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Maximum principles, entire solutions and removable singularities of fully nonlinear second order equations

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"Do not worry about your difficulties in Mathematics.
I can assure you mine are still greater."
Albert Einstein

Introduction

This PhD thesis is devoted to some qualitative aspect of viscosity solutions of nonlinear second order elliptic partial differential equations of the form

$$F(x, u(x), Du(x), D^2u(x)) = f(x), \tag{1}$$

where $F : \Omega \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ and $f : \Omega \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ are prescribed functions, Ω is a domain, $\mathcal{S}^n$ is the linear space of symmetric $n \times n$ real matrices and u is the real valued unknown function. The gradient Du and the Hessian matrix D^2u do not have a classical meaning, but they are understood in a weak sense.

The notion of viscosity solution firstly appeared in the early 80s in the works of M.G. Crandall, L.C. Evans, P.L. Lions [16]-[17]-[18] relatively to the first order Hamilton-Jacobi equations and afterwards extended by Jensen [40] to the second order case.

Many other authors, among whom Cabré, Caffarelli, Hishii, Świech, Trudinger, have contributed to the development of this theory showing existence, uniqueness, regularity, approximation and stability results so emphasizing the flexibility and the usefulness of viscosity solutions to handle nonlinear elliptic (and parabolic) problems. Moreover the range of applications is plentiful: optimal control, differential game, front propagation, Mathematical Finance are only some of the research fields in which viscosity theory applies.

In this spirit Chapter 1 is a brief introduction to viscosity solutions. First we deal with the continuous case (F and f in (1) are assumed to be continuous functions) and present some results: comparison principle, Perron method, stability and half-relaxed limits, Jensen approximation. In the last section we consider L^p-viscosity solutions, a suitable notion of viscosity solutions to treat equations with measurable dependence on x.

Chapter 2 is concerned with the existence and uniqueness of entire solutions (i.e. solutions in the whole space $\mathbb{R}^n$) of (1). The operator F is uniformly elliptic and $F(\cdot, r, \cdot, \cdot)$ satisfies a superlinear monotonicity assumption. No growth condition at infinity is assumed. Such results are known [4], in distributional sense, for the equation

$$\Delta u - |u|^{d-1}u = f(x), \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),\ d > 1 \tag{2}$$

and [27] for

$$F(D^2u) - |u|^{d-1}u = f(x), \quad f \in L^n_{\text{loc}}(\mathbb{R}^n),\ d > 1 \tag{3}$$

in the L^n-viscosity case. Our aim is to consider a larger class of equations than (3), allowing the dependence on x, on the gradient Du and going below the exponent n. The technique used to prove the existence of entire solutions allows us to solve the Dirichlet boundary problem in regular domains (even unbounded). In the last section a non-existence result is proved for the equation

$$F(x, Du, D^2u) - e^u = 0 \quad \text{in } \mathbb{R}^n.$$

Blow-up solutions are the content of Chapter 3. We extend to the nonlinear viscosity setting some known results of [47]-[48] concerning the Laplace operator.
Chapter 4 is devoted to the Extended Maximum Principle (EMP). Every subsolution on the uniform elliptic equation $F(D^2u) = 0$ in a bounded domain Ω satisfies the condition

$$\limsup_{x\to\partial\Omega} u(x) \leq 0 \Rightarrow u \leq 0 \quad \text{in} \;\; \Omega$$

in view of maximum principle. In other words the sign of u on the boundary $\partial\Omega$ propagate inside Ω. We show that the boundary condition can be weakened without altering the validity of the maximum principle: any bounded subsolution u of $F(D^2u) = 0$ is non-positive in Ω assuming $u \leq 0$ on $\partial\Omega$ up to a set E of null α-Riesz capacity (for a suitable α). So the set of zero capacity can be ignored in the maximum principle. The key point to establish this result is to construct a supersolution of a maximal equation which blows up on E and is finite outside E. The α-Riesz potential works in this case.
Next we generalize such results to the class of uniformly elliptic equations depending on the gradient variable $F(Du, D^2u) = 0$. As application of EMP we present a removable singularities result: every bounded viscosity solution of $F(Du, D^2u) = f(x)$ in $\Omega\backslash E$, $E \Subset \Omega$, can be extended to a solution of the same equation in all Ω if E has zero α-Riesz capacity.
Finally a larger class of pure second order degenerate elliptic operator is considered by showing that EMP continues to work in this case.

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Giulio Galise

Contents

Chapter 1

Basic facts on viscosity solutions

1.1 Degenerate and uniform elliptic operators

Let Ω be a domain (open connected set) of $\mathbb{R}^n$, $n \geq 2$. By $\mathcal{S}^n$ we denote the set of symmetric $n \times n$ real matrices equipped with its usual partial order: if M, $N \in \mathcal{S}^n$ the condition

$$M \geq N$$

means that

$$\langle M\xi, \xi \rangle \geq \langle N\xi, \xi \rangle \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. This implies that $\mathrm{Trace}(M) \geq \mathrm{Trace}(N)$. In $\mathcal{S}^n$ we use the norm

$$\begin{aligned} \|M\| &= \max\{|Mx| : |x| = 1\} = \max\{|\langle Mx, x\rangle| : |x| = 1\} \\ &= \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}. \end{aligned}$$

The following Lemma states that every symmetric matrix can be uniquely decomposed as a sum of two nonnegative symmetric matrices whose product is null.

