k}} \left(\int_{Q(a,b)} |F(y+he_j)|^k\, p\, dt\, dy\right)^{\frac{1}{k}} \times \\
& \times \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))} \\
& + \|q\|_{L^\infty(Q(a,b))}^{\frac{k-2}{k}} \left(\int_{Q(a,b)} |\tau_h F|^{\frac{k}{2}}\, q\, dt\, dy\right)^{\frac{2}{k}} \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|I_4| \leq & \|\tau_h q\|_\infty^{\frac{k-2}{k}} \left(\int_{Q(a,b)} |V(y+he_j)|^k\, p\, dt\, dy\right)^{\frac{1}{k}} \left(\int_{Q(a,b)} \frac{|\tau_h p(t,y)|^2}{p}\, dt\, dy\right)^{\frac{1}{k}} \times \\
& \times \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))} \\
& + \|q\|_{L^\infty(Q(a,b))}^{\frac{k-2}{k}} \left(\int_{Q(a,b)} |\tau_h V|^{\frac{k}{2}}\, p\, dt\, dy\right)^{\frac{2}{k}} \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))}.
\end{aligned}$$

Finally,

$$|I_5| \leq c\, \|\tau_h p\|_{L^{k/2}(Q(a,b))} \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))}.$$

Hence,

$$\begin{aligned}
\left|\int_{Q(a,b)} (\tau_h q)\partial_t \varphi\, dt\, dy\right| \leq c \Bigg[& \|\nabla \tau_h q\|_{L^{k/2}(Q(a,b))} + \|\nabla q\|_{L^{k/2}(Q(a,b))} \\
& + \|\tau_h q\|_\infty^{\frac{k-2}{k}} \left\|\frac{\tau_h p}{\sqrt{p}}\right\|_{L^2(Q(a,b))}^{\frac{2}{k}} \left(\int_{Q(a,b)} |F(y+he_j)|^k\, p(t,y)\, dt\, dy\right)^{\frac{1}{k}} \\
& + \|\tau_h q\|_\infty^{\frac{k-2}{k}} \left\|\frac{\tau_h p}{\sqrt{p}}\right\|_{L^2(Q(a,b))}^{\frac{2}{k}} \left(\int_{Q(a,b)} |V(y+he_j)|^k\, p(t,y)\, dt\, dy\right)^{\frac{1}{k}} \\
& + \|q\|_{L^\infty(Q(a,b))}^{\frac{k-2}{k}} \Bigg\{ \left(\int_{Q(a,b)} |\tau_h F|^{\frac{k}{2}}\, q(t,y)\, dt\, dy\right)^{\frac{2}{k}} \\
& + \left(\int_{Q(a,b)} |\tau_h V|^{\frac{k}{2}}\, q(t,y)\, dt\, dy\right)^{\frac{2}{k}} \Bigg\} + \|\tau_h p\|_{L^{k/2}(Q(a,b))} \Bigg] \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))}.
\end{aligned}$$

As $p \in W^{1,2}_{k/2}(Q(a,b))$ by Theorem 3.1.7, it follows that $\nabla \tau_h q \to \nabla D_j q$ in $L^{\frac{k}{2}}(Q(a,b))$ as $h \to 0$, which implies the boundedness of $\|\nabla \tau_h q\|_{L^{\frac{k}{2}}(Q(a,b))}$. Similarly, we may infer the boundedness of $\|\tau_h p/\sqrt{p}\|_{L^2(Q(a,b))}$ from Theorem 3.1.5. As $\nabla p \in L^\infty(Q(a,b))$ by Lemma 3.1.6, the difference quotients $\tau_h q$ converge weak* in $L^\infty(Q(a,b))$ to $D_j q$, where also $\|\tau_h q\|_\infty$ is bounded. Boundedness of the integrals involving F can easily be deduced from the fact that $F \in C^{1+\zeta}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, $V \in C^{\zeta}_{\text{loc}}(\mathbb{R}^d)$ and the mean value theorem. All together, we see that for a certain constant C, we have

$$\left| \int_{Q(a,b)} (\tau_h q)\partial_t \varphi \, dt \, dy \right| \leq C \, \|\varphi\|_{W^{0,1}_{(k/2)'}(Q(a,b))} ,$$

for all $\varphi \in C^{1,2}_c(Q(a,b))$.

By density, this estimate extends to $\varphi \in W^{0,1}_{(k/2)'}(Q(a,b))$ and it follows that the elements $\tau_h q$ are uniformly bounded in $(W^{0,1}_{(k/2)'}(Q(a,b)))'$. Thus, by reflexivity, we see that as $h \to 0$ we find cluster-points in $(W^{0,1}_{(k/2)'}(Q(a,b)))'$. But testing against functions in $C^\infty_c(Q(a,b))$, we find that the only possible cluster point is $D_j q$. This yields $\partial_t D_j p \in (W^{0,1}_{(k/2)'}(Q(a,b)))'$ and finishes the proof. $\square$

3.2 Estimates for the derivatives of the kernel in case of bounded diffusion coefficients

With the help of Theorem 2.3.6, we can now prove an upper bound for $|w\nabla p|$ that does not depend on the $\|\cdot\|_\infty$-bound of the diffusion coefficients.

Theorem 3.2.1. *Assume Hypothesis 3.1.4 and that q_{ij}, $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for $i,j,h = 1,\ldots,d$. Then there is a constant $C > 0$ depending only on d, k and η (but not depending on $\|Q\|_\infty$) such that*

$$\begin{aligned}
&|w(t,y)\nabla p(t,x,y)| \\
&\leq C \Bigg\{ B_1 \, \Xi_1(a_0,b_0)^{\frac{1}{2}} \, \|wp\|^{\frac{1}{2}}_{L^\infty(Q(a,b))} \\
&\quad + \left(B_2 \, \Xi_1(a_0,b_0)^{\frac{2}{k}} + B_3 \, \Xi_2(a_0,b_0)^{\frac{2}{k}} \right) \|wp\|^{\frac{k-2}{k}}_{L^\infty(Q(a,b))} \\
&\quad + \left[B_4 \, \Xi_1(a_0,b_0)^{\frac{1}{k}} + B_5 \, \Xi_2(a_0,b_0)^{\frac{1}{k}} \right] \|wp\|^{\frac{k-1}{k}}_{L^\infty(Q(a,b))} \\
&\quad + \left(B_6 \, \Xi_1(a_0,b_0)^{\frac{1}{2}} + B_7 \, \Xi_2(a_0,b_0)^{\frac{1}{2}} \right) \left(\int_{Q(a,b)} \frac{|\nabla p|^2}{p} \, dt \, dy \right)^{\frac{1}{2}} \Bigg\}, \quad (3.19)
\end{aligned}$$

for all $(t,y) \in (a_1,b_1) \times \mathbb{R}^d$ and fixed $x \in \mathbb{R}^d$, where B_i, $i = 1,\ldots,7$ are positive constants depending only on c_i, $i = 1,\ldots,11$, b, b_1 and k and, for $i = 1,2$, we *define*

$$\xi_{W_i}(t,x) := \int_{\mathbb{R}^d} p(t,x,y)W_i(t,y)\,dy, \quad \Xi_i(a_0,b_0) := \int_{a_0}^{b_0} \xi_{W_i}(t,x)\,dt.$$

Proof. We first prove the theorem assuming that the weight function w, along with its first order partial derivatives and its second order partial derivatives of the form $D_{ij}w$ and $\partial_t D_i w$ are bounded. We fix $a_0 < a < a_2 < a_1 < b_1 < b_2 < b < b_0$.

We show that

$$\nabla(wp) \in \mathcal{H}^{\frac{k}{2},1}(Q(a_2,b_2)) \cap L^\infty(Q(a_2,b_2)). \tag{3.20}$$

We apply Lemma 3.1.6 and Theorem 3.1.8 to infer that that

$$\nabla p \in \mathcal{H}^{\frac{k}{2},1}(Q(a_2,b_2)) \cap L^\infty(Q(a_2,b_2)).$$

By Corollary 3.1.3 we have $p \in L^\infty(Q(a_2,b_2))$. Moreover, arguing as in the proof of Theorem 3.1.5, Hypothesis 3.1.4 and Proposition 1.6.3 lead to $\Gamma_1(k/2,x,a_0,b_0)$, $\Gamma_2(k/2,x,a_0,b_0) < \infty$. As a consequence, Lemma 2.3.1 and Remark 2.3.3 imply that $p \in \mathcal{H}^{\frac{k}{2},1}(Q(a_2,b_2))$. Thus, we get (3.20).

Let $\vartheta \in C^\infty(\mathbb{R})$ be such that

- $\vartheta(t) = 1$ for $t \in [a_1,b_1]$,
- $\vartheta(t) = 0$ for $t \le a_2$, $t \ge b_2$,
- $0 \le \vartheta \le 1$ and $|\vartheta'| \le \frac{2}{b_2-b_1}$.

We define

$$q(t,y) := \vartheta^{k/2}(t)p(t,x,y)$$

and we note that $\nabla(wq) \in \mathcal{H}^{\frac{k}{2},1}(Q(a_2,b_2)) \cap L^\infty(Q(a_2,b_2))$. Furthermore, given $\varphi \in C_c^\infty(Q(a_2,b_2))$, we write

$$\psi(t,y) := \vartheta^{k/2}(t)w(t,y)D_h\varphi(t,y),$$

with $h = 1,\dots,d$. Applying Lemma 1.3.3 for each $h = 1,\dots,d$ yields

$$\int_{Q(a_2,b_2)} (\partial_t\psi(t,y) + A\psi(t,y))p(t,x,y)\,dt\,dy = 0.$$

Integrating by parts, we get

$$\int_{Q(a_2,b_2)} \big[p\partial_t\psi - \langle Q\nabla\psi, \nabla p\rangle + \langle F, \nabla\psi\rangle p - V\psi p\big]\,dt\,dy = 0.$$

Replacing the expression of the functions ψ and q, after some computations we derive that

$$\int_{Q(a_2,b_2)} \Big[\frac{k}{2}\vartheta'\vartheta^{\frac{k-2}{2}} wp(D_h\varphi) + wq(\partial_t D_h\varphi) - \langle Q\nabla w, \nabla q\rangle(D_h\varphi) \\ - \langle Q\nabla D_h\varphi, w\nabla q\rangle + \langle F, q\nabla w\rangle(D_h\varphi) + \langle F, \nabla D_h\varphi\rangle wq \\ - Vwq(D_h\varphi) + q(\partial_t w)(D_h\varphi)\Big]\, dt\, dy = 0.$$

Integrating by parts again in order to remove the derivative D_h in front of φ, we have that

$$\int_{Q(a_2,b_2)} \Big[-\frac{k}{2}\vartheta'\vartheta^{\frac{k-2}{2}} w(D_h p)\varphi - \frac{k}{2}\vartheta'\vartheta^{\frac{k-2}{2}} p(D_h w)\varphi + (\partial_t D_h(wq))\varphi \\ + \langle (D_hQ)\nabla w, \nabla q\rangle\varphi + \langle Q(D_h\nabla w), \nabla q\rangle\varphi + \langle Q\nabla w, D_h\nabla q\rangle\varphi \\ + w\langle (D_hQ)\nabla q, \nabla\varphi\rangle + \langle QD_h(w\nabla q), \nabla\varphi\rangle - q\langle F, D_h\nabla w\rangle\varphi \\ - q\langle D_hF, \nabla w\rangle\varphi - (D_hq)\langle F, \nabla w\rangle\varphi - w(D_hq)\langle F, \nabla\varphi\rangle \\ - q(D_hw)\langle F, \nabla\varphi\rangle - wq\langle D_hF, \nabla\varphi\rangle + Vw(D_hq)\varphi + Vq(D_hw)\varphi \\ + (D_hV)wq\varphi - (\partial_t D_h w)q\varphi - (\partial_t w)(D_hq)\varphi\Big]\, dt\, dy = 0. \tag{3.21}$$

Since

$$\int_{Q(a_2,b_2)} \langle Q\nabla w, D_h\nabla q\rangle\varphi\, dt\, dy = -\int_{Q(a_2,b_2)} \Big[(D_hq)\mathrm{div}(Q\nabla w)\varphi \\ + (D_hq)\langle Q\nabla w, \nabla\varphi\rangle\Big]\, dt\, dy$$

and

$$\int_{Q(a_2,b_2)} \langle QD_h(w\nabla q), \nabla\varphi\rangle\, dt\, dy \\ = \int_{Q(a_2,b_2)} \Big[\langle Q\nabla D_h(wq), \nabla\varphi\rangle - q\langle QD_h(\nabla w), \nabla\varphi\rangle \\ - (D_hq)\langle Q\nabla w, \nabla\varphi\rangle\Big]\, dt\, dy,$$

we can adjust the terms in (3.21) to obtain that

$$\int_{Q(a_2,b_2)} [\langle Q\nabla u, \nabla\varphi\rangle + \varphi\partial_t u]\, dtdy = \int_{Q(a_2,b_2)} f\varphi\, dtdy + \int_{Q(a_2,b_2)} \langle h, \nabla\varphi\rangle\, dtdy,$$

where

$$
\begin{aligned}
u =& D_h(wq),\\
f =& \frac{k}{2}\vartheta'\vartheta^{\frac{k-2}{2}} w(D_h p) + \frac{k}{2}\vartheta'\vartheta^{\frac{k-2}{2}} p(D_h w) - \langle (D_h Q)\nabla w, \nabla q\rangle - \langle Q(D_h\nabla w), \nabla q\rangle\\
&+ (D_h q)\operatorname{div}(Q\nabla w) + q\langle D_h \nabla w, F\rangle + q\langle \nabla w, D_h F\rangle + (D_h q)\langle \nabla w, F\rangle\\
&- Vw(D_h q) - Vq(D_h w) - wq(D_h V) + (\partial_t D_h w)q + (\partial_t w)(D_h q),\\
h =& 2(D_h q)Q\nabla w - w(D_h Q)(\nabla q) + qQD_h\nabla w + wF(D_h q) + qF(D_h w)\\
&+ wq(D_h F).
\end{aligned}
$$

We now aim to apply Theorem 2.3.6 to the function u and infer that there exists a constant C, depending only on d, η and k, but not on $\|Q\|_\infty$, such that

$$
\begin{aligned}
&\|D_h(wq)\|_\infty\\
&\leq C\Big[\|D_h(wq)\|_2 + \frac{k}{b_2-b_1}\left\|\vartheta^{\frac{k-2}{2}} w(D_h p)\right\|_{\frac{k}{2}} + \frac{k}{b_2-b_1}\left\|\vartheta^{\frac{k-2}{2}} p(D_h w)\right\|_{\frac{k}{2}}\\
&\quad + \|\langle (D_h Q)\nabla w, \nabla q\rangle\|_{\frac{k}{2}} + \|\langle Q(D_h\nabla w), \nabla q\rangle\|_{\frac{k}{2}} + \|(D_h q)\operatorname{div}(Q\nabla w)\|_{\frac{k}{2}}\\
&\quad + \|q\langle D_h\nabla w, F\rangle\|_{\frac{k}{2}} + \|q\langle \nabla w, D_h F\rangle\|_{\frac{k}{2}} + \|(D_h q)\langle \nabla w, F\rangle\|_{\frac{k}{2}}\\
&\quad + \|Vw(D_h q)\|_{\frac{k}{2}} + \|Vq(D_h w)\|_{\frac{k}{2}} + \|wq(D_h V)\|_{\frac{k}{2}} + \|(\partial_t D_h w)q\|_{\frac{k}{2}}\\
&\quad + \|(\partial_t w)(D_h q)\|_{\frac{k}{2}} + \|(D_h q)Q\nabla w\|_k + \|w(D_h Q)(\nabla q)\|_k + \|qQD_h\nabla w\|_k\\
&\quad + \|wF(D_h q)\|_k + \|qF(D_h w)\|_k + \|wqD_h F\|_k\Big].
\end{aligned}
$$

Summing over $h = 1, \ldots, d$ and since $\|\nabla(wq)\|_\infty \geq \|w\nabla q\|_\infty - \|q\nabla w\|_\infty$ yields

$$
\begin{aligned}
&\|w\nabla q\|_\infty\\
&\leq C\Bigg[\|w\nabla q\|_2 + \|q\nabla w\|_2 + \frac{k}{b_2-b_1}\left\|\vartheta^{\frac{k-2}{2}} w\nabla p\right\|_{\frac{k}{2}} + \frac{k}{b_2-b_1}\left\|\vartheta^{\frac{k-2}{2}} p\nabla w\right\|_{\frac{k}{2}}\\
&\quad + \|\langle \nabla Q\nabla w, \nabla q\rangle\|_{\frac{k}{2}} + \left\|QD^2 w\nabla q\right\|_{\frac{k}{2}} + \|(\nabla q)\operatorname{div}(Q\nabla w)\|_{\frac{k}{2}}\\
&\quad + \left\|q(D^2 w)F\right\|_{\frac{k}{2}} + \|q\langle \nabla w, \nabla F\rangle\|_{\frac{k}{2}} + \|(\nabla q)\langle \nabla w, F\rangle\|_{\frac{k}{2}} + \|Vw\nabla q\|_{\frac{k}{2}}\\
&\quad + \|Vq\nabla w\|_{\frac{k}{2}} + \|wq\nabla V\|_{\frac{k}{2}} + \|(\partial_t \nabla w)q\|_{\frac{k}{2}} + \|(\partial_t w)(\nabla q)\|_{\frac{k}{2}}\\
&\quad + \|\langle Q\nabla w, \nabla q\rangle\|_k + \|w(\nabla Q)(\nabla q)\|_k + \left\|qQD^2 w\right\|_k + \|w\langle \nabla q, F\rangle\|_k\\
&\quad + \|q\langle \nabla w, F\rangle\|_k + \|wq\nabla F\|_k\Bigg] + \|q\nabla w\|_\infty. \qquad (3.22)
\end{aligned}
$$