Lemma 1.1.1. *If $M \in \mathcal{S}^n$ then $M = M^+ - M^-$, where $M^\pm \in \mathcal{S}^n$, $M^\pm \geq 0$ and $M^+M^- = 0$.*

The matrices M^+ and M^- are called respectively the *positive* and *negative* part of M.

Proof. Let us suppose first M diagonal matrix: $M = D$ with $D_{ij} = 0$ for $i \neq j$. The diagonal matrices E and F defined for $i = 1, \dots, n$ by

$$E_{ii} = (D_{ii})^+ = \max(D_{ii}, 0), \quad F_{ii} = (D_{ii})^- = -\min(D_{ii}, 0) \tag{1.1}$$

give the decomposition required. In order to prove the uniqueness of decomposition let us assume that $D = \overline{E} - \overline{F}$ with $\overline{E}, \overline{F} \geq 0$, $\overline{E}\,\overline{F} = 0$ and prove that $E = \overline{E}$, $F = \overline{F}$. In fact for $i = 1, \dots, n$

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \overline{E}_{ij}\overline{F}_{ij} = \sum_{j=1}^{n} (D_{ij} + \overline{F}_{ij})\overline{F}_{ij} \\ &= D_{ii}\overline{F}_{ii} + \overline{F}_{ii}^2 + \sum_{j \neq i} \overline{F}_{ij}^2 = \overline{F}_{ii}\overline{E}_{ii} + \sum_{j \neq i} \overline{F}_{ij}^2 \\ &\Rightarrow 0 \geq -\overline{F}_{ii}\overline{E}_{ii} = \sum_{j \neq i} \overline{F}_{ij}^2 \Rightarrow \overline{F}_{ij} = 0 \quad \forall i \neq j. \end{aligned}$$

Then $\overline{E}$ and $\overline{F}$ are diagonal matrices and $D_{ii} = \overline{E}_{ii} - \overline{F}_{ii}$. Taking into account that $\overline{E}\,\overline{F} = 0$, if $D_{ii} > 0$ it follows $\overline{E}_{ii} > \overline{F}_{ii}$, $\overline{F}_{ii} = 0$ and $\overline{E}_{ii} = D_{ii}$. Similarly in $D_{ii} < 0$ then $D_{ii} = -\overline{F}_{ii}$, $\overline{E}_{ii} = \overline{F}_{ii}$ if $D_{ii} = 0$. This prove the assert in the diagonal case. The positive and negative part of D are

$$D^+ = (D_{ii})^+ \quad \text{and } D^- = (D_{ii})^-. \tag{1.2}$$

Now we treat the general case. Let us consider an orthogonal matrix O such that $M = O^T DO$ with D diagonal matrix (D_{ii} are the eigenvalues of M), then $M = O^T D^+ O - O^T D^- O$ and $O^T D^+ O$, $O^T D^- O$ satisfy the thesis. Finally we prove the uniqueness. Let us assume that $M = P - Q$ with $P, Q \geq 0$ and $PQ = 0$, we have $D = OMO^T = OPO^T - OQO^T$ and from the uniqueness in the diagonal case $D^+ = OPO^T$, $D^- = OQO^T$. We conclude $P = O^T D^+ O$ and $Q = O^T D^- O$. □

Remark 1.1.1. From previous Lemma it follows that any nonnegative (nonpositive) matrix coincide with its positive (− negative) part. Moreover Trace(M^+) (−Trace(M^-)) is the sum of positive (negative) eigenvalues of M.

We will consider second order partial differential equations of general form

$$F(x, u, Du, D^2u) = f(x), \tag{1.3}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ and $f : \Omega \mapsto \mathbb{R}$ are prescribed functions, Du and D^2u correspond respectively to the gradient and the Hessian matrix of the unknown real valued function $u : \Omega \mapsto \mathbb{R}$. Equations of this type are said to be *fully nonlinear* to emphasize the fact that the operator F can also be nonlinear in its matrix variable.

Definition 1.1.1. The operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ is *degenerate elliptic* if it is nondecreasing in the matrix variable, namely

$$M \geq N \Rightarrow F(x, r, p, M) \geq F(x, r, p, N),$$

and *proper* if it is also nonincreasing in r.

A stronger structure assumption on F is the uniform ellipticity.

Definition 1.1.2. The operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ is uniformly elliptic if there exists two constants $\Lambda \geq \lambda > 0$, said ellipticity constants, such that

$$\lambda \mathrm{Trace}(N) \leq F(x, r, p, M + N) - F(x, r, p, M) \leq \Lambda \mathrm{Trace}(N)$$

for all $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $M, N \in \mathcal{S}^n$ and $N \geq 0$. The ratio $\frac{\Lambda}{\lambda} \geq 1$ is called the ellipticity of F. The set of second order uniformly elliptic operators, with $\Lambda \geq \lambda > 0$ as ellipticity constants, satisfying the condition $F(x, \cdot, \cdot, \cdot) = 0$ for any $x \in \Omega$ is denoted by $\mathcal{F}_{\lambda,\Lambda}$.
Moreover we say $F \in \mathcal{F}_{\lambda,\Lambda,b}$, with $b > 0$, if

$$\lambda \mathrm{Trace}(N) - b|p - q| \leq F(x, r, p, M + N) - F(x, r, q, M) \leq \Lambda \mathrm{Trace}(N) + b|p - q|,$$

so that $F \in \mathcal{F}_{\lambda,\Lambda}$ is b-Lipschitz in the gradient variable.

According these definitions equations of type (1.3) are called fully nonlinear degenerate, respectively uniformly, elliptic equations if F is degenerate, respectively uniformly, elliptic.

1.1.1 Examples

We give examples of fully nonlinear elliptic equations. Some of them arise in problem of applied Mathematics, as financial Mathematics, stochastic optimal control problems and differential games.

Example 1.1.1 (First order equations).

$$F(x, u(x), Du(x)) = f(x). \tag{1.4}$$

First order operators are trivially degenerate elliptic and never uniformly elliptic. In particular Hamilton-Jacobi equations

$$u_t + H(x, t, u, D_x u(x, t)) = 0,$$

where $t \in \mathbb{R}$, $x \in \Omega$ and $H : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ is the Hamiltonian, are of type (1.4) by considering $y = (x, t)$ and

$$F(y, r, p) = p_{n+1} + H(y, r, p_1, \ldots, p_n).$$

Example 1.1.2 (Linear equations).

$$\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u = f(x) \tag{1.5}$$

Equations of type (1.5) may be written in the form (1.3) by setting

$$F(x, r, p, X) = \mathrm{Trace}(A(x)X) + \langle b(x), p \rangle + c(x) r \tag{1.6}$$

where $A(x) = (a_{ij}(x))_{ij} \in \mathcal{S}^n$ and $b(x) = (b_i(x))_i$. The operator (1.6) is degenerate elliptic if $A(x) \geq 0$ and it is uniformly elliptic if $\lambda I \leq A(x) \leq \Lambda I$. Here and in the sequel I is the identity matrix in $\mathcal{S}^n$.

Example 1.1.3 (Quasilinear equations).

$$\sum_{i,j=1}^{n} a_{ij}(x,u,Du)u_{x_ix_j} + b(x,u,Du) = f(x) \tag{1.7}$$

The corresponding operator is

$$F(x,r,p,X) = \mathrm{Trace}(A(x,t,p)X) + b(x,r,p) \tag{1.8}$$

which is degenerate elliptic if $A(x,r,p) = (a_{ij}(x,r,p))_{ij} \geq 0$, uniformly elliptic if $\lambda I \leq A(x,r,p) \leq \Lambda I$ for any $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

Example 1.1.4 (Bellman and Isaacs equations).

Consider a family of linear second order operators depending on a parameter α

$$\mathcal{L}^{\alpha}u = \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i^{\alpha}(x)u_{x_i} + c^{\alpha}(x)u - f^{\alpha}(x),$$

where $\alpha \in \mathcal{A}$, arbitrary set, f^{α} is a real function in Ω for any $\alpha \in \mathcal{A}$. The equation

$$\sup_{\alpha} \{\mathcal{L}^{\alpha}u\} = 0 \tag{1.9}$$

is degenerate (uniformly) elliptic if $A^{\alpha}(x) = (a_{ij}^{\alpha}(x))_{ij} \geq 0$ ($\lambda I \leq A^{\alpha}(x) \leq \Lambda I$) for each $x \in \Omega$ and $\alpha \in \mathcal{A}$. The Bellman equation (1.9) is the fundamental partial differential equation of stochastic control.

In the theory of differential games arise a more general equation than (1.9) said Isaacs equation:

$$\sup_{\alpha} \inf_{\beta} \left\{\mathcal{L}^{\alpha\beta}u\right\} = 0 \tag{1.10}$$

where

$$\mathcal{L}^{\alpha\beta}u = \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i^{\alpha\beta}(x)u_{x_i} + c^{\alpha\beta}(x)u - f^{\alpha\beta}(x)$$

is a family of linear second order operators indexed by α and β in two arbitrary set $\mathcal{A}$ and $\mathcal{B}$. Under the assumption $A^{\alpha\beta}(x) = (a_{ij}^{\alpha\beta}(x))_{ij} \geq 0$ ($\lambda I \leq A^{\alpha\beta}(x) \leq \Lambda I$) for each $x \in \Omega$, $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ Isaacs equation in degenerate (uniformly) elliptic.

Example 1.1.5 (Pucci's equations).

Pucci's extremal operators are defined for any $M \in \mathcal{S}^n$ by

$$\mathcal{P}^{+}_{\lambda,\Lambda}(M) = \sup_{\lambda I \leq A \leq \Lambda I} \mathrm{Trace}(AM), \quad \mathcal{P}^{-}_{\lambda,\Lambda}(M) = \inf_{\lambda I \leq A \leq \Lambda I} \mathrm{Trace}(AM). \tag{1.11}$$

$\mathcal{P}^{\pm}_{\lambda,\Lambda} \in \mathcal{F}_{\lambda,\Lambda}$ and the equations associated are called Pucci's equations:

$$\mathcal{P}^{+}_{\lambda,\Lambda}(D^2u) = f, \quad \mathcal{P}^{-}_{\lambda,\Lambda}(D^2u) = f. \tag{1.12}$$

We collect the main properties of Pucci's operator in the following Proposition.