We set

$$P := \int_{Q(a_2,b_2)} \frac{|\nabla p|^2}{p}\, dt\, dy$$

and, for a sake of simplicity, we write Ξ_i instead of $\Xi_i(a_2, b_2)$ to refer to $\int_{a_2}^{b_2} \xi_{W_i}(t,x)\, dt$ for $i = 1, 2$. We observe that Ξ_1, $\Xi_2 < \infty$ by Proposition

1.6.3. Moreover, thanks to Theorem 3.1.5, we know that $P < \infty$. Finally, we estimate the terms in the right hand side of (3.22). We start with $\|w\nabla q\|_2$. Using Hölder's inequality and Hypothesis 3.1.4(c) one obtains

$$\begin{aligned}
\|w\nabla q\|_2^2 &= \int_{Q(a_2,b_2)} w^2 |\nabla q|^2 \,dt\,dy \le \|w\nabla q\|_\infty \int_{Q(a_2,b_2)} \frac{|\nabla q|}{\sqrt{q}} \sqrt{q} w \,dt\,dy \\
&\le \|w\nabla q\|_\infty \left(\int_{Q(a_2,b_2)} \frac{|\nabla q|^2}{q} \,dt\,dy \right)^{\frac{1}{2}} \left(\int_{Q(a_2,b_2)} w^2 q \,dt\,dy \right)^{\frac{1}{2}} \\
&\le c_1^{\frac{k}{2}} \|w\nabla q\|_\infty \left(\int_{Q(a_2,b_2)} \frac{|\nabla p|^2}{p} \,dt\,dy \right)^{\frac{1}{2}} \left(\int_{Q(a_2,b_2)} \xi_{W_1}(t,x) \,dt \right)^{\frac{1}{2}} \\
&= c_1^{\frac{k}{2}} \|w\nabla q\|_\infty P^{\frac{1}{2}} \Xi_1^{\frac{1}{2}}.
\end{aligned}$$

Hence, we have

$$\|w\nabla q\|_2 \le c_1^{\frac{k}{4}} P^{\frac{1}{4}} \Xi_1^{\frac{1}{4}} \|w\nabla q\|_\infty^{\frac{1}{2}}.$$

Similarly, we get

$$\left\|\vartheta^{\frac{k-2}{2}} w\nabla p\right\|_{\frac{k}{2}} \le c_1 P^{\frac{1}{k}} \Xi_1^{\frac{1}{k}} \|w\nabla q\|_\infty^{\frac{k-2}{k}},$$

$$\|\langle \nabla Q \nabla w, \nabla q\rangle\|_{\frac{k}{2}} \le \eta^{-1} c_2 c_7 P^{\frac{1}{k}} \Xi_1^{\frac{1}{k}} \|w\nabla q\|_\infty^{\frac{k-2}{k}},$$

$$\left\|QD^2 w\nabla q\right\|_{\frac{k}{2}} \le c_3 P^{\frac{1}{k}} \Xi_1^{\frac{1}{k}} \|w\nabla q\|_\infty^{\frac{k-2}{k}},$$

$$\|(\nabla q)\mathrm{div}(Q\nabla w)\|_{\frac{k}{2}} \le d(\eta^{-1} c_2 c_7 + c_3) P^{\frac{1}{k}} \Xi_1^{\frac{1}{k}} \|w\nabla q\|_\infty^{\frac{k-2}{k}},$$

where we have applied here (3.1). Moreover,

$$\|(\nabla q)\langle \nabla w, F\rangle\|_{\frac{k}{2}} \le \eta^{-1} c_2 c_6 P^{\frac{1}{k}} \Xi_2^{\frac{1}{k}} \|w\nabla q\|_\infty^{\frac{k-2}{k}},$$

$$\|Vw\nabla q\|_{\frac{k}{2}} \le c_5^2 P^{\frac{1}{k}} \Xi_2^{\frac{1}{k}} \|w\nabla q\|_\infty^{\frac{k-2}{k}},$$

$$\|(\partial_t w)(\nabla q)\|_{\frac{k}{2}} \le c_4 P^{\frac{1}{k}} \Xi_1^{\frac{1}{k}} \|w\nabla q\|_\infty^{\frac{k-2}{k}},$$

$$\|\langle Q\nabla w, \nabla q\rangle\|_k \le c_2 P^{\frac{1}{2k}} \Xi_1^{\frac{1}{2k}} \|w\nabla q\|_\infty^{\frac{k-1}{k}},$$

$$\|w(\nabla Q)(\nabla q)\|_k \le c_7 P^{\frac{1}{2k}} \Xi_1^{\frac{1}{2k}} \|w\nabla q\|_\infty^{\frac{k-1}{k}},$$

$$\|w\langle \nabla q, F\rangle\|_k \le c_6 P^{\frac{1}{2k}} \Xi_2^{\frac{1}{2k}} \|w\nabla q\|_\infty^{\frac{k-1}{k}}.$$

In addition, we estimate $\|q\nabla w\|_2^2$ as follows

$$\begin{aligned}\|q\nabla w\|_2^2 &= \int_{Q(a_2,b_2)} q^2 |\nabla w|^2 \,dt\,dy \le \eta^{-2} c_2^2 \|wq\|_\infty \int_{Q(a_2,b_2)} W_1^{\frac{1}{k}} q\,dt\,dy \\ &\le \eta^{-2} c_2^2 \|wq\|_\infty \,\Xi_1.\end{aligned}$$

Thus, we have

$$\|q\nabla w\|_2 \le \eta^{-1} c_2 \Xi_1^{\frac{1}{2}} \|wq\|_\infty^{\frac{1}{2}}.$$

In a similar way, we obtain

$$\left\|\vartheta^{\frac{k-2}{2}} p\nabla w\right\|_{\frac{k}{2}} \le \eta^{-1} c_2 \Xi_1^{\frac{2}{k}} \|wq\|_\infty^{\frac{k-2}{k}},$$

$$\left\|q(D^2 w)F\right\|_{\frac{k}{2}} \le \eta^{-1} c_3 c_6 \Xi_2^{\frac{2}{k}} \|wq\|_\infty^{\frac{k-2}{k}},$$

$$\|q\langle\nabla w, \nabla F\rangle\|_{\frac{k}{2}} \le \eta^{-1} c_2 c_8 \Xi_2^{\frac{2}{k}} \|wq\|_\infty^{\frac{k-2}{k}},$$

$$\|Vq\nabla w\|_{\frac{k}{2}} \le \eta^{-1} c_2 c_5^2 \Xi_2^{\frac{2}{k}} \|wq\|_\infty^{\frac{k-2}{k}},$$

$$\|wq\nabla V\|_{\frac{k}{2}} \le c_9 \Xi_2^{\frac{2}{k}} \|wq\|_\infty^{\frac{k-2}{k}},$$

$$\|(\partial_t \nabla w)q\|_{\frac{k}{2}} \le c_{11} \Xi_1^{\frac{2}{k}} \|wq\|_\infty^{\frac{k-2}{k}},$$

$$\left\|qQD^2 w\right\|_k \le c_3 \Xi_1^{\frac{1}{k}} \|wq\|_\infty^{\frac{k-1}{k}},$$

$$\|q\langle\nabla w, F\rangle\|_k \le \eta^{-1} c_2 c_6 \Xi_2^{\frac{1}{k}} \|wq\|_\infty^{\frac{k-1}{k}},$$

$$\|wq\nabla F\|_k \le c_8 \Xi_2^{\frac{1}{k}} \|wq\|_\infty^{\frac{k-1}{k}}.$$

Finally, we get

$$\|q\nabla w\|_\infty \le \|q\|_\infty^{\frac{k-1}{k}} \left\|q(1+|\nabla w|^2)^{\frac{k}{2}}\right\|_\infty^{\frac{1}{k}}.$$

We now estimate $\|q(1+|\nabla w|^2)^{\frac{k}{2}}\|_\infty$ by applying Theorem 3.1.1 with w replaced by $\tilde{w} = (1+|\nabla w|^2)^{\frac{k}{2}}$. First, we check the assumptions using Hypothesis 3.1.4(c):

$$\tilde{w}^{\frac{2}{k}} = 1 + |\nabla w|^2 \le 1 + \eta^{-2} c_2^2 W_1^{\frac{1}{k}} \le (1+\eta^{-2}c_2^2) W_1^{\frac{2}{k}},$$

$$|Q\nabla\tilde{w}| = k(1+|\nabla w|^2)^{\frac{k-2}{2}} |(QD^2 w)\nabla w| \le k\tilde{w}^{\frac{k-2}{k}} |QD^2 w||\nabla w| \le kc_3 \tilde{w}^{\frac{k-1}{k}} W_1^{\frac{1}{k}},$$

$$\begin{aligned}|\mathrm{div}(Q\nabla\tilde{w})| \le & d|\nabla(Q\nabla\tilde{w})| \le d|\nabla Q||\nabla\tilde{w}| + d|QD^2\tilde{w}| \\ \le & d|\nabla Q|k\tilde{w}^{\frac{k-2}{k}}|D^2w||\nabla w| + d\Big[(k-2)\tilde{w}^{-\frac{2}{k}}|\nabla\tilde{w}||QD^2w||\nabla w| \\ & + k\tilde{w}^{\frac{k-2}{k}}|D^3w||Q\nabla w| + k\tilde{w}^{\frac{k-2}{k}}|QD^2w||D^2w|\Big] \\ \le & kd[\eta^{-2}c_2c_3c_7 + (k-1)\eta^{-1}c_3^2 + c_2c_{10}]\tilde{w}^{\frac{k-2}{k}}W_1^{\frac{2}{k}},\end{aligned}$$

$$|\partial_t\tilde{w}| \le k(1+|\nabla w|^2)^{\frac{k-2}{2}}|\nabla w||\partial_t\nabla w| \le k\eta^{-1}c_2c_{11}\tilde{w}^{\frac{k-2}{k}}W_1^{\frac{2}{k}},$$

$$\tilde{w}^{\frac{1}{k}}V^{\frac{1}{2}} \le (1+|\nabla w|)V^{\frac{1}{2}} \le (c_5+\eta^{-1}c_2c_5)W_2^{\frac{1}{k}},$$

$$\tilde{w}^{\frac{1}{k}}|F| \le (1+|\nabla w|)|F| \le (c_6+\eta^{-1}c_2c_6)W_2^{\frac{1}{k}}.$$

Moreover, $\tilde{w}^{-2}\nabla\tilde{w}$ and $\tilde{w}^{-2}\partial_t\tilde{w}$ are bounded on $Q(a_0,b_0)$ as we assumed in Hypothesis 3.1.4(b) that the functions $|\nabla w|^{-k-1}D^2w$ and $|\nabla w|^{-k-1}\partial_t\nabla w$ are bounded. Hence, the assumptions of Theorem 3.1.1 hold true with w replaced by $\tilde{w}$ and with the constants $c_1,\dots,c_6$ replaced, respectively, by $1+\eta^{-2}c_2^2$, kc_3, $kd[\eta^{-2}c_2c_3c_7+(k-1)\eta^{-1}c_3^2+c_2c_{10}]$, $k\eta^{-1}c_2c_{11}$, $c_5+\eta^{-1}c_2c_5$ and $c_6+\eta^{-1}c_2c_6$. Thus, we obtain that

$$\begin{aligned}\left\|q(1+|\nabla w|^2)^{\frac{k}{2}}\right\|_\infty \le C\Bigg[& \left(c_2^k + \frac{c_2^k}{(b_2-b_1)^{\frac{k}{2}}} + c_3^k + c_2^{\frac{k}{2}}c_3^{\frac{k}{2}}c_7^{\frac{k}{2}} + c_2^{\frac{k}{2}}c_{10}^{\frac{k}{2}} + c_2^{\frac{k}{2}}c_{11}^{\frac{k}{2}}\right)\Xi_1 \\ & + \left(c_6^k + c_2^kc_6^k + c_3^{\frac{k}{2}}c_6^{\frac{k}{2}} + c_2^{\frac{k}{2}}c_3^{\frac{k}{2}}c_6^{\frac{k}{2}} + c_5^k + c_2^kc_5^k\right)\Xi_2\Bigg].\end{aligned}$$

Consequently, we estimate the last term in the right hand side of (3.22) as follows

$$\begin{aligned}\|q\nabla w\|_\infty \le C\Bigg[& \left(c_2 + \frac{c_2}{(b_2-b_1)^{\frac{1}{2}}} + c_3 + c_2^{\frac{1}{2}}c_3^{\frac{1}{2}}c_7^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_{10}^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_{11}^{\frac{1}{2}}\right)\Xi_1^{\frac{1}{k}} \\ & + \left(c_6 + c_2c_6 + c_3^{\frac{1}{2}}c_6^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_3^{\frac{1}{2}}c_6^{\frac{1}{2}} + c_5 + c_2c_5\right)\Xi_2^{\frac{1}{k}}\Bigg]\|wq\|_\infty^{\frac{k-1}{k}}.\end{aligned}$$

Combining (3.22) with the above estimates yields

$$
\begin{aligned}
&\|w\nabla q\|_\infty \\
&\quad \leq C c_1^{\frac{k}{4}} P^{\frac{1}{4}} \,\Xi_1^{\frac{1}{4}} \,\|w\nabla q\|_\infty^{\frac{1}{2}} + C P^{\frac{1}{2k}} \left[(c_2 + c_7)\,\Xi_1^{\frac{1}{2k}} + c_6\,\Xi_2^{\frac{1}{2k}} \right] \|w\nabla q\|_\infty^{\frac{k-1}{k}} \\
&\qquad + C P^{\frac{1}{k}} \left[\left(\frac{c_1}{b_2 - b_1} + c_2 c_7 + c_3 + c_4 \right) \Xi_1^{\frac{1}{k}} + (c_2 c_6 + c_5^2)\,\Xi_2^{\frac{1}{k}} \right] \|w\nabla q\|_\infty^{\frac{k-2}{k}} \\
&\qquad + C c_2 \,\Xi_1^{\frac{1}{2}} \,\|wq\|_\infty^{\frac{1}{2}} \\
&\qquad + C \left[\left(\frac{c_2}{b_2 - b_1} + c_{11} \right) \Xi_1^{\frac{2}{k}} + (c_2 c_5^2 + c_3 c_6 + c_2 c_8 + c_9)\,\Xi_2^{\frac{2}{k}} \right] \|wq\|_\infty^{\frac{k-2}{k}} \\
&\qquad + C \Bigg[\left(c_2 + \frac{c_2}{(b_2 - b_1)^{\frac{1}{2}}} + c_3 + c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_7^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_{10}^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_{11}^{\frac{1}{2}} \right) \Xi_1^{\frac{1}{k}} \\
&\qquad + \left(c_6 + c_2 c_6 + c_3^{\frac{1}{2}} c_6^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_6^{\frac{1}{2}} + c_5 + c_2 c_5 + c_8 \right) \Xi_2^{\frac{1}{k}} \Bigg] \|wq\|_\infty^{\frac{k-1}{k}} .
\end{aligned}
$$

We observe that, by Young's inequality, we find

$$
C c_1^{\frac{k}{4}} P^{\frac{1}{4}} \Xi_1^{\frac{1}{4}} \,\|w\nabla q\|_\infty^{\frac{1}{2}} \leq C^2 c_1^{\frac{k}{2}} P^{\frac{1}{2}} \Xi_1^{\frac{1}{2}} + \frac{1}{4} \|w\nabla q\|_\infty .
$$

Then, setting

$$
\begin{aligned}
X &:= \|w\nabla q\|_\infty^{\frac{1}{k}} , \\
\alpha &:= C^2 c_1^{\frac{k}{2}} P^{\frac{1}{2}} \,\Xi_1^{\frac{1}{2}} + C c_2 \,\Xi_1^{\frac{1}{2}} \,\|wq\|_\infty^{\frac{1}{2}} + C \Bigg[\left(\frac{c_2}{b_2 - b_1} + c_{11} \right) \Xi_1^{\frac{2}{k}} \\
&\qquad\quad + (c_2 c_5^2 + c_3 c_6 + c_2 c_8 + c_9)\,\Xi_2^{\frac{2}{k}} \Bigg] \|wq\|_\infty^{\frac{k-2}{k}} \\
&\qquad + C \Bigg[\left(c_2 + \frac{c_2}{(b_2 - b_1)^{\frac{1}{2}}} + c_3 + c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_7^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_{10}^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_{11}^{\frac{1}{2}} \right) \Xi_1^{\frac{1}{k}} \\
&\qquad\quad + \left(c_6 + c_2 c_6 + c_3^{\frac{1}{2}} c_6^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_6^{\frac{1}{2}} + c_5 + c_2 c_5 + c_8 \right) \Xi_2^{\frac{1}{k}} \Bigg] \|wq\|_\infty^{\frac{k-1}{k}} , \\
\beta &:= C P^{\frac{1}{2k}} \left[(c_2 + c_7)\,\Xi_1^{\frac{1}{2k}} + c_6\,\Xi_2^{\frac{1}{2k}} \right], \\
\gamma &:= C P^{\frac{1}{k}} \left[\left(\frac{c_1}{b_2 - b_1} + c_2 c_7 + c_3 + c_4 \right) \Xi_1^{\frac{1}{k}} + (c_2 c_6 + c_5^2)\,\Xi_2^{\frac{1}{k}} \right],
\end{aligned}
$$

we derive that

$$
X^k \leq \frac{4}{3}\alpha + \frac{4}{3}\beta X^{k-1} + \frac{4}{3}\gamma X^{k-2}. \tag{3.23}
$$

We now prove that it leads to

$$
X \leq \frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + \left(\frac{4}{3}\alpha \right)^{\frac{1}{k}} . \tag{3.24}
$$

We consider the function

$$f(r) := r^k - \frac{4}{3}\beta r^{k-1} - \frac{4}{3}\gamma r^{k-2} - \frac{4}{3}\alpha = r^{k-2}\left(r^2 - \frac{4}{3}\beta r - \frac{4}{3}\gamma\right) - \frac{4}{3}\alpha$$
$$=: r^{k-2}g(r) - \frac{4}{3}\alpha.$$