Proposition 1.1.1. *For any $M \in \mathcal{S}^n$*

1. $\mathcal{P}^-_{\lambda,\Lambda}(M) \leq \mathcal{P}^+_{\lambda,\Lambda}(M)$.
2. $\mathcal{P}^-_{\lambda,\Lambda}(M) = -\mathcal{P}^+_{\lambda,\Lambda}(-M)$.
3. $\mathcal{P}^\pm_{\lambda,\Lambda}(\alpha M) = \alpha\mathcal{P}^\pm_{\lambda,\Lambda}(M)$ *for any* $\alpha \geq 0$
4. $\mathcal{P}^+_{\lambda,\Lambda}$ $(\mathcal{P}^-_{\lambda,\Lambda})$ *is superadditive (subadditive)*
5. $\mathcal{P}^+_{\lambda,\Lambda}(M) = \Lambda\mathrm{Trace}(M^+) - \lambda\mathrm{Trace}(M^-)$
   $= \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$ *with* e_i *eigenvalues of* M
6. $\mathcal{P}^-_{\lambda,\Lambda}(M) = \lambda\mathrm{Trace}(M^+) - \Lambda\mathrm{Trace}(M^-)$
   $= \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i$

Proof. We prove (5)-(6), the other ones follow directly from definition (1.11). For any $\lambda I \leq A \leq \Lambda I$

$$\mathrm{Trace}(AM) = \mathrm{Trace}(AM^+) - \mathrm{Trace}(AM^-) \leq \Lambda\mathrm{Trace}(M^+) - \lambda\mathrm{Trace}(M^-)$$

and taking the supremum

$$\mathcal{P}^+_{\lambda,\Lambda}(M) \leq \Lambda\mathrm{Trace}(M^+) - \lambda\mathrm{Trace}(M^-). \tag{1.13}$$

Since $M = O^T DO$, where O is an orthogonal matrix and D the diagonal matrix of the eigenvalues of M, if we consider the diagonal matrix X with

$$X_{ii} = \begin{cases} \Lambda & \text{if } D_{ii} \geq 0 \\ \lambda & \text{if } D_{ii} < 0, \end{cases}$$

then the matrix $O^T XO$ satisfies the condition $\lambda I \leq O^T XO \leq \Lambda I$ and

$$\mathrm{Trace}(O^T XOM) = \mathrm{Trace}(XD) = \Lambda\mathrm{Trace}(M^+) - \lambda\mathrm{Trace}(M^-).$$

This prove (5). From (2) and (5) we obtain

$$\begin{aligned} \mathcal{P}^-_{\lambda,\Lambda}(M) &= -\mathcal{P}^+_{\lambda,\Lambda}(-M) = -\Lambda\mathrm{Trace}((-M)^+) + \lambda\mathrm{Trace}((-M)^-) \\ &= \lambda\mathrm{Trace}(M^+) - \Lambda\mathrm{Trace}(M^-). \end{aligned}$$

□

Using Pucci's operators it is possible to characterize the uniform ellipticity.

Proposition 1.1.2. *Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$, the following conditions are equivalent:*

1. *F is uniformly elliptic.*

2. $F(x,r,p,M+N) - F(x,r,p,M) \le \Lambda\mathrm{Trace}(N^+) - \lambda\mathrm{Trace}(N^-) \quad \forall (x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $M, N \in \mathcal{S}^n$.

3. $\mathcal{P}^-_{\lambda,\Lambda}(M-N) \le F(x,r,p,M) - F(x,r,p,N) \le \mathcal{P}^+_{\lambda,\Lambda}(M-N) \quad \forall (x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $M, N \in \mathcal{S}^n$.

Proof. 1) $\Rightarrow$ 2). Using the uniform ellipticity of F we have for any $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $M, N \in \mathcal{S}^n$

$$\begin{aligned} F(x,r,p,M+N) &- F(x,r,p,M) \\ &= F(x,r,p,M-N^- + N^+) - F(x,r,p,M-N^-) \\ &\quad + F(x,r,p,M-N^-) - F(x,r,p,M) \\ &\le \Lambda\mathrm{Trace}(N^+) - \lambda\mathrm{Trace}(N^-). \end{aligned}$$

2) $\Rightarrow$ 3). It follows easily from (5)-(6) of Proposition 1.1.1.
3) $\Rightarrow$ 1). For any $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $M, N \in \mathcal{S}^n$, $N \ge 0$

$$\lambda\mathrm{Trace}(N) = \mathcal{P}^-_{\lambda,\Lambda}(N) \le F(x,r,p,M+N) - F(x,r,p,M) \le \mathcal{P}^+_{\lambda,\Lambda}(N) = \Lambda\mathrm{Trace}(N).$$

□

In the following Proposition we compute the Pucci's operator of Hessian matrices in the radial case.