First, we show that f is increasing in $\left(\frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + (\frac{4}{3}\alpha)^{\frac{1}{k}}, \infty\right)$. This can be seen by computing the first derivative:

$$f'(r) = (k-2)r^{k-3}g(r) + r^{k-2}g'(r).$$

Since the function g in positive and increasing in $\left(\frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + (\frac{4}{3}\alpha)^{\frac{1}{k}}, \infty\right)$, it follows that $f'(r) \geq 0$ in the given interval, so f is increasing.
Second, as in (2.55), we have that

$$f\left(\frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + \left(\frac{4}{3}\alpha\right)^{\frac{1}{k}}\right) > 0.$$

On one hand, from the previous observations we deduce that $f(r) > 0$ if $r > \frac{4}{3}\beta + \sqrt{\frac{4}{3}\gamma} + (\frac{4}{3}\alpha)^{\frac{1}{k}}$. On the other hand, by (3.23), $f(X) \leq 0$. Thus, we conclude that (3.24) holds true. Consequently, there exists a positive constant K_1 such that

$$\|w\nabla q\|_\infty \leq K_1\left(\alpha + \beta^k + \gamma^{\frac{k}{2}}\right).$$

By plugging in the previous inequality the definition of α, β, γ we get

$$\begin{aligned}
&\|w\nabla q\|_{L^\infty(Q(a_2,b_2))} \\
&\leq C\Bigg\{ c_2\Xi_1^{\frac{1}{2}}\|wq\|^{\frac{1}{2}}_{L^\infty(Q(a_2,b_2))} + \Bigg[\left(\frac{c_2}{b_2-b_1} + c_{11}\right)\Xi_1^{\frac{2}{k}} \\
&\quad + (c_2c_5^2 + c_3c_6 + c_2c_8 + c_9)\,\Xi_2^{\frac{2}{k}}\Bigg]\|wq\|^{\frac{k-2}{k}}_{L^\infty(Q(a_2,b_2))} \\
&\quad + \Bigg[\left(c_2 + \frac{c_2}{(b_2-b_1)^{\frac{1}{2}}} + c_3 + c_2^{\frac{1}{2}}c_3^{\frac{1}{2}}c_7^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_{10}^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_{11}^{\frac{1}{2}}\right)\Xi_1^{\frac{1}{k}} \\
&\qquad + \left(c_6 + c_2c_6 + c_3^{\frac{1}{2}}c_6^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_3^{\frac{1}{2}}c_6^{\frac{1}{2}} + c_5 + c_2c_5 + c_8\right)\Xi_2^{\frac{1}{k}}\Bigg]\|wq\|^{\frac{k-1}{k}}_{L^\infty(Q(a_2,b_2))} \\
&\quad + \Bigg[\left(c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b_2-b_1)^{\frac{k}{2}}} + c_2^k + c_2^{\frac{k}{2}}c_7^{\frac{k}{2}} + c_3^{\frac{k}{2}} + c_7^k + c_4^{\frac{k}{2}}\right)\Xi_1^{\frac{1}{2}} \\
&\qquad + (c_6^k + c_2^{\frac{k}{2}}c_6^{\frac{k}{2}} + c_5^k)\,\Xi_2^{\frac{1}{2}}\Bigg]P^{\frac{1}{2}}\Bigg\}.
\end{aligned}$$

Letting $a_2 \downarrow a$ and $b_2 \uparrow b$ and considering that $\int_a^b \xi_{W_j}(t,x)\,dt \le \int_{a_0}^{b_0} \xi_{W_j}(t,x)\,dt$ for $j = 1, 2$, we gain

$$
\begin{aligned}
&|w(t,y)\nabla p(t,x,y)| \\
&\le C\Bigg\{ c_2 \Xi_1(a_0,b_0)^{\frac{1}{2}} \|wp\|^{\frac{1}{2}}_{L^\infty(Q(a,b))} + \Bigg[\left(\frac{c_2}{b-b_1} + c_{11}\right) \Xi_1(a_0,b_0)^{\frac{2}{k}} \\
&\quad + (c_2c_5^2 + c_3c_6 + c_2c_8 + c_9)\Xi_2(a_0,b_0)^{\frac{2}{k}} \Bigg] \|wp\|^{\frac{k-2}{k}}_{L^\infty(Q(a,b))} \\
&\quad + \Bigg[\left(c_2 + \frac{c_2}{(b-b_1)^{\frac{1}{2}}} + c_3 + c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_7^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_{10}^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_{11}^{\frac{1}{2}} \right) \Xi_1(a_0,b_0)^{\frac{1}{k}} \\
&\quad + \left(c_6 + c_2c_6 + c_3^{\frac{1}{2}} c_6^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_6^{\frac{1}{2}} + c_5 + c_2c_5 + c_8 \right) \Xi_2(a_0,b_0)^{\frac{1}{k}} \Bigg] \|wp\|^{\frac{k-1}{k}}_{L^\infty(Q(a,b))} \\
&\quad + \Bigg[\left(c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b-b_1)^{\frac{k}{2}}} + c_2^k + c_2^{\frac{k}{2}} c_7^{\frac{k}{2}} + c_3^{\frac{k}{2}} + c_7^k + c_4^{\frac{k}{2}} \right) \Xi_1(a_0,b_0)^{\frac{1}{2}} \\
&\quad + (c_6^k + c_2^{\frac{k}{2}} c_6^{\frac{k}{2}} + c_5^k)\Xi_2(a_0,b_0)^{\frac{1}{2}} \Bigg] \left(\int_{Q(a,b)} \frac{|\nabla p|^2}{p}\,dt\,dy \right)^{\frac{1}{2}} \Bigg\}, \qquad (3.25)
\end{aligned}
$$

for all $(t,y) \in (a_1,b_1) \times \mathbb{R}^d$ and fixed $x \in \mathbb{R}^d$.

To finish the proof, it remains to remove the additional assumption on the weight w. For $\varepsilon > 0$, we define the function

$$w_\varepsilon := \frac{w}{1+\varepsilon w}.$$

We have

$$
\begin{aligned}
D_i w_\varepsilon &= (1+\varepsilon w)^{-2} D_i w, \\
\partial_t w_\varepsilon &= (1+\varepsilon w)^{-2} \partial_t w, \\
D_{ij} w_\varepsilon &= (1+\varepsilon w)^{-2} D_{ij} w - 2\varepsilon(1+\varepsilon w)^{-3}(D_i w)(D_j w), \\
\partial_t D_i w_\varepsilon &= (1+\varepsilon w)^{-2} \partial_t D_i w - 2\varepsilon(1+\varepsilon w)^{-3}(\partial_t w)(D_i w),
\end{aligned}
$$

for all $i, j = 1, \dots, d$. Then by Hypothesis 3.1.4(b) it follows that w_ε, along with its first order partial derivatives and its second order partial derivatives of the form $D_{ij} w_\varepsilon$ and $\partial_t D_i w_\varepsilon$ are bounded. If we now check Hypothesis 3.1.4(c) we have that

$$w_\varepsilon \le w \le c_1^{\frac{k}{2}} W_1^{\frac{1}{2}},$$

$$|Q\nabla w_\varepsilon| = (1+\varepsilon w)^{-2} |Q\nabla w| \le c_2 W_1^{\frac{1}{2k}},$$

$$|QD^2 w_\varepsilon| \le (1+\varepsilon w)^{-2}|QD^2 w| + 2\varepsilon(1+\varepsilon w)^{-3}|Q\nabla w||\nabla w| \le (c_3 + 2\eta^{-1}c_2^2) W_1^{\frac{1}{k}},$$

$$\begin{aligned}|D^3 w_\varepsilon| &\le (1+\varepsilon w)^{-2}|D^3 w| + 6\varepsilon(1+\varepsilon w)^{-3}|\nabla w||D^2 w| + 6\varepsilon^2(1+\varepsilon w)^{-4}|\nabla w|^3\\ &\le (c_{10} + 6\eta^{-2}c_2c_3 + 6\eta^{-3}c_2^3)W_1^{\frac{3}{2k}},\end{aligned}$$

$$|\partial_t w_\varepsilon| = (1+\varepsilon w)^{-2}|\partial_t w| \le c_4 w^{\frac{k-2}{k}} W_1^{\frac{1}{2k}}$$

and

$$|\partial_t \nabla w_\varepsilon| \le (1+\varepsilon w)^{-2}|\partial_t\nabla w| + 2\varepsilon(1+\varepsilon w)^{-3}|\nabla w||\partial_t w| \le (c_{11} + 2\eta^{-1}c_2c_4)W_1^{\frac{1}{k}}.$$

This shows that w_ε satisfies Hypothesis 3.1.4(c) with the same constants $c_1, c_2, c_4, c_5, c_6, c_7, c_8, c_9$ and with the constants c_3, c_{10}, c_{11} replaced, respectively, by $c_3 + 2\eta^{-1}c_2^2$, $c_{10} + 6\eta^{-2}c_2c_3 + 6\eta^{-3}c_2^3$ and $c_{11} + 2\eta^{-1}c_2c_4$.

Thus, the estimate (3.25) shown in the first part of the proof holds true with the function w_ε instead of w and with the constants on the right hand side that do not depend on ε. We finally let $\varepsilon \to 0$ to gain inequality (3.19). □

Remark 3.2.2. *From the above proof one can see that the constants B_i, $i = 1, \ldots, 6$ are given by*

$$\begin{aligned}
&B_1 = c_2,\\
&B_2 = \frac{c_2}{b-b_1} + c_2c_4 + c_{11},\\
&B_3 = c_2c_5^2 + c_3c_6 + c_2^2c_6 + c_2c_8 + c_9,\\
&B_4 = c_2 + c_3 + c_2^2 + \frac{c_2}{(b-b_1)^{\frac12}} + c_2^{\frac12}c_3^{\frac12}c_7^{\frac12} + c_2^{\frac32}c_7^{\frac12} + c_2^{\frac12}c_{10}^{\frac12} + c_2c_3^{\frac12} + c_2c_4^{\frac12} + c_2^{\frac12}c_{11}^{\frac12},\\
&B_5 = c_6 + c_2c_6 + c_3^{\frac12}c_6^{\frac12} + c_2c_6^{\frac12} + c_2^{\frac12}c_3^{\frac12}c_6^{\frac12} + c_2^{\frac32}c_6^{\frac12} + c_5 + c_2c_5 + c_8,\\
&B_6 = c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b-b_1)^{\frac{k}{2}}} + c_2^k + c_2^{\frac{k}{2}}c_7^{\frac{k}{2}} + c_3^{\frac{k}{2}} + c_7^k + c_4^{\frac{k}{2}},\\
&B_7 = c_6^k + c_2^{\frac{k}{2}}c_6^{\frac{k}{2}} + c_5^k.
\end{aligned}\tag{3.26}$$

3.3 Estimates for the derivatives of the kernel for general diffusion coefficients

In this section we proceed by approximation as in Chapter 2, Section 2.5. We approximate the operator A as in Chapter 1. We consider the function φ_n defined by (1.31), where W_1 is the time dependent Lyapunov function from Hypothesis 3.1.4 and the constant $t_0 \in (0, T)$ will be chosen later on. Then, we take the matrix $Q_n := (q_{ij}^{(n)})$ defined by (1.32) and the following family of operators A_n with bounded diffusion coefficients

$$A_n = \mathrm{div}(Q_n\nabla) + F\cdot\nabla - V.$$

As a consequence of the Lemma 1.6.4, for every $n \in \mathbb{N}$ the semigroup generated by A_n in $C_b(\mathbb{R}^d)$ is given by a kernel $p_n(t, x, y)$.

We now make the following assumptions.

Hypothesis 3.3.1. *Fix $T > 0$, $x \in \mathbb{R}^d$ and $0 < a_0 < a < b < b_0 < T$. Let us consider two time dependent Lyapunov functions $1 \leq W_1$, W_2 for the operators $L := \partial_t + A$ and $\partial_t + \eta\Delta + F \cdot \nabla - V$ with $W_1 \leq W_2$, $|\nabla W_1|, |\nabla W_2|$ bounded on $(0, T) \times B_R$ for all $R > 0$ and a weight function $1 \leq w \in C^{1,2}((0, T) \times \mathbb{R}^d)$ such that*

(a) there is $t_0 \in (0, T)$ such that

$$|Q||\nabla W_1(t_0, \cdot)| \leq c_{12} W_1(t_0, \cdot) w^{-1/k} W_1^{1/2k};$$

(b) there are $c_0 > 0$ and $\sigma \in (0, 1)$ such that

$$W_2 \leq c_0 Z^{1-\sigma},$$

where Z is the function introduced in Hypothesis 3.0.1(b);

(c) there is a nonnegative function f such that

$$\nabla Z(x) = f(x) \nabla W_1(t_0, x),$$

for all $x \in \mathbb{R}^d$.

As a consequence of the previous assumptions, the operators A_n inherit Hypothesis 3.1.4 from A.

Lemma 3.3.2. *Assume that the operator A satisfies Hypotheses 3.1.4(c) and 3.3.1(a). Then the operator A_n satisfies Hypothesis 3.1.4(c) with the same constants $c_1, c_4, c_5, c_6, c_8, c_9, c_{10}, c_{11}$ and with c_2, c_3, c_7 being replaced, respectively, by $2c_2, 2c_3$ and $\sqrt{3}(c_7 + 2(1 + \sqrt{d})c_{12})$.*

Proof. The constants $c_1, c_4, c_5, c_6, c_8, c_9, c_{10}, c_{11}$ remain the same because the corresponding inequalities do not depend on the diffusion coefficients. Let us note that Hypothesis 3.1.4(c)-(ii) implies that

$$|\nabla w| \leq \eta^{-1} c_2 W_1^{\frac{1}{2k}}.$$

So, it follows that

$$|Q_n \nabla w| = |\varphi_n Q \nabla w + (1 - \varphi_n)\eta \nabla w| \leq |Q \nabla w| + \eta |\nabla w| \leq 2c_2 W_1^{\frac{1}{2k}}.$$

Similarly, we get

$$|Q_n D^2 w| = |\varphi_n Q D^2 w + (1 - \varphi_n)\eta D^2 w| \leq |Q D^2 w| + \eta |D^2 w| \leq 2c_3 W_1^{\frac{1}{k}}.$$

Finally, for $(t,y) \in [a_0,b_0] \times \mathbb{R}^d$, given that the function φ satisfies $|s\varphi'(s)| \le 2$ as defined in Section 1.6 and using Hypothesis 3.3.1(a), we have

$$
\begin{aligned}
&|\nabla Q_n|^2 \\
&= \sum_{i,j,h=1}^d \left| \varphi_n D_h q_{ij} + \frac{\varphi'(W_1(t_0,\cdot)/n)}{n} D_h W_1(t_0,\cdot)(q_{ij} - \eta\delta_{ij}) \right|^2 \\
&\le 3 \sum_{i,j,h=1}^d \left[|\varphi_n D_h q_{ij}|^2 + \frac{|\varphi'(W_1(t_0,\cdot)/n)|^2}{n^2} |D_h W_1(t_0,\cdot)|^2 (q_{ij}^2 + \eta^2 \delta_{ij}) \right] \\
&\le 3|\varphi_n|^2 |\nabla Q|^2 \\
&\quad + 3(W_1(t_0,\cdot)/n)^2 |\varphi'(W_1(t_0,\cdot)/n)|^2 (W_1(t_0,\cdot))^{-2} \sum_{i,j,h=1}^d |q_{ij} D_h W_1(t_0,\cdot)|^2 \\
&\quad + 3d\eta^2 (W_1(t_0,\cdot)/n)^2 |\varphi'(W_1(t_0,\cdot)/n)|^2 (W_1(t_0,\cdot))^{-2} \sum_{h=1}^d |D_h W_1(t_0,\cdot)|^2 \\
&\le 3(c_7^2 + 4c_{12}^2 + 4dc_{12}^2) w^{-\frac{2}{k}} W_1^{\frac{1}{k}}.
\end{aligned}
$$

Then,

$$
|\nabla Q_n| \le \sqrt{3}(c_7 + 2c_{12} + 2\sqrt{d}c_{12}) w^{-\frac{1}{k}} W_1^{\frac{1}{2k}}. \qquad \square
$$

We can now obtain estimates for the gradients of the kernels p_n.

Lemma 3.3.3. *Assume that Hypothesis 3.3.1 holds and that the operator A satisfies Hypothesis 3.1.4. For $i = 1,2$, we set*

$$
\xi_{W_i,n}(t,x) := \int_{\mathbb{R}^d} p_n(t,x,y) W_i(t,y)\, dy \quad \text{and} \quad \Xi_{i,n}(a_0,b_0) := \int_{a_0}^{b_0} \xi_{W_i,n}(t,x)\, dt.
$$

Then for any $n \in \mathbb{N}$ we have

$$
|w(t,y) \nabla p_n(t,x,y)| \le K_n,
$$

for all $(t,y) \in (a_1,b_1) \times \mathbb{R}^d$ and fixed $x \in \mathbb{R}^d$, where

$$
\begin{aligned}
K_n = C\Bigg\{ &\left(B_1 \tilde{A}_1^{\frac{1}{2}} + B_2 \tilde{A}_1^{\frac{k-2}{k}} + \tilde{B}_4 \tilde{A}_1^{\frac{k-1}{k}} \right) \Xi_{1,n}(a_0,b_0) + \Big[B_1 A_2^{\frac{1}{2}} + (B_2 + B_3) A_2^{\frac{k-2}{k}} \\
&+ B_3 \tilde{A}_1^{\frac{k-2}{k}} + (\tilde{B}_4 + B_5) A_2^{\frac{k-1}{k}} + B_5 \tilde{A}_1^{\frac{k-1}{k}} + \tilde{B}_6 B_8 + B_7 B_8 \Big] \Xi_{2,n}(a_0,b_0) \\
&+ \left(\tilde{B}_6\, \Xi_{1,n}(a_0,b_0)^{\frac{1}{2}} + B_7\, \Xi_{2,n}(a_0,b_0)^{\frac{1}{2}} \right) \left(\int_{Q(a,b)} p_n \log^2 p_n \, dt\, dy \right)^{\frac{1}{2}} \\
&- \left(\tilde{B}_6\, \Xi_{1,n}(a_0,b_0)^{\frac{1}{2}} + B_7\, \Xi_{2,n}(a_0,b_0)^{\frac{1}{2}} \right) \left(\int_{\mathbb{R}^d} [p_n \log p_n]_{t=a}^{t=b} dy \right)^{\frac{1}{2}} \Bigg\},
\end{aligned}
$$

and the constants $A_2, B_1, \ldots, B_8, \tilde{A}_1, \tilde{B}_4, \tilde{B}_6$ are defined as in (3.3), (3.26), (3.30), (3.28) and (3.29).