Proposition 1.1.3. *Let $u \in C^2(\mathbb{R}^n \setminus \{x_0\})$ be a radial function: $u(x) = \tilde{u}(r)$, $r = |x - x_0| \ne 0$. Then*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u(x)) = \theta(\tilde{u}''(r))\tilde{u}''(r) + \theta\left(\frac{\tilde{u}'(r)}{r}\right)(n-1)\frac{\tilde{u}'(r)}{r} \tag{1.14}$$

where $\theta(s) = \begin{cases} \Lambda & \textit{if } s \ge 0 \\ \lambda & \textit{otherwise}. \end{cases}$

Proof. A straightforward computation yields

$$D^2u(x) = \tilde{u}''(r)P + \frac{\tilde{u}'(r)}{r}(I - P)$$

with $P = \frac{(x-x_0)\otimes(x-x_0)}{r^2}$ the orthogonal projection along $x - x_0$. Any vector orthogonal to $x - x_0$ is an eigenvector of $D^2u(x)$ with $\frac{\tilde{u}'(r)}{r}$ as the eigenvalue. Moreover $x - x_0$ is itself an eigenvector with eigenvalue $\tilde{u}''(r)$. Thus the Hessian matrix $D^2u(x)$ posses two eigenvalues: $\tilde{u}''(r)$ and $\frac{\tilde{u}'(r)}{r}$ with multiplicity 1 and $n-1$ respectively. We conclude from Proposition 1.1.1. □

Remark 1.1.2. Assuming $\tilde{u} \in C^2([0,+\infty))$ and $\tilde{u}'(0) = 0$, a direct computation shows that $D^2u(x_0) = \tilde{u}''(0)I$. In this way the eigenvalue $\tilde{u}''(0)$ is computable as the limit of the eigenvalues $\tilde{u}''(r)$ and $\frac{\tilde{u}'(r)}{r}$ of $D^2u(x)$ as r goes to 0. The condition $\tilde{u}'(0) = 0$ is fulfilled for instance by functions depending on r^2, $\tilde{u}(r) = \bar{u}(r^2)$ with $\bar{u} \in C^2([0,+\infty)$.

1.2 C-Viscosity solutions

In this section we will introduce a suitable notion of solution for fully nonlinear elliptic equations (1.3) assuming the continuity of the functions $F(x, r, p, M)$ and $f(x)$.

1.2.1 Definitions and properties

A function $u : \Omega \to \mathbb{R}$ is *lower semicontinuous* at a point $x \in \Omega$ if $u > -\infty$ and

$$u(x) \leq \liminf_{y \to x} u(y).$$

We say that u is *lower semicontinuous* in Ω if it is lower semicontinuous at each point $x \in \Omega$. Conversely u is *upper semicontinuous* if $-u$ is lower semicontinuous. We will denote by $LSC(\Omega)$ ($USC(\Omega)$) the space of lower (upper) semicontinuous functions in Ω noting that $u \in LSC(\Omega)$ ($USC(\Omega)$) if, and only if, the super (sub) level sets

$$\{x \in \Omega \text{ such that } u(x) > (<)\, t\}$$

are open in Ω for all $t \in \mathbb{R}$.

Theorem 1.2.1 (Weierstrass). *An upper (lower) semicontinuous function on a compact set attains its maximum (minimum)*

Definition 1.2.1. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ be a degenerate elliptic operator. An upper semicontinuous function $u : \Omega \mapsto \mathbb{R}$ is said to be a C-viscosity subsolution of the equation $F(x, u, Du, D^2u) = f(x)$ if

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq f(x)$$

for all $x \in \Omega$ and $\varphi \in C^2(B_\delta(x))$ such that $u - \varphi$ has a local maximum at x. Similarly $u : \Omega \mapsto \mathbb{R}$ is a C-viscosity supersolution of $F(x, u, Du, D^2u) = f(x)$ if u is lower semicontinuous and for all $x \in \Omega$ and $\varphi \in C^2(B_\delta(x))$ such that $u - \varphi$ has a local minimum at x we have

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq f(x).$$

Finally $u \in C(\Omega)$ is a C-viscosity solution if is both subsolution and supersolution.

Remark 1.2.1. Sub(super)solution of $F = f$ or solution of $F \geq (\leq) f$ have the same meaning.

It easy to check that a classical solution of $F \geq (\leq) f$, that is a function $u \in C^2(\Omega)$ satisfying pointwise the equation, is a viscosity solution of $F \geq (\leq) f$. Conversely a viscosity solution $u \in C^2(\Omega)$ is a classical one.
As shown in the next Proposition we may suppose in Definition 1.2.1 that test functions φ touch u at x from above (below) and that the maximum (minimum) is strict.

Proposition 1.2.1. *For $u : \Omega \mapsto \mathbb{R}$ the following conditions are equivalent:*

1. *u is a viscosity subsolution (supersolution) of (1.3).*

2. *If $\varphi \in C^2(B_\delta(x))$, $u(x) = \varphi(x)$ and $u(y) \leq \varphi(y)$ $(u(y) \geq \varphi(y))$ for $y \in B_\delta(x)$ then*

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq (\leq) f(x).$$

3. *If $\varphi \in C^2(B_\delta(x))$, $u(x) = \varphi(x)$ and $u(y) < \varphi(y)$ $(u(y) > \varphi(y))$ for $y \in B_\delta(x) \setminus \{x\}$ then*

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq (\leq) f(x).$$

4. *u satisfies 3) with $\varphi(x) = P(x)$, a paraboloid (by paraboloid $P(x)$ we mean a polynomial of degree two: $P(x) = a + \langle b, x\rangle + \frac{1}{2}\langle Cx, x\rangle$, with $a \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $C \in \mathcal{S}^n$),*

Proof. 1) $\Rightarrow$ 2), 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 4) are trivial. We prove 4) $\Rightarrow$ 1) in the subsolutions case. Let φ be a C^2 function in a neighborhood of x such that $u - \varphi$ as a local maximum at x. By using the second order Taylor expansion

$$u(y) \leq u(x) + \varphi(y) - \varphi(x) = u(x) + \langle D\varphi(x), y - x\rangle + \frac{1}{2}\left\langle D^2\varphi(x)(y - x), y - x\right\rangle + o(|y - x|^2) \quad \text{as } y \to x.$$

Fix $\varepsilon > 0$, for $\delta > 0$ small enough the paraboloid

$$P(y) = u(x) + \langle D\varphi(x), y - x\rangle + \frac{1}{2}\left\langle D^2\varphi(x)(y - x), y - x\right\rangle + \varepsilon|y - x|^2$$

touches u at x and $u(y) < P(y)$ in $B_\delta(x) \setminus \{x\}$. Then

$$F(x, u(x), D\varphi(x), D^2\varphi(x) + 2\varepsilon I) \geq f(x)$$

and letting $\varepsilon \to 0$ we get $F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq f(x)$. □

Example. The function $u(x, y) = e^{x^{\frac{4}{3}} - y^{\frac{4}{3}}}$ is a viscosity solution in $\mathbb{R}^2$ of $F(x, u, Du, D^2u) = 0$, where

$$F(x, r, p, M) = \langle (|r|M - p \otimes p)p, p\rangle$$

is a degenerate elliptic operator.
In fact $u(x, y)$ is C^∞ in $A = \mathbb{R}^2 \setminus (\{(x, 0),\ x \in \mathbb{R}\} \cup \{(0, y),\ y \in \mathbb{R}\})$ and a direct computation shows that

$$\left\langle (uD^2u - Du \otimes Du)Du, Du\right\rangle = 0,$$

so $u(x, y)$ is a classical solution in A and then solution in the viscosity sense. Let us prove that $u(x, y)$ is a subsolution in the viscosity sense, similar arguments hold in the supersolution case. First of all we note that the set of test functions φ touching u from above at $(0, y)$, $y \in \mathbb{R}$, is empty: by contradiction if such φ exists we have

$$\begin{aligned}\varphi_{xx}(0, y) &= \lim_{h \to 0} \frac{\varphi_x(h, y)}{h} = 2 \lim_{h \to 0} \frac{\varphi(h, y) - e^{-y^{\frac{4}{3}}}}{h^2} \\ &\geq 2 \lim_{h \to 0} \frac{e^{h^{\frac{4}{3}} - y^{\frac{4}{3}}} - e^{-y^{\frac{4}{3}}}}{h^2} = +\infty.\end{aligned}$$

Conversely the set of test functions touching u from above at $(x,0)$, $x\in\mathbb{R}\setminus\{0\}$, is not empty: for example $\varphi(x,y)=e^{x^{\frac{4}{3}}}$ belongs to this set. Now let φ be a C^2-function such that

$$\begin{aligned} u(x_0,0)&=\varphi(x_0,0)\\ u(x,y)&\le\varphi(x,y)\quad\forall(x,y)\in B_\delta((x_0,0)),\ x_0\neq0,\ \delta>0.\end{aligned}$$

In particular

$$\begin{aligned} u(x,0)&\le\varphi(x,0)\qquad\text{for } |x-x_0|<\delta,\\ u(x_0,y)&\le\varphi(x_0,y)\qquad\text{for } |y|<\delta\end{aligned}$$

and

$$u_x(x_0,0)-\varphi_x(x_0,0)=\frac{4}{3}x_0^{\frac{1}{3}}e^{x_0^{\frac{4}{3}}}-\varphi_x(x_0,0)=0, \tag{1.15}$$

$$u_y(x_0,0)-\varphi_y(x_0,0)=\varphi_y(x_0,0)=0, \tag{1.16}$$

$$u_{xx}(x_0,0)-\varphi_{xx}(x_0,0)=\frac{4}{9}e^{x_0^{\frac{4}{3}}}\left(x_0^{-\frac{2}{3}}+4x_0^{\frac{2}{3}}\right)-\varphi_{xx}(x_0,0)\le0. \tag{1.17}$$

Taking into account (1.15)-(1.16)-(1.17) we conclude

$$\begin{aligned}&\big\langle\big(u(x_0,0)D^2\varphi(x_0,0)-D\varphi(x_0,0)\otimes D\varphi(x_0,0)\big)\,D\varphi(x_0,0),D\varphi(x_0,0)\big\rangle\\ &=\varphi_x^2(x_0,0)\left(u(x_0,0)\varphi_{xx}(x_0,0)-\varphi_x^2(x_0,0)\right)\\ &\ge\frac{4}{9}\varphi_x^2(x_0,0)e^{2x_0^{\frac{4}{3}}}x_0^{-\frac{2}{3}}\ge0.\end{aligned}$$