Proof. Since the operator A satisfies Hypotheses 3.1.4 and 3.3.1, then for any $n \in \mathbb{N}$ the operator A_n satisfies Hypotheses 3.0.1 and 3.1.4 by Lemma 1.6.4 with slightly different constants given by Lemma 3.3.2. Consequently, applying (3.2) to p_n we get

$$w(t,y)p_n(t,x,y) \le C\left(\tilde{A}_1\,\Xi_{1,n}(a_0,b_0) + A_2\,\Xi_{2,n}(a_0,b_0)\right), \tag{3.27}$$

where

$$\tilde{A}_1 = c_1^{\frac{k}{2}} + c_2^k + \frac{c_1^{\frac{k}{2}}}{(b_0-b)^{\frac{k}{2}}} + c_2^{\frac{k}{2}}c_7^{\frac{k}{2}} + c_2^{\frac{k}{2}}c_{12}^{\frac{k}{2}} + c_3^{\frac{k}{2}} + c_4^{\frac{k}{2}}. \tag{3.28}$$

Moreover, applying (3.19) to p_n, we obtain

$$\begin{aligned}
&|w(t,y)\nabla p_n(t,x,y)| \\
&\le C\Bigg\{B_1\,\Xi_{1,n}(a_0,b_0)^{\frac{1}{2}}\,\|wp_n\|_{L^\infty(Q(a,b))}^{\frac{1}{2}} \\
&\quad + \left(B_2\,\Xi_{1,n}(a_0,b_0)^{\frac{2}{k}} + B_3\,\Xi_{2,n}(a_0,b_0)^{\frac{2}{k}}\right)\|wp_n\|_{L^\infty(Q(a,b))}^{\frac{k-2}{k}} \\
&\quad + \left[\tilde{B}_4\,\Xi_{1,n}(a_0,b_0)^{\frac{1}{k}} + B_5\,\Xi_{2,n}(a_0,b_0)^{\frac{1}{k}}\right]\|wp_n\|_{L^\infty(Q(a,b))}^{\frac{k-1}{k}} \\
&\quad + \left(\tilde{B}_6\,\Xi_{1,n}(a_0,b_0)^{\frac{1}{2}} + B_7\,\Xi_{2,n}(a_0,b_0)^{\frac{1}{2}}\right)\left(\int_{Q(a,b)}\frac{|\nabla p_n|^2}{p_n}\,dt\,dy\right)^{\frac{1}{2}}\Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{B}_4 =& c_2 + c_3 + c_2^2 + \frac{c_2}{(b-b_1)^{\frac{1}{2}}} + c_2^{\frac{1}{2}}c_3^{\frac{1}{2}}c_7^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_3^{\frac{1}{2}}c_{12}^{\frac{1}{2}} + c_2^{\frac{3}{2}}c_7^{\frac{1}{2}} \\
&+ c_2^{\frac{3}{2}}c_{12}^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_{10}^{\frac{1}{2}} + c_2c_3^{\frac{1}{2}} + c_2c_4^{\frac{1}{2}} + c_2^{\frac{1}{2}}c_{11}^{\frac{1}{2}}, \\
\tilde{B}_6 =& c_1^{\frac{k}{2}} + \frac{c_1^{\frac{k}{2}}}{(b-b_1)^{\frac{k}{2}}} + c_2^k + c_2^{\frac{k}{2}}c_7^{\frac{k}{2}} + c_2^{\frac{k}{2}}c_{12}^{\frac{k}{2}} + c_3^{\frac{k}{2}} + c_7^k + c_{12}^k + c_4^{\frac{k}{2}}.
\end{aligned} \tag{3.29}$$

Finally, by Theorem 3.1.5 we have

$$\int_{Q(a,b)}\frac{|\nabla p_n|^2}{p_n}\,dt\,dy \le C\left[(c_6^2 + c_5^4)\,\Xi_{2,n}(a_0,b_0) + \int_{Q(a,b)} p_n\log^2 p_n\,dt\,dy - \int_{\mathbb{R}^d}[p_n\log p_n]_{t=a}^{t=b}dy\right].$$

Combining them yields

$$
\begin{aligned}
&|w(t,y)\nabla p_n(t,x,y)| \\
&\le C\Bigg\{B_1\,\Xi_{1,n}(a_0,b_0)^{\frac12}\Big(\tilde A_1^{\frac12}\,\Xi_{1,n}(a_0,b_0)^{\frac12}+A_2^{\frac12}\,\Xi_{2,n}(a_0,b_0)^{\frac12}\Big) \\
&\quad+\Big(B_2\Xi_{1,n}(a_0,b_0)^{\frac2k}+B_3\Xi_{2,n}(a_0,b_0)^{\frac2k}\Big)\Big(\tilde A_1^{\frac{k-2}{k}}\,\Xi_{1,n}(a_0,b_0)^{\frac{k-2}{k}} \\
&\quad+A_2^{\frac{k-2}{k}}\,\Xi_{2,n}(a_0,b_0)^{\frac{k-2}{k}}\Big)+\Big(\tilde B_4\,\Xi_{1,n}(a_0,b_0)^{\frac1k}+B_5\,\Xi_{2,n}(a_0,b_0)^{\frac1k}\Big)\times \\
&\quad\times\Big(\tilde A_1^{\frac{k-1}{k}}\,\Xi_{1,n}(a_0,b_0)^{\frac{k-1}{k}}+A_2^{\frac{k-1}{k}}\,\Xi_{2,n}(a_0,b_0)^{\frac{k-1}{k}}\Big)+\Big(\tilde B_6\,\Xi_{1,n}(a_0,b_0)^{\frac12} \\
&\quad+B_7\,\Xi_{2,n}(a_0,b_0)^{\frac12}\Big)\Bigg[B_8\,\Xi_{2,n}(a_0,b_0)^{\frac12}+\Big(\int_{Q(a,b)}p_n\log^2 p_n\,dt\,dy\Big)^{\frac12} \\
&\quad-\Big(\int_{\mathbb{R}^d}[p_n\log p_n]_{t=a}^{t=b}dy\Big)^{\frac12}\Bigg]\Bigg\},
\end{aligned}
$$

where

$$B_8=c_6+c_5^2. \tag{3.30}$$

Considering that $\Xi_{1,n}(a_0,b_0)\le\Xi_{2,n}(a_0,b_0)$, the statement follows. □

Lemma 3.3.4. *Assume that Hypothesis 3.3.1 holds and that the operator A satisfies Hypothesis 3.1.4. Then, for $n\to\infty$, we have*

$$\int_{\mathbb{R}^d}[p_n(t,x,y)\log p_n(t,x,y)]_{t=a}^{t=b}dy\to\int_{\mathbb{R}^d}[p(t,x,y)\log p(t,x,y)]_{t=a}^{t=b}dy$$

and

$$\int_{Q(a,b)}p_n(t,x,y)\log^2 p_n(t,x,y)\,dt\,dy\to\int_{Q(a,b)}p(t,x,y)\log^2 p(t,x,y)\,dt\,dy.$$

In particular, the latter integrals are finite.

Proof. We observe that, by Lemma 1.6.6, we have that $p_n(t,x,\cdot)\to p(t,x,\cdot)$ locally uniformly in $\mathbb{R}^d$ as $n\to\infty$. Moreover, Lemma 1.6.7 implies that $\xi_{W_j,n}(\cdot,x)\to\xi_{W_j}(\cdot,x)$ uniformly in (a_0,b_0) as $n\to\infty$ for $j=1,2$. Then, it follows from inequality (3.27) that $p_n\le C_nw^{-1}$ for a certain constant C_n with $\sup C_n<\infty$. Making use of Hypothesis 3.1.4(a), we find integrable majorants for $p_n\log p_n$ and $p_n\log^2 p_n$. At this point, the statements follows by means of the dominated convergence theorem. □

Corollary 3.3.5. *Assume Hypotheses 3.1.4 and 3.3.1. Then*

$$\sqrt{p}\in W_2^{0,1}(Q(a,b)).$$

Proof. As a consequence of Hypothesis 3.1.4(b) and Lemma 3.3.4,

$$
\begin{aligned}
C := \sup_{n\in\mathbb{N}} \frac{1}{\eta^2} \int_{Q(a,b)} & \left(|F(y)|^2 + V^2(y)\right) p_n(t,x,y)\, dt\, dy \\
& + \int_{Q(a,b)} p_n(t,x,y) \log^2 p_n(t,x,y)\, dt\, dy \\
& - \frac{2}{\eta} \int_{\mathbb{R}^d} [p_n(t,x,y) \log p_n(t,x,y)]_{t=a}^{t=b} dy < \infty.
\end{aligned}
$$

It follows from Theorem 3.1.5 that $\sqrt{p_n}$ is bounded in $W_2^{0,1}(Q(a,b))$. As this space is reflexive, a subsequence of p_n converges weakly to some element q of $W_2^{0,1}(Q(a,b))$. However, as $p_n \to p$ pointwise and with an integrable majorant, testing against a function in $C_c^\infty(\mathbb{R}^d)$, we see that $q = p$. $\square$

We can now prove our main result.

Theorem 3.3.6. *Assume that the operator A satisfies Hypotheses 3.1.4 and 3.3.1. Then we have*

$$|w(t,y)\nabla p(t,x,y)| \le K,$$

for all $(t,y) \in (a,b)\times\mathbb{R}^d$ and fixed $x \in \mathbb{R}^d$, where

$$
\begin{aligned}
K = C\Bigg\{ & \left(B_1\tilde{A}_1^{\frac{1}{2}} + B_2\tilde{A}_1^{\frac{k-2}{k}} + \tilde{B}_4\tilde{A}_1^{\frac{k-1}{k}}\right)\Xi_1(a_0,b_0) + \Big[B_1A_2^{\frac{1}{2}} + (B_2+B_3)A_2^{\frac{k-2}{k}} \\
& + B_3\tilde{A}_1^{\frac{k-2}{k}} + (\tilde{B}_4 + B_5)A_2^{\frac{k-1}{k}} + B_5\tilde{A}_1^{\frac{k-1}{k}} + \tilde{B}_6B_8 + B_7B_8\Big]\Xi_2(a_0,b_0) \\
& + \left(\tilde{B}_6\,\Xi_1(a_0,b_0)^{\frac{1}{2}} + B_7\,\Xi_2(a_0,b_0)^{\frac{1}{2}}\right)\left(\int_{Q(a,b)} p\log^2 p\, dt\, dy\right)^{\frac{1}{2}} \\
& - \left(\tilde{B}_6\,\Xi_1(a_0,b_0)^{\frac{1}{2}} + B_7\,\Xi_2(a_0,b_0)^{\frac{1}{2}}\right)\left(\int_{\mathbb{R}^d} [p\log p]_{t=a}^{t=b} dy\right)^{\frac{1}{2}}\Bigg\}, \qquad (3.31)
\end{aligned}
$$

and the constants $A_2, B_1, \ldots, B_8, \tilde{A}_1, \tilde{B}_4, \tilde{B}_6$ are defined as in (3.3), (3.26), (3.30), (3.28) and (3.29).

Proof. By Lemmas 3.3.3 and 3.3.4 we infer that

$$\limsup_{n\to\infty} |w(t,y)\nabla p_n(t,x,y)| \le K.$$

Then, for $|h|$ small, we have

$$
\begin{aligned}
w(t,y)\left|\frac{p(t,x,y+h) - p(t,x,y)}{h}\right| & \\
&= \limsup_{n\to\infty} w(t,y)\left|\frac{p_n(t,x,y+h) - p_n(t,x,y)}{h}\right| \\
&\le \limsup_{n\to\infty} w(t,y)\int_0^1 |\nabla p_n(t,x,y+sh)|\, ds \\
&\le K\int_0^1 \frac{w(t,y)}{w(t,y+sh)}\, ds.
\end{aligned}
$$

If we now let $|h| \to 0$, the statement follows. $\square$

As a simple consequence one obtains the following Sobolev regularity for p.

Corollary 3.3.7. *Assume in addition to Hypotheses 3.1.4 and 3.3.1, that* $\int_{Q(a,b)} w(t,x)^{-r}\,dtdx < \infty$ *for some* $r \in (1,\infty)$. *Then* $p \in W_r^{0,1}(Q(a,b))$.

3.4 Application to the case of polynomial coefficients

Here we apply the results of the previous sections to the case of operators with polynomial diffusion coefficients, drift and potential terms.

Consider $Q(x) = (1+|x|_*^m)I$, $F(x) = -|x|^{p-1}x$ and $V(x) = |x|^s$ with $p > (m-1)\vee 1$, $s > |m-2|$ and $m > 0$. To apply Theorem 3.3.6 we set

$$w(t,x) = e^{\varepsilon t^\alpha |x|_*^\beta} \text{ and } W_j(t,x) = e^{\varepsilon_j t^\alpha |x|_*^\beta},$$

for $(t,y) \in (0,1)\times\mathbb{R}^d$, where $j = 1,2$, $\beta = \frac{s-m+2}{2}$, $0 < 2k\varepsilon < \varepsilon_1 < \varepsilon_2 < \frac{1}{\beta}$ and $\alpha > \frac{\beta}{\beta+m-2}$.

Theorem 3.4.1. *Let p be the integral kernel associated with the operator A with $Q(x) = (1+|x|_*^m)I$, $F(x) = -|x|^{p-1}x$ and $V(x) = |x|^s$, where $p > (m-1)\vee 1$, $s > |m-2|$ and $m > 0$. Then*

$$p(t,x,y) \le Ct^{1-\frac{\alpha(2m\vee 2p\vee s)}{s-m+2}k} e^{-\varepsilon t^\alpha |y|_*^{\frac{s-m+2}{2}}}$$

and

$$|\nabla p(t,x,y)| \le C(1-\log t)t^{\frac{3}{2}-\frac{3\alpha(m\vee p\vee\frac{s}{2})k+\alpha}{s-m+2}} e^{-\varepsilon t^\alpha |y|_*^\beta} \tag{3.32}$$

for $k > 2(d+2)$ and any $t \in (0,1)$, $x,y \in \mathbb{R}^d$.

Proof. Step 1. We show that W_1 and W_2 are time dependent Lyapunov functions for $L = \partial_t + A$ and $\partial_t + \eta\Delta + F\cdot\nabla - V$ with respect to the function

$$Z(x) = e^{\varepsilon_2 |x|_*^\beta}.$$

For that, we take into account Remark 2.2.2. Let $|x| \ge 1$ and set $G_j = \sum_{i=1}^d D_i q_{ij} = m|x|^{m-2}x_j$. Since $s > |m-2|$, we have $\beta > (2-m)\vee 0$. Moreover,

$$\begin{aligned}
&|x|^{1-\beta-m}\left((G+F)\cdot\frac{x}{|x|} - \frac{V}{\varepsilon_j\beta|x|^{\beta-1}}\right)\\
&= |x|^{1-\beta-m}\left(m|x|^{m-1} - |x|^p - \frac{|x|^s}{\varepsilon_j\beta|x|^{\beta-1}}\right)\\
&\le m|x|^{-\beta} - \frac{1}{\varepsilon_j\beta}.
\end{aligned}$$

If $|x|$ is large enough, for example $|x| \geq K$ with

$$K > \left(\frac{m}{\frac{1}{\varepsilon_j \beta} - 1}\right)^{\frac{1}{\beta}},$$

we get

$$\begin{aligned} |x|^{1-\beta-m}\left((G+F)\cdot\frac{x}{|x|} - \frac{V}{\varepsilon_j\beta|x|^{\beta-1}}\right) &\leq m|x|^{-\beta} - \frac{1}{\varepsilon_j\beta} \\ &\leq mK^{-\beta} - \frac{1}{\varepsilon_j\beta} < -1, \end{aligned}$$

where we have used that $\varepsilon_j < \frac{1}{\beta}$. In addition, we have

$$\lim_{|x|\to\infty} V(x)\,|x|^{2-2\beta-m} = \lim_{|x|\to\infty} |x|^{2-2\beta-m+s} = 1.$$

Hence, $\lim_{|x|\to\infty} V(x)\,|x|^{2-2\beta-m} > c$ for any $c < 1$. Consequently, by Proposition 2.2.1 and Remark 2.2.2 we obtain that W_1 and W_2 are time dependent Lyapunov functions for $L = \partial_t + A$ and $\partial_t + \eta\Delta + F\cdot\nabla - V$. Similar computations show that the function $Z(x)$ satisfies Hypothesis 3.0.1(b).

Step 2. We now show that A satisfies Hypotheses 3.1.4 and 3.3.1. Fix $T = 1$, $x \in \mathbb{R}^d$, $0 < a_0 < a < b < b_0 < T$ and $k > 2(d+2)$. Let $(t,y) \in [a_0, b_0] \times \mathbb{R}^d$. Clearly, Hypothesis 3.1.4(a)-(b) and Hypothesis 3.3.1(b) are satisfied. We assume that $|y| \geq 1$; otherwise, in a neighborhood of the origin, all the quantities we are going to estimate are obviously bounded.