Proposition 1.2.2. *Let $u,v\in USC(\Omega)$ be respectively subsolutions of $F(x,u,Du,D^2u)=f(x)$ and $G(x,v,Dv,D^2v)=g(x)$. Then the function $w=\sup(u,v)\in USC(\Omega)$ is a subsolution of*

$$\sup(F(x,w,Dw,D^2w),G(x,w,Dw,D^2w))=\inf(f(x),g(x)).$$

Proof. Let φ be a test function touching w at $x_0\in\Omega$ from above:

$$w(x_0)=\varphi(x_0)\quad\text{ and }\quad w(x)\le\varphi(x)\quad\text{ in } B_\delta(x_0)\subseteq\Omega$$

for some $\delta>0$. Then φ touches from above one of two functions u and v, u for instance. It follows

$$\begin{aligned}\sup(F(x_0,w(x_0),D\varphi(x_0),D^2\varphi(x_0)),G(x_0,w(x_0),D\varphi(x_0),D\varphi(x_0)))\ge\\ F(x_0,w(x_0),D\varphi(x_0),D^2\varphi(x_0)\ge f(x_0)\ge\inf(f(x_0),g(x_0))\end{aligned}$$

as claimed. □

Proposition 1.2.3. *If $u\in C(\Omega)$ is a viscosity solution of (1.3) and if $z:\mathbb{R}\mapsto\mathbb{R}$ is a C^2-function such that $z'>0$ in $\mathbb{R}$, then the function $v=z(u)$ is a viscosity solution of*

$$K(x,v,Dv,D^2v)=f(x)$$

where

$$K(x,r,p,M)=F(x,w(r),w'(r)p,w''(r)p\otimes p+w'(r)M) \tag{1.18}$$

and $w=z^{-1}$.

Proof. First of all we note that the operator K is degenerate elliptic.
Let φ be a test function touching v from above at $x_0 \in \Omega$. Then $w(\varphi)$ satisfies

$$u(x_0) = w(\varphi(x_0)) \quad \text{and } u(x) \leq w(\varphi(x)) \quad \forall x \in B_\delta(x_0)$$

for some $\delta > 0$. Taking into account that $Dw(\varphi) = w'(\varphi)D\varphi$ and $D^2w(\varphi) = w''(\varphi)D\varphi \otimes D\varphi + w'(\varphi)D^2\varphi$ we have

$$\begin{aligned}&K(x_0, v(x_0), D\varphi(x_0), D^2\varphi(x_0)) =\\ &F(x_0, u(x_0), w'(\varphi(x_0))D\varphi(x_0), w''(\varphi(x_0))D\varphi(x_0) \otimes D\varphi(x_0) + w'(\varphi(x_0))D^2\varphi(x_0)) \geq f(x).\end{aligned}$$

This prove that v is a subsolution. The proof in the case of supersolution is similar. □

With similar arguments of those of Proposition 1.2.1 one can prove that if $z' < 0$, then $v = z(u)$ is a solution of

$$H(x, v, Dv, D^2v) = -f(x),$$

where

$$H(x, r, p, M) = -F(x, w(r), w'(r)p, w''(r)p \otimes p + w'(r)M). \tag{1.19}$$

In the particular case $z(u) = -u$, then $v = -u$ is a solution of

$$-F(x, -v, -Dv, -D^2v) = -f(x).$$

Moreover if $F \in \mathcal{F}_{\lambda,\Lambda}$, in order to preserve the ellipticity of K (H) we need to assume the existence of two constants c_1, c_2 such that $0 < c_1 < z' < c_2$ ($c_1 < z' < c_2 < 0$). In this way $K \in \mathcal{F}_{\frac{\lambda}{c_2},\frac{\Lambda}{c_1}}$ ($H \in \mathcal{F}_{-\frac{\lambda}{c_1},-\frac{\Lambda}{c_2}}$).

We present a first stability result for viscosity solutions. We need the following Lemma.

Lemma 1.2.1. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of upper semicontinuous function on Ω which converges to u uniformly in any compact sets $K \subset \Omega$. If $\widehat{x} \in \Omega$ is a strict local maximum point of u, there exists a sequence of local maximum points of u_k which converges to $\widehat{x}$.*

Proof. By assumption there exists $\delta > 0$ such that

$$u(x) < u(\widehat{x}) \quad \forall x \in B_\delta(\widehat{x}) \setminus \{\widehat{x}\}\,. \tag{1.20}$$

Let $K \subset B_\delta(\widehat{x})$ be a compact set and

$$\max_K u_k = u_k(x_k),$$

we claim that $x_k \to \widehat{x}$. By contradiction, using the boundedness of $(x_k)_{k\in\mathbb{N}}$, we find a subsequence $(x_{k_h})_{h\in\mathbb{N}}$ and a point $\overline{x} \neq \widehat{x}$, $\overline{x} \in K$, such that $x_{k_h} \to \overline{x}$ as $h \to +\infty$. It follows from uniform convergence

$$\lim_{h\to+\infty} u_{k_h}(x_{k_h}) = u(\overline{x})$$

and

$$|u_k(x_k) - u(\widehat{x})| = |\max_K u_k - \max_K u| \leq \max_K |u_k - u| \to 0 \quad \text{as } k \to +\infty.$$