First, since $2\varepsilon < \varepsilon_1$, we infer that

$$w \leq c_1 w^{\frac{k-2}{k}} W_1^{\frac{1}{k}},$$

with $c_1 = 1$. Second, we have

$$\begin{aligned} \frac{|Q(y)\nabla w(t,y)|}{W_1(t,y)^{\frac{1}{2k}}} &= \varepsilon\beta t^\alpha |y|^{\beta-1}(1+|y|^m)e^{-\frac{1}{2k}(\varepsilon_1-2k\varepsilon)t^\alpha|y|^\beta} \\ &\leq 2\varepsilon\beta t^\alpha |y|^{\beta+m-1}e^{-\frac{1}{2k}(\varepsilon_1-2k\varepsilon)t^\alpha|y|^\beta}. \end{aligned} \tag{3.33}$$

We make use of the following remark: since the function $t \mapsto t^p e^{-t}$ on $(0,\infty)$ attains its maximum at the point $t = p$, then for $\tau, \gamma, z > 0$ we have

$$z^\gamma e^{-\tau z^\beta} = \tau^{-\frac{\gamma}{\beta}}(\tau z^\beta)^{\frac{\gamma}{\beta}} e^{-\tau z^\beta} \leq \tau^{-\frac{\gamma}{\beta}}\left(\frac{\gamma}{\beta}\right)^{\frac{\gamma}{\beta}} e^{-\frac{\gamma}{\beta}} =: C(\gamma,\beta)\tau^{-\frac{\gamma}{\beta}}. \tag{3.34}$$

Applying (3.34) to the inequality (3.33) with $z = |y|$, $\tau = \frac{1}{2k}(\varepsilon_1 - 2k\varepsilon)t^\alpha$, $\beta = \beta$ and $\gamma = \beta + m - 1 > 0$ yields

$$\begin{aligned} \frac{|Q(y)\nabla w(t,y)|}{W_1(t,y)^{\frac{1}{2k}}} &\leq 2C(\beta+m-1,\beta)\varepsilon\beta t^\alpha \left[\frac{1}{2k}(\varepsilon_1 - 2k\varepsilon)t^\alpha\right]^{-\frac{\beta+m-1}{\beta}} \\ &\leq \overline{c}t^{-\frac{\alpha(m-1)^+}{\beta}} \leq \overline{c}a_0^{-\frac{\alpha(m-1)^+}{\beta}}. \end{aligned}$$

Thus, we choose $c_2 = \bar{c}a_0^{-\frac{\alpha(m-1)^+}{\beta}}$, where $\bar{c}$ is a universal constant. In a similar way,

$$\begin{aligned}\frac{|Q(y)D^2w(t,y)|}{W_1(t,y)^{\frac{1}{k}}} &= \frac{(1+|y|^m)|D^2w(t,y)|}{W_1(t,y)^{\frac{1}{k}}}\\ &\leq 2\sqrt{3}\varepsilon\beta t^\alpha \left[\left((\beta-2)^+ + \sqrt{d}\right)|y|^{\beta+m-2} + \varepsilon\beta t^\alpha |y|^{2\beta+m-2}\right] e^{-\frac{1}{k}(\varepsilon_1 - k\varepsilon)t^\alpha|y|^\beta}.\end{aligned}$$

Applying (3.34) to each term, we get

$$\begin{aligned}\frac{|Q(y)D^2w(t,y)|}{W_1(t,y)^{\frac{1}{k}}} \leq & C(\beta,m)\varepsilon\beta t^\alpha \left\{\left((\beta-2)^+ + \sqrt{d}\right)\left[\frac{1}{k}(\varepsilon_1 - k\varepsilon)t^\alpha\right]^{-\frac{\beta+m-2}{\beta}}\right.\\ &\left. + \varepsilon\beta t^\alpha \left[\frac{1}{k}(\varepsilon_1 - k\varepsilon)t^\alpha\right]^{-\frac{2\beta+m-2}{\beta}}\right\} \leq \bar{c}t^{-\frac{\alpha(m-2)}{\beta}} \leq \bar{c}a_0^{-\frac{\alpha(m-2)^+}{\beta}}.\end{aligned}$$

Therefore, we pick $c_3 = \bar{c}a_0^{-\frac{\alpha(m-2)^+}{\beta}}$. Furthermore, if we consider $t_0 \in (0,t)$, we have

$$\begin{aligned}\frac{|Q(y)||\nabla W_1(t_0,y)|}{W_1(t_0,y)w(t,y)^{-1/k}W_1(t,y)^{1/2k}} &= \sqrt{d}\beta\varepsilon_1 t_0^\alpha(1+|y|^m)|y|^{\beta-1}e^{-\frac{1}{2k}(\varepsilon_1-2\varepsilon)t^\alpha|y|^\beta}\\ &\leq \bar{c}t^{-\alpha\frac{m-1}{\beta}} \leq \bar{c}a_0^{-\alpha\frac{(m-1)^+}{\beta}} =: c_{12},\end{aligned}$$

where we used (3.34). We can proceed in the same way to check the remaining inequalities. To sum up, the constants $c_1,\ldots,c_{12}$ are the following:

$$\begin{array}{lll} c_1 = 1, & c_2 = c_7 = c_{12} = \bar{c}a_0^{-\frac{\alpha(m-1)^+}{\beta}}, & c_3 = \bar{c}a_0^{-\frac{\alpha(m-2)^+}{\beta}},\\ c_4 = c_{11} = \bar{c}a_0^{-1}, & c_5 = \bar{c}a_0^{-\frac{\alpha s}{2\beta}}, & c_6 = \bar{c}a_0^{-\frac{\alpha p}{\beta}},\\ c_8 = \bar{c}a_0^{-\frac{\alpha(p-1)}{\beta}}, & c_9 = \bar{c}a_0^{-\frac{\alpha(s-1)^+}{\beta}}, & c_{10} = \bar{c}. \end{array}$$

Step 3. We are now ready to apply Theorem 3.3.6. To that end, we choose $a_0 = t/4$, $a = t/2$, $b = (t+1)/2$ and $b_0 = (t+3)/4$. If we now set $\lambda = m \vee p \vee \frac{s}{2}$, since $\alpha > \frac{\beta}{\beta+m-2}$, $s > |m-2|$ and $\beta = \frac{s-m+2}{2}$, we have

$$\frac{\alpha\lambda}{\beta} > \frac{s}{2(\beta+m-2)} = \frac{s}{s+m-2} > \frac{1}{2}.$$

Hence we can estimate the constant A_2 in (3.3) as follows

$$A_2 = c_6^k + c_2^{\frac{k}{2}}c_6^{\frac{k}{2}} + c_5^k = \bar{c}\left(t^{-\frac{\alpha pk}{\beta}} + t^{-\frac{\alpha((m-1)^+ + p)k}{2\beta}} + t^{-\frac{\alpha sk}{2\beta}}\right) \leq \bar{c}t^{-\frac{\alpha\lambda k}{\beta}}. \tag{3.35}$$

Similarly, if we consider the remaining constants in the right hand side of (3.31) we obtain that

$$\begin{array}{lll} \tilde{A}_1 \leq \bar{c}\left(t^{-\frac{\alpha\lambda k}{\beta}} + t^{-\frac{k}{2}}\right), & B_1 \leq \bar{c}t^{-\frac{\alpha\lambda}{\beta}}, & B_2 \leq \bar{c}t^{-\frac{\alpha\lambda}{\beta}-1},\\ B_3 \leq \bar{c}t^{-\frac{3\alpha\lambda}{\beta}}, & \tilde{B}_4 \leq \bar{c}t^{-\frac{2\alpha\lambda}{\beta}}, & B_5 \leq \bar{c}t^{-\frac{2\alpha\lambda}{\beta}},\\ \tilde{B}_6 \leq \bar{c}t^{-\frac{\alpha\lambda k}{\beta}}, & B_7 \leq \bar{c}t^{-\frac{\alpha\lambda k}{\beta}}, & B_8 \leq \bar{c}t^{-\frac{\alpha\lambda}{\beta}}. \end{array} \tag{3.36}$$

Moreover, by Proposition 2.2.1, there are two constants H_1 and H_2 not depending on a_0 and b_0 such that $\xi_{W_j}(t,x) \le H_j$ for all $(s,x) \in [0,1]\times\mathbb{R}^d$, so for $j=1,2$ we have

$$\Xi_j(a_0,b_0) = \int_{a_0}^{b_0} \xi_{W_j}(t,x)\,dt \le H_j(b_0-a_0) = \frac{3t}{4}H_j. \tag{3.37}$$

Furthermore, by Corollary 3.1.3, the approximation procedure used to obtain p_n, (3.27) and Lemma 1.6.7 we obtain

$$p(t,x,y) \le Ct^{1-\frac{\alpha\lambda k}{\beta}} e^{-\varepsilon t^\alpha |y|_*^\beta}. \tag{3.38}$$

Then,

$$\begin{aligned} p(t,x,y)\log p(t,x,y) &\le Ct^{1-\frac{\alpha\lambda k}{\beta}}\left[\log C + \left(1-\frac{\alpha\lambda k}{\beta}\right)\log t - \varepsilon t^\alpha |y|_*^\beta\right] e^{-\varepsilon t^\alpha |y|_*^\beta} \\ &\le C(1-\log t)t^{1-\frac{\alpha\lambda k}{\beta}} e^{-\varepsilon t^\alpha |y|_*^\beta}. \end{aligned}$$

Considering that $a=\frac{t}{2}$ and $b=(t+1)/2$, it leads to

$$\begin{aligned} \int_{\mathbb{R}^d} [p(t,x,y)\log p(t,x,y)]_{t=a}^{t=b}\,dy &\le C(1-\log t)t^{1-\frac{\alpha\lambda k+\alpha}{\beta}} \int_{\mathbb{R}^d} e^{-\varepsilon |z|_*^\beta}\,dz \\ &\le C(1-\log t)t^{1-\frac{\alpha\lambda k+\alpha}{\beta}}, \end{aligned} \tag{3.39}$$

where in the integral we performed the change of variables $z = a^{\frac{\alpha}{\beta}}y$ and $z = b^{\frac{\alpha}{\beta}}y$. We also get

$$\int_{Q(a,b)} p(t,x,y)\log^2 p(t,x,y)\,dt\,dy \le C(1-\log t)^2 t^{2-\frac{\alpha\lambda k+\alpha}{\beta}}. \tag{3.40}$$

Putting (3.35)-(3.40) in (3.31) yields

$$K \le C(1-\log t)t^{\frac{3}{2}-\frac{3\alpha\lambda k+\alpha}{2\beta}}.$$

Thus, Estimate (3.32) follows from Theorem 3.3.6. □

Similar estimates as in the symmetric case, see Theorem 2.6.1, can be also obtained for operators with drift term.

Remark 3.4.2. *If in addition to the assumptions of Theorem 3.4.1 one assumes that $s > (p-1)\vee(2p-m)^+$, then*

$$p(t,x,y) \le Ct^{1-\frac{\alpha(2m\vee 2p\vee s)}{s-m+2}k} e^{-\frac{\varepsilon}{2}t^\alpha |x|_*^{\frac{s-m+2}{2}}} e^{-\frac{\varepsilon}{2}t^\alpha |y|_*^{\frac{s-m+2}{2}}} \tag{3.41}$$

holds for $k > d+2$ and any $t\in(0,1)$, $x,y\in\mathbb{R}^d$. Indeed, the formal adjoint of A is $A^* = A - 2F\cdot\nabla + (d+p-1)|x|^{p-1}$. The associated minimal semigroup is given by the kernel $p(t,x,y)^* = p(t,y,x)$. Since $s > (p-1)\vee(2p-m)^+$, one can see that the condition in Remark 2.2.2 is satisfied and since $s > p-1$, it follows that $\lim_{|x|\to\infty}\widetilde{V}(x)\,|x|^{2-2\beta-m} = 1$, where $\widetilde{V}(x) := |x|^s - (d+p-1)|x|^{p-1}$, $x\in\mathbb{R}^d$. So, $p^*(t,x,y)$ satisfies (3.38). Arguing as in Step 4 of the proof of Theorem 2.6.1 one obtains (3.41).

Appendix A

Function spaces

In this appendix we collect all the function spaces that we consider in this manuscript. In the following we will deal with real-valued or complex-valued functions. However, all the definitions apply for vector-valued functions: for example we say that $F = (F_j)$ belongs to the space $C_b(\mathbb{R}^d; \mathbb{R}^d)$ if each component F_j belongs to $C_b(\mathbb{R}^d)$.

A.1 Spaces of continuous functions

Let Ω be a domain or its closure and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition A.1.1. • *We denote by $C(\Omega)$ the set of all continuous functions $f\colon \Omega \to \mathbb{K}$.*

- *For $\alpha \in (0,1)$, $C^\alpha(\Omega)$ is the subset of $C(\Omega)$ consisting of functions $f\colon \Omega \to \mathbb{K}$ which are α-Hölder continuous in Ω, namely such that*

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

- *In general, for $\alpha \in (0,\infty)$, $C^\alpha(\Omega)$ is the subset of $C(\Omega)$ of functions $f\colon \Omega \to \mathbb{K}$ which admit derivatives up to the order $[\alpha]$ and their derivatives of order $[\alpha]$ are $(\alpha - [\alpha])$-Hölder continuous in Ω (if $\alpha \notin \mathbb{N}$).*

- *We denote by $C_b(\Omega)$ the set of all functions $f\colon \Omega \to \mathbb{K}$ which are bounded and continuous in Ω. It is a Banach space when endowed with the sup-norm*

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|, \quad f \in C_b(\Omega).$$

- *If Ω is bounded, then $C_0(\Omega)$ denotes the set of all continuous functions $f\colon \overline{\Omega} \to \mathbb{K}$ which vanish on the boundary of Ω. If Ω is unbounded, then sometimes we also require that f vanishes as $|x| \to \infty$. It is a Banach space when endowed with the sup-norm.*

- *For* $\alpha \in (0,1)$, $C_b^\alpha(\Omega)$ *is the space of bounded* α*-Hölder continuous functions in* Ω*, namely the subset of* $C_b(\Omega)$ *consisting of functions* $f\colon \Omega \to \mathbb{K}$ *such that*
$$[f]_{C_b^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$
It is a Banach space when endowed with the norm
$$\|f\|_{C_b^\alpha(\Omega)} = \|f\|_\infty + [f]_{C_b^\alpha(\Omega)}, \quad f \in C_b^\alpha(\Omega).$$

- *In general, for* $\alpha \in (0,\infty)$, $C_b^\alpha(\Omega)$ *is the subset of* $C_b(\Omega)$ *of functions* $f\colon \Omega \to \mathbb{K}$ *which admit bounded derivatives up to the order* $[\alpha]$ *and their derivatives of order* $[\alpha]$ *are* $(\alpha-[\alpha])$*-Hölder continuous in* Ω *(if* $\alpha \notin \mathbb{N}$*). It is a Banach space when endowed with the norm*
$$\|f\|_{C_b^\alpha(\Omega)} := \sum_{|\beta|\le[\alpha]} \left\|D^\beta f\right\|_\infty + \sum_{|\beta|=[\alpha]} [D^\beta f]_{C_b^{\alpha-[\alpha]}(\Omega)}, \quad f \in C_b^\alpha(\Omega).$$

- *For* $\alpha \in (0,\infty)$*, we denote by* $C_{\mathrm{loc}}^\alpha(\Omega)$ *the set of all functions* $f\colon \Omega \to \mathbb{K}$ *which belong to* $C_b^\alpha(K)$ *for each compact subset* K *of* Ω*.*

- $C_c(\Omega)$ *denotes the space of continuous functions with compact support in* $\Omega \subset \mathbb{R}^d$*.*

- *We denote by* $C_c^\infty(\Omega)$ *the space of smooth functions with compact support in* Ω*.*

A.2 Parabolic Hölder spaces

Let $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$ be, respectively, an interval and a domain, or a closure of a domain. Moreover, let $\alpha, \beta \in (0,1)$ and $k \in \mathbb{N}$ be fixed.

Definition A.2.1.

- $C_b^{\alpha,0}(I\times\Omega)$ *denotes the space of all the bounded continuous functions* $f\colon I\times\Omega \to \mathbb{K}$ *such that the function* $f(\cdot,x)$ *is* α*-Hölder continuous in* I *for each* $x \in \Omega$*. It is a Banach space with the norm*
$$\|f\|_{C_b^{\alpha,0}(I\times\Omega)} = \sup_{x\in\Omega} \|f(\cdot,x)\|_{C_b^\alpha(I)}, \quad f \in C_b^{\alpha,0}(I\times\Omega).$$

- *By* $C_b^{0,\beta}(I\times\Omega)$ *we denote the space of all the bounded continuous functions* $f\colon I\times\Omega \to \mathbb{K}$ *such that the function* $f(t,\cdot)$ *is* β*-Hölder continuous in* Ω *for each* $t \in I$*. It is a Banach space with the norm*
$$\|f\|_{C_b^{0,\beta}(I\times\Omega)} = \sup_{t\in I} \|f(t,\cdot)\|_{C_b^\beta(\Omega)}, \quad f \in C_b^{0,\beta}(I\times\Omega).$$

- $C_b^{\alpha,\beta}(I\times\Omega) := C_b^{\alpha,0}(I\times\Omega)\cap C_b^{0,\beta}(I\times\Omega)$. *It is a Banach space with the norm*

$$\|f\|_{C_b^{\alpha,\beta}(I\times\Omega)} = \|f\|_{C_b^{0,\beta}(I\times\Omega)} + \sup_{x\in\Omega}[f(\cdot,x)]_{C_b^{\alpha}(I)}, \quad f\in C_b^{\alpha,\beta}(I\times\Omega).$$

- $C^{1,2}(I\times\Omega)$ *denotes the space of all functions* $f\colon I\times\Omega\to\mathbb{K}$ *which are once continuously differentiable with respect to time and twice continuously differentiable with respect to the spatial variables in* $I\times\Omega$ *with continuous derivatives.*

- $C_b^{1+\alpha/2,2+\alpha}(I\times\Omega)$ *is the subspace of* $C^{1,2}(I\times\Omega)$ *consisting of all the bounded functions* $f\colon I\times\Omega\to\mathbb{K}$ *with* $\partial_t f$ *and* D_{ij} *in* $C_b^{\alpha/2,\alpha}(I\times\Omega)$ *for each* $i,j=1,\dots,d$. *It is a Banach space with the norm*

$$\|f\|_{C_b^{1+\alpha/2,2+\alpha}(I\times\Omega)} = \|f\|_\infty + \sum_{j=1}^d \|D_j f\|_\infty + \sum_{i,j=1}^d \|D_{ij}f\|_{C_b^{\alpha/2,\alpha}(I\times\Omega)} + \|\partial_t f\|_{C_b^{\alpha/2,\alpha}(I\times\Omega)}, \quad f\in C_b^{1+\alpha/2,2+\alpha}(I\times\Omega).$$

- $C_{\rm loc}^{1+\alpha/2,2+\alpha}(I\times\Omega)$ *is the local Hölder space consisting of functions* $f\colon I\times\Omega\to\mathbb{K}$ *which belong to* $C_b^{1+\alpha/2,2+\beta}(K)$ *for every compact subset* K *of* $I\times\Omega$.

- *If* $I=(a,b)$ *for* $0\le a<b$ *and* $\Omega=\mathbb{R}^d$, *we denote by* $C_c^{1,2}((a,b)\times\mathbb{R}^d)$ *the space of all functions* $f\colon (a,b)\times\mathbb{R}^d\to\mathbb{K}$ *compactly supported in* $(a,b)\times\mathbb{R}^d$, *which belong to* $C^{1,2}((a,b)\times\mathbb{R}^d)$. *Notice that we are not requiring that* $f\in C_c^{1,2}((a,b)\times\mathbb{R}^d)$ *vanishes at* $t=a$, $t=b$.

- $C_c^\infty(I\times\Omega)$ *denotes the space of smooth functions* $f\colon I\times\Omega\to\mathbb{K}$ *with compact support in* $I\times\Omega$.

We skip the subscript "b" to define the sets $C^{\alpha,0}(I\times\Omega)$, $C^{0,\beta}(I\times\Omega)$, $C^{\alpha,\beta}(I\times\Omega)$ and $C^{1+\alpha/2,2+\alpha}(I\times\Omega)$ when the boundedness is not required.

A.3 L^p and Sobolev spaces

Let Ω be a domain of $\mathbb{R}^d$.

Definition A.3.1. • *For every* $p\in[1,\infty)$, $L^p(\Omega)$ *denotes the space of all the (equivalence classes of) measurable functions* $f\colon\Omega\to\mathbb{K}$ *such that*

$$\int_\Omega |f|^p\,dx<\infty.$$

It is a Banach space when endowed with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\,dx\right)^{\frac1p}, \quad f\in L^p(\Omega).$$

- *We denote by* $L^\infty(\Omega)$ *the space of all the (equivalence classes of) measurable functions* $f\colon \Omega \to \mathbb{K}$ *such that*

$$\operatorname*{ess\,sup}_{\Omega} f = \inf\{C > 0\colon |f(x)| \le C, \text{ for almost every } x \in \Omega\}.$$

  It is a Banach space when endowed with the norm $\|f\|_\infty = \operatorname{ess\,sup}_\Omega f$ *for every* $f \in L^\infty(\Omega)$.

- *For* $p \in [1, \infty]$, $L^p_{\mathrm{loc}}(\Omega)$ *denotes the set of all the (equivalence classes of) measurable functions* $f\colon \Omega \to \mathbb{K}$ *which belong to* $L^p(K)$ *for every bounded domain* K *whose closure is contained in* Ω.

If Ω is clear from the context, then for $p \in [1, \infty]$ we simply write $\|\cdot\|_p$ for the norm in $L^p(\Omega)$.

We now introduce Sobolev spaces of integer order. Let Ω be a domain of $\mathbb{R}^d$, $k \in \mathbb{N}$ and $p \in [1, \infty)$.

Definition A.3.2. • *We denote by* $W^{k,p}(\Omega)$ *the subspace of* $L^p(\Omega)$ *of all the (equivalence classes of) measurable functions* $f\colon \Omega \to \mathbb{K}$ *with distributional derivatives up to the order* k *belonging to* $L^p(\Omega)$. *It is a Banach space when endowed with the norm*

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, \quad f \in W^{k,p}(\Omega).$$

- *We set* $H^1(\Omega) := W^{1,2}(\Omega)$.

- *By* $W^{k,p}_0(\Omega)$ *we denote the closure of the set of test functions* $C^\infty_c(\Omega)$ *into* $W^{k,p}(\Omega)$ *with respect to the norm of* $W^{k,p}(\Omega)$.

- *We denote by* $H^1_0(\Omega)$ *the closure of the set of test functions* $C^\infty_c(\Omega)$ *with respect to the norm of* $H^1(\Omega)$.

- $W^{k,p}_{\mathrm{loc}}(\Omega)$ *denotes the set of all the (equivalence classes of) measurable functions* $f\colon \Omega \to \mathbb{K}$ *which belong to* $W^{k,p}(K)$ *for every bounded domain* K *whose closure is contained in* Ω.

A.4 Parabolic L^p and Sobolev spaces

Let $0 \le a < b < \infty$ and consider the set $Q(a, b) = (a, b) \times \mathbb{R}^d$. We define the L^p-spaces in $Q(a, b)$ as in Definition A.3.1.

Definition A.4.1. • *For every* $p \in [1, \infty)$, $L^p(Q(a, b))$ *denotes the space of all the (equivalence classes of) measurable functions* $f\colon Q(a, b) \to \mathbb{K}$ *such that*

$$\int_{Q(a,b)} |f|^p \, dt \, dx < \infty.$$

It is a Banach space when endowed with the norm

$$\|f\|_{L^p(Q(a,b))} = \left(\int_{Q(a,b)} |f|^p \, dt \, dx\right)^{\frac{1}{p}}, \quad f \in L^p(Q(a,b)).$$

- *We denote by $L^\infty(Q(a,b))$ the space of all the (equivalence classes of) measurable functions $f\colon Q(a,b) \to \mathbb{K}$ such that*

$$\operatorname*{ess\,sup}_{Q(a,b)} f = \inf\{C > 0\colon |f(t,x)| \le C, \text{ for almost every } (t,x) \in Q(a,b)\}.$$

It is a Banach space when endowed with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{Q(a,b)} f$ for every $f \in L^\infty(Q(a,b))$.

We now define parabolic Sobolev spaces for $p \in (1,\infty)$ as follows.

Definition A.4.2. • *By $W_p^{0,1}(Q(a,b))$ we denote the space of functions $f \in L^p(Q(a,b))$ having weak space derivatives $D_i f \in L^p(Q(a,b))$ for $i = 1,\dots,d$ equipped with the norm*

$$\|f\|_{W_p^{0,1}(Q(a,b))} := \|f\|_{L^p(Q(a,b))} + \|\nabla f\|_{L^p(Q(a,b);\mathbb{R}^d)}.$$

- *We denote by $W_p^{1,2}(Q(a,b))$ the space of functions $f \in L^p(Q(a,b))$ having weak space derivatives $D_i^\alpha f \in L^p(Q(a,b))$ for $|\alpha| \le 2$ and weak time derivative $\partial_t f \in L^p(Q(a,b))$ equipped with the norm*

$$\|f\|_{W_p^{1,2}(Q(a,b))} := \|f\|_{L^p(Q(a,b))} + \|\partial_t f\|_{L^p(Q(a,b))} + \sum_{1\le|\alpha|\le 2} \|D^\alpha f\|_{L^p(Q(a,b))}.$$

We shall also define the space $\mathcal{H}^{p,1}(Q(a,b))$ and provide some properties.

Definition A.4.3. *For $1 < p < \infty$, we denote by $\mathcal{H}^{p,1}(Q(a,b))$ the space of all functions $f \in W_p^{0,1}(Q(a,b))$ with $\partial_t f \in (W_{p'}^{0,1}(Q(a,b)))'$, the dual space of $W_{p'}^{0,1}(Q(a,b))$, endowed with the norm*

$$\|f\|_{\mathcal{H}^{p,1}(Q(a,b))} := \|\partial_t f\|_{(W_{p'}^{0,1}(Q(a,b)))'} + \|f\|_{W_p^{0,1}(Q(a,b))},$$

where $1/p + 1/p' = 1$.

Lemma A.4.4. *[41, Lemmas 7.1, 7.2] There exists a linear, continuous extension operator $E\colon \mathcal{H}^{p,1}(Q(a,b)) \to \mathcal{H}^{p,1}(\mathbb{R}^{d+1})$. Moreover, the restrictions of functions in $C_c^\infty(\mathbb{R}^{d+1})$ to $Q(a,b)$ are dense in $\mathcal{H}^{p,1}(Q(a,b))$.*

Theorem A.4.5. *[41, Theorem 7.1] If $p > d+2$, then $\mathcal{H}^{p,1}(Q(a,b))$ is continuously embedded in $C_0(Q(a,b))$.*

Lemma A.4.6. *[30, Lemma 12.3] Let $u \in \mathcal{H}^{p,1}(Q(a,b)) \cap C_b(\overline{Q(a,b)})$ for some $p \in (1,\infty)$. Then, there exists a sequence $(u_n) \subset C_c^\infty(\mathbb{R}^{d+1})$ of smooth functions such that u_n tends to u in $W_p^{0,1}(Q(a,b))$ and locally uniformly in $\overline{Q(a,b)}$, and $\partial_t u_n$ converges to $\partial_t u$ weakly* in $(W_{p'}^{0,1}(Q(a,b)))'$ as $n \to \infty$.*

We conclude this section by defining the space $L^p(a,b;L^q(\mathbb{R}^d))$ as follows.

Definition A.4.7. • *For every $p,q \in (1,\infty)$ denote by $L^p(a,b;L^q(\mathbb{R}^d))$ the space of all the (equivalence classes of) measurable functions $f\colon Q(a,b) \to \mathbb{R}$ such that*

$$\int_a^b \|f(t,\cdot)\|_{L^q(\mathbb{R}^d)}^p \, dt < \infty.$$

It is a Banach space when endowed with the norm

$$\|f\|_{p,q} = \left(\int_a^b \|f(t,\cdot)\|_{L^q(\mathbb{R}^d)}^p \, dt\right)^{\frac{1}{p}}.$$

• *We denote by $L^\infty(a,b;L^2(\mathbb{R}^d))$ the space of all the (equivalence classes of) measurable functions $f\colon Q(a,b) \to \mathbb{R}$ such that*

$$\sup_{t\in(a,b)} \|f(t,\cdot)\|_{L^2(\mathbb{R}^d)} < \infty.$$

It is a Banach space when endowed with the norm

$$\|f\|_{\infty,2} = \sup_{t\in(a,b)} \|f(t,\cdot)\|_{L^2(\mathbb{R}^d)}.$$

Lemma A.4.8. *[33, Chapter 2, § 3] Let $d \geq 2$, p and q be given such that $\frac{1}{p} + \frac{d}{2q} = \frac{d}{4}$. Here we have $p \in [2,\infty]$ and $q \in [2, 2d/(d-2)]$ in the case where $d \geq 3$ and $p \in (2,\infty]$, $q \in [2,\infty]$ in the case where $d = 2$. Then every function in $W_2^{0,1}(Q(a,b)) \cap L^\infty(a,b;L^2(\mathbb{R}^d))$ belongs to $L^p(a,b;L^q(\mathbb{R}^d))$. Moreover, there is a constant c_S, which is independent of a,b in bounded subsets of $\mathbb{R}$, such that for $f \in W_2^{0,1}(Q(a,b)) \cap L^\infty(a,b;L^2(\mathbb{R}^d))$ we have*

$$\|f\|_{p,q} \leq c_S(\|f\|_{\infty,2} + \|\nabla f\|_2).$$

Appendix B

Introduction to semigroup theory

Semigroup theory has been widely studied and nowadays it is well understood. We refer for example to K.J. Engel and R. Nagel [19], T. Kato [28], A. Lunardi [40], L. Lorenzi and A. Rhandi [38]. In the following we provide a brief survey on semigroups of bounded linear operators on a Banach space $(X, \|\cdot\|)$. In particular, we first introduce strongly continuous semigroups and analytic semigroups. Subsequently, we deal with sub-Markovian and ultracontractive C_0-semigroups on L^2-spaces.

Now, we take a step back and we give the definition of semigroup.

Definition B.0.1. *A family $\{T(t)\colon t \geq 0\}$ of bounded and linear operators on X is called a* semigroup *(or semigroup of bounded operators) if it satisfies the semigroup property, i.e.,*

(a) $T(0) = I$,

(b) $T(t+s) = T(t)T(s)$ *for every* $t, s \geq 0$.

In order to simplify the notation, in the following we will write $T(\cdot)$. Moreover, we say that a semigroup $T(\cdot)$ is contractive if $\|T(t)\| \leq 1$ for any $t \geq 0$.

B.1 Spectrum and resolvent

Let $A\colon D(A) \subset X \to X$ be an operator on X. We define spectrum and resolvent set of A the following sets

$$\rho(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A\colon D(A) \to X \text{ is bijective with bounded inverse}\},$$
$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

Moreover, we call the resolvent of A the operator $R(\lambda, A) \in \mathcal{L}(X)$ defined by

$$R(\lambda, A) = (\lambda I - A)^{-1},$$

for any $\lambda \in \rho(A)$.

If $A : D(A) \subset X \to X$ is a closed linear operator, then the family $\{R(\lambda, A) \mid \lambda \in \rho(A)\}$ satisfies the resolvent identity

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A),$$

for any $\lambda, \mu \in \rho(A)$. Actually, if a family of operators satisfies the resolvent identity, then it's a resolvent family as the next proposition states.

Proposition B.1.1. *[38, Proposition A.4.6] Let $\Omega \subset \mathbb{C}$ be an open set, and let $\{F(\lambda) \colon \lambda \in \Omega\} \subset \mathcal{L}(X)$ be a family of linear operators verifying the resolvent identity*

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu),$$

for any $\lambda, \mu \in \Omega$. If the operator $F(\lambda_0)$ is injective, for some $\lambda_0 \in \Omega$, then there exists a closed linear operator $A \colon D(A) \subset X \to X$ such that $\rho(A)$ contains Ω and $R(\lambda, A) = F(\lambda)$ for each $\lambda \in \Omega$.

B.2 Strongly continuous semigroups

In this section we deal with the first important class of semigroups characterized by the strong continuity property.

Definition B.2.1. *A family of bounded operators $T(\cdot)$ on X, which satisfies the semigroup property, is a* strongly continuous semigroup *(or C_0-semigroup) if the function*

$$t \in [0, +\infty) \mapsto T(t)x \in X$$

is continuous for every $x \in X$.

The next result shows that the function $t \mapsto \|T(t)\|$ grows at most exponentially at infinity.

Proposition B.2.2. *Let $T(\cdot)$ be a C_0-semigroup. Then there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t},$$

for any $t \geq 0$.

Moreover, the following characterization of strong continuity holds.

Corollary B.2.3. *A semigroup of bounded operators $T(\cdot)$ on X is strongly continuous if and only if the function $t \mapsto T(t)x$ is continuous at $t = 0$ for any $x \in X$.*

It is possible to associate to the C_0-semigroup $T(\cdot)$ a linear operator, the infinitesimal generator, defined as follows

$$D(A) = \left\{ x \in X \mid \exists \lim_{t \to 0^+} \frac{T(t)x - x}{t} \in X \right\},$$
$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad x \in D(A).$$

It turns out that A is a closed linear operator whose domain $D(A)$ is dense in X. Moreover, $T(t)D(A) \subseteq D(A)$ and $AT(t)f = T(t)Af$, for all $t \geq 0$, $f \in D(A)$. For every $t > 0$ and $f \in X$ we have

$$\int_0^t T(s)f\,ds \in D(A) \quad \textit{and} \quad T(t)f - f = A\int_0^t T(s)f\,ds.$$

In particular, if $f \in D(A)$ then

$$A\int_0^t T(s)f\,ds = \int_0^t T(s)Af\,ds.$$

Proposition B.2.4. *Let $M \geq 1$ and $\omega \in \mathbb{R}$ be such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Then*

(a) $\rho(A) \supset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > \omega\}$*;*

(b) the resolvent operator is given by the Laplace trasform of $T(t)$, namely

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t}T(t)f\,dt,$$

for any $f \in X$, $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \omega$;

(c) For any $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \omega$ we have

$$\|(R(\lambda, A))^n\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^n}.$$

There is a close connection between C_0-semigroups and the abstract Cauchy problem
$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = f. \end{cases}$$
Indeed, for every $f \in D(A)$ the function $T(\cdot)f$ is differentiable and it is the unique solution of the previous Cauchy problem. For this reason, it is interesting to establish if A is the generator of a C_0-semigroup.

The first of this kind of results is the Hille-Yosida theorem.

Theorem B.2.5. *Let $A\colon D(A) \subset X \to X$ be a closed and densely defined operator ($\overline{D(A)} = X$). Then A is the generator of a C_0-semigroup on X if and only if there exist $\omega \geq 0$, $M > 0$ such that* *(a)* $\rho(A) \supset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > \omega\}$*;*

(b) *for any* $n \in \mathbb{N}$

$$\|(R(\lambda, A))^n\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^n}.$$

In this case, the semigroup $T(\cdot)$ *generated by* A *satisfies* $\|T(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$*.*

The second result we aim to state is the Lumer-Phillips theorem.

Theorem B.2.6. *Let* $A : D(A) \subset X \to X$ *be a densely defined operator. Moreover assume that* $\rho(A) \cap (0, +\infty) \neq \varnothing$ *and* A *is dissipative, i.e.*

$$\|\lambda f - Af\| \geq \lambda \|f\|,$$

for any $\lambda > 0$ *and* $f \in D(A)$*. Then* A *generates a* C_0*-semigroup of contractions on* X *(i.e.* $\|T(t)\| \leq 1$ *for any* $t \geq 0$*).*

B.3 Analytic semigroups

In this section we introduce another relevant class of semigroups of bounded operators: the analytic semigroups. For $\omega \in \mathbb{R}$ and $\theta_0 \in (\pi/2, \pi)$ we denote by

$$\Sigma_{\omega,\theta_0} := \{\lambda \in \mathbb{C} \mid \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta_0\}$$

the sector in $\mathbb{C}$ of angle θ_0. We now define sectorial operators; they are deeply connected to analytic semigroups.

Definition B.3.1. *Let* $A : D(A) \subset X \to X$ *be a closed linear operator.* A *is called* sectorial *in* X *if there exist* $\omega \in \mathbb{R}$*,* $\theta_0 \in (\pi/2, \pi)$ *and* $M > 0$ *such that* $\rho(A) \supseteq \Sigma_{\omega,\theta_0}$ *and*

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq M|\lambda - \omega|^{-1},$$

for any $\lambda \in \Sigma_{\omega,\theta_0}$*.*

Moreover, $S(\omega, \theta_0, M)$ denotes the set of sectorial operators which satisfy the previous definition. Then, if $A \in S(\omega, \theta_0, M)$, we define

$$T(t) = \frac{1}{2\pi i} \int_{\gamma_{r,\eta,\omega}} e^{t\lambda} R(\lambda, A)\, d\lambda, \tag{B.1}$$

for any $t > 0$, where $\gamma_{r,\eta,\omega}$ is the union of the following curves

$$\gamma_{1,r,\eta,\omega} : \rho \in [r, \infty) \mapsto \omega + \rho\, e^{-i\eta} \in \mathbb{C},$$
$$\gamma_{2,r,\eta,\omega} : \theta \in [-\eta, \eta] \mapsto \omega + r\, e^{i\theta} \in \mathbb{C},$$
$$\gamma_{3,r,\eta,\omega} : \rho \in [r, \infty) \mapsto \omega + \rho\, e^{i\eta} \in \mathbb{C},$$

with $r > 0$ and $\eta \in (\pi/2, \theta_0)$ fixed. It turns out that the above expression does not depend on r and η and the following theorem holds.

Theorem B.3.2. *Let $A \in S(\omega, \theta_0, M)$ and $T(t)$ be defined as in (B.1) for any $t > 0$. Then the following statements hold.*

(a) *For any $x \in X$, $k \in \mathbb{N}$ and $t > 0$, $T(t)x \in D(A^k)$. Moreover, if $x \in D(A^k)$, then $A^k T(t)x = T(t)A^k x$ for all $t \geq 0$.*

(b) *If we set $T(0) = I$, the family $T(\cdot)$ defines a semigroup of bounded operators.*

(c) *There exists $M_k > 0$ ($k \in \mathbb{N} \cup \{0\}$) such that*

$$\left\| t^k (A - \omega I)^k T(t) \right\| \leq M_k e^{\omega t},$$

for any $t > 0$ and $k \in \mathbb{N} \cup \{0\}$.

(d) *The function $t \mapsto T(t)$ belongs to $C^\infty((0, \infty), L(X))$.*

(e) *We have $D_t^k T(t) = A^k T(t)$ for any $t > 0$.*

(f) *The function $t \mapsto T(t)$ admits an analytic extention to the sector $\Sigma_{0,\theta_0 - \pi/2}$ given by*

$$T(z) = \frac{1}{2\pi i} \int_{\gamma_{r,\theta'_z,\omega}} e^{\lambda z} R(\lambda, A)\, d\lambda,$$

for any $z \in \Sigma_{0,\theta_0-\pi/2}$, where θ'_z is arbitrarily fixed in $(\pi/2, \theta_0 - \arg(z))$.

By means of the previous results we define an analytic semigroup as follows.

Definition B.3.3. *Let $A : D(A) \subset X \to X$ be a sectorial operator. Then the family $T(\cdot)$ defined by (B.1) for $t > 0$ such that $T(0) = I$ is called analytic semigroup generated by A in X.*

We now state some properties of the semigroup $T(\cdot)$.

Proposition B.3.4. *Let $T(\cdot)$ be the analytic semigroup generated by $A \in S(\omega, \theta_0, M)$. Then the following properties hold true.*

(a) *For each $x \in X$ and $t > 0$, $\int_0^t T(s)x\, ds \in D(A)$ and*

$$A \int_0^t T(s)x\, ds = T(t)x - x.$$

If, in addition, $x \in D(A)$, then

$$T(t)x - x = \int_0^t T(s)Ax\, ds,$$

for any $t \geq 0$.

(b) *If $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, then*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\, dt.$$

One can wonder if A generates a C_0-semigroup. In general the answer is negative. However, if $D(A)$ is dense in X, then $T(\cdot)$ is a C_0-semigroup whose infinitesimal generator is the sectorial operator A.

B.4 Sub-Markovian and ultracontractive C_0-semigroups on L^2

In this section we deal with C_0-semigroups on $L^2(\Omega)$, where Ω is a subset of $\mathbb{R}^d$. We start by giving some definitions.

Definition B.4.1. *Let $\Omega \subset \mathbb{R}^d$ and $T(\cdot)$ be a C_0-semigroup on $L^2(\Omega)$. We say that*

- $T(\cdot)$ *is* real *if, given a real-valued function f, then $T(t)f$ is real-valued for all $t \geq 0$;*
- $T(\cdot)$ *is* positive *if $T(t)f \geq 0$ for all $t \geq 0$ and $f \geq 0$;*
- $T(\cdot)$ *is* L^∞-contractive *if $\|T(t)f\|_\infty \leq \|f\|_\infty$ for all $t \geq 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$;*
- $T(\cdot)$ *is* sub-Markovian *if it is positive and L^∞-contractive;*
- $T(\cdot)$ *is* symmetric *if $T^*(\cdot) = T(\cdot)$;*
- $T(\cdot)$ *is* ultracontractive *if there is a constant $c > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ct^{-\frac{d}{2}},$$

  for all $t > 0$.

To establish ultracontractivity we use the following useful result, see [4, Proposition 1.5], where we replace the H^1-norm with the L^2-norm of the gradient. The proof remains the same and is based on Nash's inequality

$$\|u\|_2^{1+\frac{2}{d}} \leq c_d \||\nabla u|\|_2 \|u\|_1^{\frac{2}{d}}, \tag{B.2}$$

for all $u \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$.

Proposition B.4.2. *Let $T(\cdot)$ be a C_0-semigroup on $L^2(\mathbb{R}^d)$ such that $T(\cdot)$ and $T^*(\cdot)$ are sub-Markovian. Assume that, for $\delta > 0$, the generator A of $T(\cdot)$ satisfies*

(a) $D(A) \subset H^1(\mathbb{R}^d)$;

(b) $\langle -Au, u\rangle \geq \delta \||\nabla u|\|_2^2, \quad \forall u \in D(A)$;

(c) $\langle -A^*u, u\rangle \geq \delta \||\nabla u|\|_2^2, \quad \forall u \in D(A^*)$.

Then there is $c_\delta > 0$ such that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c_\delta t^{-d/2}, \quad \forall t > 0,$$

i.e. $T(\cdot)$ is ultracontractive.

Proof. Since $T^*(\cdot)$ is sub-Markovian, the L^∞-contractivity of $T^*(\cdot)$ implies that $T(\cdot)$ is contractive on $L^1(\mathbb{R}^d)$, that is

$$\|T(t)f\|_1 \leq \|f\|_1\,, \tag{B.3}$$

for any $t \geq 0$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Therefore, $T(\cdot)$ extrapolates to a C_0-semigroup on $L^1(\mathbb{R}^d)$ (see [5, Section 7.2]). Hence for $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\lambda R(\lambda, A)f \to f$ in $L^1(\mathbb{R}^d)$ and in $L^2(\mathbb{R}^d)$ as $\lambda \to \infty$. Given that $\lambda R(\lambda, A)f \in D(A)$, it follows that $D(A) \cap L^1(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

We now prove that

$$\|T(t)f\|_2 \leq \left(\frac{dc_d^2}{4\delta}\right)^{d/4} t^{-d/4}\,\|f\|_1\,, \tag{B.4}$$

for every $t \geq 0$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let $f \in D(A) \cap L^1(\mathbb{R}^d)$. Since $T(t)f \in D(A)$, by (b) we have

$$\frac{d}{dt}\|T(t)f\|_2^2 = 2\langle AT(t)f, T(t)f\rangle_2 \leq -2\delta\,\|\nabla T(t)f\|_2^2\,.$$

Thus, considering that $T(t)f \in H^1(\mathbb{R}^d)$ by (a), we apply Nash's inequality (B.2) and we deduce that

$$\frac{d}{dt}\|T(t)f\|_2^2 \leq -\frac{2\delta}{c_d^2}\frac{\|T(t)f\|_2^{2+4/d}}{\|T(t)f\|_1^{4/d}}.$$

As a result, we derive

$$\begin{aligned}
\frac{d}{dt}(\|T(t)f\|_2^2)^{-\frac{2}{d}} &= -\frac{2}{d}(\|T(t)f\|_2^2)^{-\frac{2}{d}-1}\frac{d}{dt}\|T(t)f\|_2^2 \\
&\geq \frac{4\delta}{dc_d^2}(\|T(t)f\|_2)^{-\frac{4}{d}-2}\frac{\|T(t)f\|_2^{2+4/d}}{\|T(t)f\|_1^{4/d}} \\
&= \frac{4\delta}{dc_d^2}\|T(t)f\|_1^{-4/d} \geq \frac{4\delta}{dc_d^2}\|f\|_1^{-4/d}\,,
\end{aligned}$$

where the last inequality follows by (B.3). Integrating, we obtain

$$\|T(t)f\|_2^{-4/d} = \int_0^t \frac{d}{ds}(\|T(s)f\|_2^2)^{-\frac{2}{d}}ds + (\|f\|_2^2)^{-\frac{2}{d}} \geq \frac{4\delta}{dc_d^2}t\,\|f\|_1^{-4/d}\,.$$

Since $D(A)\cap L^1(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, the previous inequality holds true for any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Thus, (B.4) follows. If we now repeat the same computations for the adjoint semigroup $T^*(\cdot)$ applying (c) instead of (b), we get

$$\|T^*(t)f\|_2 \leq \left(\frac{dc_d^2}{4\delta}\right)^{d/4} t^{-d/4}\,\|f\|_1\,,$$

for every $t \geq 0$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then

$$\|T(t)f\|_\infty \leq \left(\frac{dc_d^2}{4\delta}\right)^{d/4} t^{-d/4}\,\|f\|_2\,, \tag{B.5}$$

for every $t \geq 0$ and $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Combining (B.4) with (B.5) yields

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq \|T(t/2)\|_{\mathcal{L}(L^1,L^2)} \|T(t/2)\|_{\mathcal{L}(L^2,L^\infty)} \leq \left(\frac{dc_d^2}{4\delta}\right)^{d/2} t^{-d/2},$$

for every $t \geq 0$. Therefore, the semigroup $T(\cdot)$ is ultracontractive with

$$c_\delta = \left(\frac{dc_d^2}{4\delta}\right)^{d/2}. \qquad \square$$

The following result displays an important feature of ultracontractive semigroups, namely they are given through an integral kernel (see for example [38, Theorem 15.1.3]).

Theorem B.4.3. *Let $\Omega \subset \mathbb{R}^d$. If a C_0-semigroup $T(\cdot)$ on $L^2(\Omega)$ is ultracontractive, then for every $t > 0$ there exists an integral kernel $p(t,\cdot,\cdot) \in L^\infty(\Omega \times \Omega)$ such that*

$$T(t)f(x) = \int_\Omega p(t,x,y)f(y)\,dy,$$

for every $t > 0$ and $f \in L^1(\Omega) \cap L^2(\Omega)$. Moreover, $\|k(t,\cdot,\cdot)\|_{L^\infty(\Omega\times\Omega)} \leq ct^{-\frac{d}{2}}$ for every $t > 0$ and some constant $c > 0$.

We now give the definition of consistent semigroup.

Definition B.4.4. *Let $\Omega \subset \mathbb{R}^d$ and $1 \leq p_1 < p_2 \leq \infty$. A semigroup $T_p(\cdot)$ which is defined on $L^p(\Omega)$ for $p \in [p_1,p_2]$ is called* consistent *if $T_p(t)f = T_q(t)f$ for all $t > 0$, $q \in [p_1,p_2]$ and $f \in L^p(\Omega) \cap L^q(\Omega)$.*

As in [18, Theorem 1.4.1], we see that a symmetric sub-Markovian semigroup on $L^2(\Omega)$ gives rise to consistent semigroups.

Theorem B.4.5. *Let $\Omega \subset \mathbb{R}^d$. If $T(\cdot)$ is a symmetric sub-Markovian semigroup on $L^2(\Omega)$, then $L^1(\Omega) \cap L^\infty(\Omega)$ is invariant under $T(\cdot)$ and $T(\cdot)$ may be extended from $L^1(\Omega) \cap L^\infty(\Omega)$ to a positive contraction semigroup $T_p(\cdot)$ on $L^p(\Omega)$ for all $1 \leq p \leq \infty$. These semigroups are strongly continuous if $1 \leq p < \infty$ and are consistent.*

Appendix C

Classical results on PDE's of elliptic and parabolic problems

In this appendix we recall some classical results we used in the previous chapters, such as interior Schauder estimates and some maximum principle.

Let Ω be an open set of $\mathbb{R}^d$. We consider the second order elliptic partial differential operator A defined by

$$A\varphi(x) = \sum_{i,j=1}^{d} q_{ij}(x) D_{ij}\varphi(x) + \sum_{i=1}^{d} F_i(x) D_i\varphi(x) - V(x)\varphi(x), \quad x \in \Omega,$$

on smooth functions, with real coefficients q_{ij}, F_i and V defined in Ω. Throughout, we keep the following assumptions.

Hypothesis C.0.1. *The matrix $Q = (q_{ij})_{i,j=1,\dots,d}$ is symmetric and uniformly elliptic, i.e. there is $\eta > 0$ such that*

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2 \quad \textit{for all } \xi \in \Omega,\ x \in \overline{\Omega}.$$

C.1 A priori estimates

In the following theorem we state some well-known interior L^p-estimates.

Theorem C.1.1. *[36, Theorem C.1.1] Let Ω be an open set and p any real number in the interval $(1, \infty)$. If the coefficients of the operator A are bounded and continuous in Ω, then for any open set $\Omega' \subset\subset \Omega$ there exists a positive constant C, depending on p, Ω, Ω', η and the moduli of continuity of q_{ij} in Ω', such that*

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_{L^p(\Omega)} + \|Au\|_{L^p(\Omega)}),$$

for any $u \in L^p(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ such that $Au \in L^p(\Omega)$.

The next two theorems provide us with Schauder estimates.

Theorem C.1.2. *[36, Theorem C.1.4] Assume that the coefficients of the operator A belong to $C_b^\zeta(\overline{\Omega})$ for some $\zeta \in (0,1)$. Further, assume that $u \in C_{\rm loc}^{1+\zeta/2,2+\zeta}((0,T)\times\Omega)$ is a bounded (with respect to the sup-norm) solution of the equation $\partial_t u(t,x) - Au(t,x) = 0$ for every $t \in (0,T)$, $x \in \Omega$. Then, for any open set $\Omega' \subset\subset \Omega$ and any $s \in (0,T)$, there exists a positive constant C depending on s, the coefficients of the operator A, Ω, Ω' and T such that*

$$\|u\|_{C^{1+\zeta/2,2+\zeta}([s,T)\times\Omega')} \le C \sup_{(0,T)\times\Omega} |u|.$$

Theorem C.1.3. *[36, Theorem C.1.5] Let Ω be an open subset of $\mathbb{R}^d$ with boundary of class $C^{2+\zeta}$ for some $\zeta \in (0,1)$, and let Ω' and Ω'' be two bounded subsets of Ω such that $\Omega' \subset \Omega'' \subset \Omega$ and $\operatorname{dist}(\Omega', \Omega\backslash\Omega'') > 0$. Moreover, assume that the coefficients of the operator A belong to $C_{\rm loc}^\zeta(\overline{\Omega})$. Finally, assume that $u \in C^{1+\zeta/2,2+\zeta}([T_1,T_2]\times\overline{\Omega}'')$ solves the differential equation $\partial_t u - Au = 0$ in $(T_1,T_2)\times\Omega''$ for some $0 \le T_1 < T_2$. Then, if $u \equiv 0$ on $(T_1,T_2)\times\partial\Omega''$, we have that for any $T^* \in (T_1,T_2)$ there exists a positive constant C depending on $T^*, T_1, T_2, \Omega', \Omega''$ such that*

$$\|u\|_{C^{1+\zeta/2,2+\zeta}([T^*,T_2]\times\overline{\Omega}')} \le C\,\|u\|_{L^\infty((T_1,T_2)\times\Omega'')}\,.$$

C.2 Classical maximum principles

In this section we collect the classical maximum principles for continuous solutions to both the elliptic equations and for the parabolic Cauchy problems. We make the following assumptions.

Hypothesis C.2.1. *(a) Ω is either an open bounded set with boundary of class C^2 or $\Omega = \mathbb{R}^d$;*

(b) q_{ij}, F_i and $0 \le V$ belong to $C_b(\overline{\Omega})$;

(c) Hypothesis C.0.1 is satisfied.

We start with the classical maximum principle for elliptic equations.

Theorem C.2.2. *[36, Theorem C.2.2] Let $\lambda > 0$ and suppose that $u \in W^{2,p}_{\rm loc}(\Omega)$ for all $1 < p < \infty$ satisfies the differential inequality $\lambda u - Au \ge 0$. Then*

(a) if $u \ge 0$ on $\partial\Omega$, then $u \ge 0$ in Ω;

(b) if $f \in C_b(\overline{\Omega})$ and $u \in W^{2,p}_{\rm loc}(\Omega)$ for all $1 \le p < \infty$ solves the problem

$$\begin{cases}\lambda u(x) - Au(x) = f(x), & x \in \Omega,\\ u(x) = 0, & x \in \partial\Omega,\end{cases}$$

then

$$\|u\|_\infty \le \frac{\|f\|_\infty}{\lambda}.$$

We now state the weak parabolic maximum principle in the form we need.

Proposition C.2.3. *[36, Proposition C.2.3] Fix $T > 0$. Let $u \in C^{1,2}((0,T] \times \Omega) \cap C_b([0,T] \times \overline{\Omega})$ be such that*

$$\begin{cases} \partial_t u(t,x) - Au(t,x) \geq 0, & t \in (0,T],\ x \in \Omega, \\ u(t,x) \geq 0, & t \in (0,T],\ x \in \partial\Omega, \\ u(0,x) \geq 0, & x \in \overline{\Omega}. \end{cases}$$

Then $u \geq 0$ in $[0,T] \times \overline{\Omega}$.

C.3 Existence of classical solution to PDE's and analytic semigroups

Here, we adopt the following assumptions on Ω and on the coefficients of the operator A.

Hypothesis C.3.1. *(a) Ω is either an open set with a boundary which is uniformly of class $C^{2+2\zeta}$ for some $\zeta \in (0,1)$ or $\Omega = \mathbb{R}^d$;*

(b) q_{ij}, F_i and $0 \leq V$ belong to $C_b^{2\zeta}(\overline{\Omega})$;

(c) The matrix $Q = (q_{ij})_{i,j=1,\ldots,d}$ is symmetric and uniformly elliptic, i.e. there is $\eta > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d,\ x \in \overline{\Omega}.$$

Proposition C.3.2. *[36, Proposition C.3.2] For every $f \in C_b(\overline{\Omega})$ the Cauchy-Dirichlet problem*

$$\begin{cases} \partial_t u(t,x) = Au(t,x), & t > 0, x \in \Omega, \\ u(t,x) = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = f(x), & x \in \Omega, \end{cases}$$

admits a unique solution $u \in C([0,\infty) \times \overline{\Omega} \setminus (\{0\} \times \partial\Omega)) \cap C^{1,2}((0,\infty) \times \Omega)$ which is bounded in $[0,T] \times \overline{\Omega}$ for any $T > 0$. Moreover, $\|u(t,\cdot)\|_\infty \leq \|f\|_\infty$ for every $t > 0$.

Moreover, the following result involving analytic semigroups holds.

Theorem C.3.3. *[36, Theorem C.3.6] The realization of the operator A with domain*

$$D(A) = \{u \in C_0(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega) \text{ for all } 1 \leq p < \infty \colon Au \in C(\overline{\Omega})\}$$

is sectorial in $C_b(\overline{\Omega})$.

Concerning the elliptic equation, we have the following result.

Proposition C.3.4. *[36, Proposition C.3.4] For every $f \in C_b(\overline{\Omega})$ and any $\lambda > 0$, there exists a unique solution $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for all $1 \le p < \infty$ to the Dirichlet problem*

$$\begin{cases} \lambda u(x) - Au(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

C.4 Local regularity of transition densities

In this section we combine the results of [8] with the Schauder estimates to obtain regularity properties of the transition kernel associated with the second order elliptic operator defined as in Chapter 3 by

$$A\varphi = \mathrm{div}(Q\nabla\varphi) + F \cdot \nabla\varphi - V\varphi.$$

We assume that the diffusion coefficients q_{ij} and their spatial derivatives $D_h q_{ij}$ are bounded on $\mathbb{R}^d$ for all $i, j, h = 1, \dots, d$, whereas the drift F and the potential V can also be unbounded. More precisely, we make the following assumptions.

Hypothesis C.4.1. *(a) We have $q_{ij} \in C_b^{1+\zeta}(\mathbb{R}^d)$, $F_i \in C^{\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$, $0 \le V \in C^{\zeta}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\zeta \in (0,1)$;*

(b) The matrix $Q = (q_{ij})_{i,j=1,\dots,d}$ is symmetric and uniformly elliptic, i.e. there is $\eta > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \ge \eta|\xi|^2 \quad \textit{for all } x,\ \xi \in \mathbb{R}^d.$$

We consider the minimal semigroup $T(\cdot)$ in $C_b(\mathbb{R}^d)$ generated by A as constructed in Chapter 1. It is given through an integral kernel p as follows

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy, \quad t > 0,\ x \in \mathbb{R}^d,\ f \in C_b(\mathbb{R}^d).$$

Then, the following result shows some regularity properties of p with respect to all the variables (t, x, y).

Proposition C.4.2. *[34, Proposition 2.1] The kernel $p(t,x,y)$ is a positive continuous function in $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ which enjoys the following properties.*

(a) For every $x \in \mathbb{R}^d$, $1 < s < \infty$, the function $p(\cdot, x, \cdot)$ belongs to $\mathcal{H}^{s,1}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$. In particular $p, D_y p \in L^s_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ for all $i = 1, \dots, d$ and $p(\cdot, x, \cdot)$ is continuous.

(b) *For every* $y \in \mathbb{R}^d$ *the function* $p(\cdot,\cdot,y)$ *belongs to* $C^{1+\zeta/2,2+\zeta}_{\rm loc}((0,\infty)\times\mathbb{R}^d)$ *and solves the equation* $\partial_t p = Ap$ *for* $t>0$. *Moreover*

$$\sup_{|y|\le R} \|p(\cdot,\cdot,y)\|_{C^{1+\zeta/2,2+\zeta}([\varepsilon,T]\times B_R)} < \infty,$$

for every $0<\varepsilon<T$ *and* $R>0$.

(c) *If, in addition,* $F \in C^1(\mathbb{R}^d;\mathbb{R}^d)$, *then* $p(\cdot,x,\cdot) \in W^{1,2}_{s,{\rm loc}}(Q(0,T))$ *for every* $x \in \mathbb{R}^d$, $1<s<\infty$ *and satisfies the equation* $\partial_t p - A^*_y p = 0$, *where*

$$A^* = {\rm div}(Q\nabla) - F\cdot\nabla - (V + {\rm div}F)$$

is the formal adjoint of A.

Appendix D

Semigroups associated with sesquilinear forms

In this appendix we give an overview of sesquilinear form theory and associated operators and semigroups. We refer to the book of Ouhabaz [49] for a wide description on this subject.

Let H be a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $D(\mathfrak{a})$ a linear subspace of H. We denote by $\langle \cdot, \cdot \rangle$ the inner product of H and by $\|\cdot\|$ the corresponding norm.

Definition D.0.1. *An application*

$$\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$$

is called unbounded sesquilinear form *if for every $\alpha \in \mathbb{K}$ and $u, v, w \in D(\mathfrak{a})$ we have*

$$\mathfrak{a}(\alpha u + v, w) = \alpha \mathfrak{a}(u, w) + \mathfrak{a}(v, w)$$

and

$$\mathfrak{a}(u, \alpha v + w) = \overline{\alpha} \mathfrak{a}(u, v) + \mathfrak{a}(u, w).$$

The space $D(\mathfrak{a})$ is the domain of $\mathfrak{a}$.

We now introduce some relevant properties a sesquilinear form may enjoy.

Definition D.0.2. *Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$ be a sesquilinear form. We say that*

(a) $\mathfrak{a}$ *is* densely defined *if*

$$D(\mathfrak{a}) \text{ is dense in } H. \tag{D.1}$$

(b) $\mathfrak{a}$ *is* accretive *if*

$$\operatorname{Re} \mathfrak{a}(u, u) \geq 0 \text{ for all } u \in D(\mathfrak{a}). \tag{D.2}$$

(c) $\mathfrak{a}$ *is* continuous *if there exists a non-negative constant* M *such that*

$$|\mathfrak{a}(u,v)| \leq M \|u\|_{\mathfrak{a}} \|v\|_{\mathfrak{a}} \text{ for all } u,v \in D(\mathfrak{a}). \tag{D.3}$$

where $\|u\|_{\mathfrak{a}} := \sqrt{\operatorname{Re}\mathfrak{a}(u,u) + \|u\|^2}$.

(d) $\mathfrak{a}$ *is* closed *if*

$$(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}) \text{ is a complete space.} \tag{D.4}$$

If the form $\mathfrak{a}$ satisfies conditions (D.1)-(D.4), then $\|\cdot\|_{\mathfrak{a}}$ is a norm on $D(\mathfrak{a})$, the norm associated with the form $\mathfrak{a}$, and $D(\mathfrak{a})$ is a Hilbert space.

Definition D.0.3. *Let* $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$ *be a sesquilinear form. The adjoint form of* $\mathfrak{a}$ *is the sesquilinear form* $\mathfrak{a}^*$ *defined by*

$$\mathfrak{a}^*(u,v) := \overline{\mathfrak{a}(v,u)},$$

with domain $D(\mathfrak{a}^*) = D(\mathfrak{a})$*. We say that* $\mathfrak{a}$ *is a symmetric form if* $\mathfrak{a}^* = \mathfrak{a}$*, that is*

$$\mathfrak{a}(u,v) := \overline{\mathfrak{a}(v,u)},$$

for all $u, v \in D(\mathfrak{a})$.

D.1 Generation result

One can associate an operator to sesquilinear forms enjoying the properties mentioned above.

Definition D.1.1. *Let* $\mathfrak{a}$ *be a densely defined, accretive, continuous and closed sesquilinear form on* H*. The unbounded operator* A *defined by*

$$\begin{aligned} D(A) &= \{u \in D(\mathfrak{a}) \mid \exists v \in H \colon \mathfrak{a}(u,\phi) = \langle v, \phi\rangle \ \forall \phi \in D(\mathfrak{a})\}, \\ Au &= v, \quad u \in D(A) \end{aligned}$$

is called the operator associated with the form $\mathfrak{a}$.

The following result clarifies the connection between sesquilinear forms and semigroups.

Proposition D.1.2. *[49, Proposition 1.51] The operator* $-A$ *is the generator of a strongly continuous contraction semigroup on* H*.*

Remark D.1.3. *Let* $(V, \|\cdot\|_V)$ *be an Hilbert space that is continuously and densely injected into* H *(we write* $V \overset{d}{\hookrightarrow} H$*), i.e.* $V \subset H$*,* V *is dense in* H *for the norm of* H *and there exists a constant* $c > 0$ *such that*

$$\|u\| \leq c \|u\|_V \text{ for all } u \in V. \tag{D.5}$$

Let $\mathfrak{a} : V \times V \to \mathbb{K}$ *be a sesquilinear form satifying the following conditions:*

(a) there exists a non-negative constant M such that

$$|\mathfrak{a}(u,v)| \leq M \|u\|_V \|v\|_V \text{ for all } u,v \in V. \tag{D.6}$$

(b) $\mathfrak{a}$ is coercive, i.e. there exists $\nu > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) \geq \nu \|u\|_V^2 \text{ for all } u \in V. \tag{D.7}$$

Then $\mathfrak{a}$ is densely defined and accretive. Moreover, we note that the norms $\|\cdot\|_V$ and $\|\cdot\|_{\mathfrak{a}}$ are equivalent. Indeed, by (D.7), we have that

$$\|u\|_{\mathfrak{a}}^2 = \operatorname{Re} \mathfrak{a}(u,u) + \|u\|^2 \geq \operatorname{Re} \mathfrak{a}(u,u) \geq \nu \|u\|_V^2$$

and, by (D.5) and (D.6),

$$\|u\|_{\mathfrak{a}}^2 = \operatorname{Re} \mathfrak{a}(u,u) + \|u\|^2 \leq |\mathfrak{a}(u,u)| + c^2 \|u\|_V^2 \leq (M + c^2) \|u\|_V^2 .$$

Consequently, the form $\mathfrak{a}$ is also continuous and closed. Then, by Proposition D.1.2, $-A$ is the generator of a strongly continuous contraction semigroup on H.

D.2 Positive and L^∞-contractive semigroups

Let $H = L^2(\Omega)$ and $\mathfrak{a}$ be a densely defined, accretive, continuous and closed sesquilinear form on $L^2(\Omega)$. Denote by A its associated operator and by $(e^{-tA})_{t\geq 0}$ the semigroup generated by $-A$ on $L^2(\Omega)$. The following criteria provide us with equivalent conditions on the sesquilinear form $\mathfrak{a}$ to check if the semigroup $(e^{-tA})_{t\geq 0}$ is positive and L^∞-contractive.

Theorem D.2.1 (First Beurling-Deny criterion). *[49, Theorem 2.6] The following assertions are equivalent.*

(a) The semigroup $(e^{-tA})_{t\geq 0}$ is positive.

(b) $u \in D(\mathfrak{a}) \implies (\operatorname{Re} u)^+ \in D(\mathfrak{a}), \mathfrak{a}(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R}$ and $\mathfrak{a}((\operatorname{Re} u)^+, (\operatorname{Re} u)^-) \leq 0$.

Theorem D.2.2 (Second Beurling-Deny criterion). *[49, Theorem 2.13] The following assertions are equivalent.*

(a) The semigroup $(e^{-tA})_{t\geq 0}$ is L^∞-contractive.

(b) $u \in D(\mathfrak{a}) \implies (1 \wedge |u|) \operatorname{sign} u \in D(\mathfrak{a})$ and $\operatorname{Re} \mathfrak{a}((1 \wedge |u|) \operatorname{sign} u, (|u| - 1)^+ \operatorname{sign} u) \geq 0$.

List of Symbols

Sets

$\mathbb{N}$	set of all positive natural numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{R}^d$	euclidean d-dimensional space
B_ρ	open disk with centre at 0 and radius $\rho > 0$
$\overline{B}_\rho$	the closure of B_ρ
$Q(a,b)$	the strip $(a,b) \times \mathbb{R}^d$
$A \subset\subset B$	given two subsets $A, B \subset \mathbb{R}^d$ with B open, it means that $\overline{A}$ is contained in B
$\varnothing$	empty set
$\mathcal{L}(X,Y)$	space of all bounded linear operators T from X into Y with $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$
$\mathcal{L}(X)$	$:= \mathcal{L}(X,X)$
$\mathcal{D}(\mathbb{R}^d)$	$:= C_c^\infty(\mathbb{R}^d)$ space of test functions
$\mathcal{D}(\mathbb{R}^d)'$	space of distributions on $\Omega \subset \mathbb{R}^d$

Matrix and linear algebra

e_j	the j-th vector of the canonical basis of $\mathbb{R}^d$
$\langle x, y\rangle$	inner euclidean product between the vectors $x, y \in \mathbb{R}^d$
$x \cdot y$	$=: \langle x, y\rangle$
$\|x\|$	euclidean norm of $x \in \mathbb{R}^d$
$\|Q\|$	euclidean norm of the $d\times d$ matrix $Q = (q_{ij})$, i.e. $\|Q\|^2 = \sum_{i,j=1}^d \|q_{ij}\|^2$
$\\|Q\\|_\infty$	norm of the $d \times d$ matrix $Q = (q_{ij})$ if the entries depend on $x \in \Omega \subset \mathbb{R}^d$, i.e. $\\|Q\\|_\infty^2 = \sum_{i,j=1}^d \\|q_{ij}\\|_\infty^2$
$\|\nabla Q\|$	euclidean norm of the gradient of the matrix $Q = (q_{ij})$ whose entries are continuously differentiable in an open set $\Omega \subset \mathbb{R}^d$, i.e. $\|\nabla Q\|^2 = \sum_{i,j,k=1}^d \|D_k q_{ij}\|^2$

Functions

χ_E	characteristic function of the set E, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$
$\mathbb{1}$	the function identically equal to 1
f^+	positive part of the real-valued function $f\colon \Omega \subset \mathbb{R}^d \to \mathbb{R}$, i.e. $f^+(x) = \max(f(x), 0)$ for every $x \in \Omega$
f^-	negative part of the real-valued function $f\colon \Omega \subset \mathbb{R}^d \to \mathbb{R}$, i.e. $f^-(x) = \min(f(x), 0)$ for every $x \in \Omega$
$\overline{f}$	complex conjugate of the complex function of $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}$
$\mathrm{Re} f$	real part of the function $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}$
$\mathrm{Im} f$	imaginary part of the function $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}$
$\mathrm{sign} f$	sign of the function $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}$ defined as $\frac{u(x)}{\lvert u(x)\rvert}$ if $u(x) \neq 0$ and 0 if $u(x) = 0$
$\partial_t f$	$= \frac{\partial f}{\partial t}$ time derivative of a function $f : I \times \mathbb{R}^d \to \mathbb{R}$, where $I \subset [0, \infty)$ is an interval
$D_i f$	$= \frac{\partial f}{\partial x_i}$ i-th spatial derivative of a function $f : I \times \mathbb{R}^d \to \mathbb{R}$, where $I \subset [0, \infty)$ is an interval
$D_{ij} f$	$= D_i D_j f$ second order spatial derivative of a function $f : I \times \mathbb{R}^d \to \mathbb{R}$, where $I \subset [0, \infty)$ is an interval
∇f	$= (D_1 f, \dots, D_d f)$ gradient of f
$\lvert\nabla f\rvert^2$	$= \sum_{j=1}^d \lvert D_j f\rvert^2$
$\lvert D^2 f\rvert^2$	$= \sum_{i,j=1}^d \lvert D_{ij} f\rvert^2$
$\mathrm{div}(F)$	$= \sum_{i=1}^d D_i F_i$ divergence of $F : \mathbb{R}^d \to \mathbb{R}^d$

Operators

I	identity operator in a Banach space X

Miscellanea

$\mathrm{supp}(f)$	support of a function f
$x \vee y$	maximum between $x, y \in \mathbb{R}$
$[x]$	integer part of $x \in \mathbb{R}$
$\lvert\alpha\rvert$	length of the multi-index α, i.e. $\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_d$
δ_{ij}	Kronecker delta, i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 1$ otherwise
$\mathrm{dist}(x, \Omega_2)$	distance of the point x from the set Ω_2, i.e. $\mathrm{dist}(x, \Omega_2) = \inf_{x \in \Omega_2} \lvert x - y\rvert$

$\mathrm{dist}(\Omega_1, \Omega_2)$	distance of the set Ω_1 from the set Ω_2, i.e. the number $\mathrm{dist}(\Omega_1, \Omega_2) = \inf_{x \in \Omega_1} \mathrm{dist}(x, \Omega_2)$
dx	Lebesgue measure in $\mathbb{R}^d$

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