Then $u(\widehat{x}) = u(\overline{x})$ which contradicts (1.20). □

Theorem 1.2.2 (Stability). *Let $(u_k)_{k\in\mathbb{N}} \subseteq USC(\Omega)$ be a sequence of subsolutions of*

$$F_k(x, u_k, Du_k, D^2u_k) = f_k(x), \tag{1.21}$$

where $(F_k)_{k\in\mathbb{N}}$ is a sequence of degenerate elliptic operator and $(f_k)_{k\in\mathbb{N}} \subseteq C(\Omega)$. If $u_k \to u$, $F_k \to F$, $f_k \to f$ as $k \to +\infty$ uniformly in any compact sets of Ω, $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$, Ω respectively, then u is a subsolution of the equation

$$F(x, u, Du, D^2u) = f(x).$$

Proof. Let φ be a test function such that $u - \varphi$ has a strict local maximum at $x \in \Omega$. By Lemma 1.2.1 there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of maximum point of $u_k - \varphi$ converging to x as $k \to +\infty$. Since u_k is a subsolution of (1.21) we have

$$F_k(x_k, u_k(x_k), D\varphi(x_k), D^2\varphi(x_k)) \geq f_k(x_k)$$

and passing to the limit as $k \to +\infty$

$$F(x, u(x), Du(x), D^2u(x)) \geq f(x)$$

because of the local uniform convergence. □

We give now a characterization of viscosity solutions based on the notion of "semijet" of a function $u : \Omega \mapsto \mathbb{R}$.

Definition 1.2.2. The second order "superjet" $J^{2,+}u(x)$ of u at $x \in \Omega$ is the set of all pairs $(p, M) \in \mathbb{R}^n \times \mathcal{S}^n$ such that

$$u(y) \leq u(x) + \langle p, y - x\rangle + \frac{1}{2}\langle M(y - x), y - x\rangle + o(|y - x|^2) \quad \text{as } y \to x.$$

The second order "subjet" $J^{2,-}u(x)$ of u at $x \in \Omega$ is the set of all pairs $(p, M) \in \mathbb{R}^n \times \mathcal{S}^n$ such that

$$u(y) \geq u(x) + \langle p, y - x\rangle + \frac{1}{2}\langle M(y - x), y - x\rangle + o(|y - x|^2) \quad \text{as } y \to x.$$

From definition it's easy to check that super(sub)jets are convex sets (possibly empty[1]) and

$$J^{2,-}u(x) = -J^{2,+}(-u)(x). \tag{1.22}$$

If $(p, M) \in J^{2,+}u(x) \cap J^{2,-}u(x)$ then

$$u(y) - u(x) - \langle p, y - x\rangle - \frac{1}{2}\langle M(y - x), y - x\rangle = o(|y - x|^2) \quad \text{as } y \to x,$$

[1]For example $J^{2,+}u(0)$ and $J^{2,-}u(0)$ are empty if

$$u(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$$

so u is twice differentiable at x and

$$(p, M) = (Du(x), D^2u(x)).$$

Conversely if u is twice differentiable at x then

$$\begin{aligned} J^{2,+}u(x) &= \left\{(Du(x), M) \in \mathbb{R}^n \times \mathcal{S}^n : \, M \geq D^2u(x)\right\} \\ J^{2,-}u(x) &= \left\{(Du(x), M) \in \mathbb{R}^n \times \mathcal{S}^n : \, M \leq D^2u(x)\right\}. \end{aligned}$$

In the next Proposition we prove a density property.

Proposition 1.2.4. *Let $u \in LSC(\Omega)$ $(USC(\Omega))$, $u < +\infty$ $(u > -\infty)$ and $x \in \Omega$. There exists a sequence $(x_k)_{k\in\mathbb{N}}$ converging to x such that*

$$J^{2,-}u(x_k) \neq \emptyset \;\; (J^{2,+}u(x_k) \neq \emptyset).$$

Proof. We give the proof in the case $u \in LSC(\Omega)$, if $u \in USC(\Omega)$ we conclude using (1.22). Fix $x \in \Omega$ and $r > 0$ small enough such that $\overline{B}_r(x) \subseteq \Omega$. For $k \in \mathbb{N}$ the function $u(y) + k|y - x|^2$ attains its minimum over $\overline{B}_r(x)$ at x_k, then

$$k|x_k - x|^2 \leq u(x) - u(x_k) \leq \max_{\overline{B}_r(x)}(u(x) - u(y)) < +\infty$$

and $x_k \to x$ as $k \to +\infty$. Moreover

$$\begin{aligned} u(y) &\geq u(x_k) + k\left(|x_k - x|^2 - |y - x|^2\right) \\ &= u(x_k) + k\left(\langle 2(x - x_k), y - x_k\rangle - \langle I(y - x_k), y - x_k\rangle\right) \end{aligned}$$

and $(2k(x - x_k), -kI) \in J^{2,-}u(x_k)$. □

Super(sub)jets are strictly related to test functions. In fact if $\varphi \in C^2(B_\delta(x))$ such that $u - \varphi$ has a local maximum (minimum) at $x \in \Omega$, then using Taylor's expansion

$$\begin{aligned} u(y) &\leq (\geq) u(x) + \varphi(y) - \varphi(x) \\ &= u(x) + \langle D\varphi(x), y - x\rangle + \frac{1}{2}\left\langle D^2\varphi(x)(y - x), y - x\right\rangle + o(|y - x|^2) \quad \text{ as } y \to x \end{aligned}$$

and $(D\varphi(x), D^2\varphi(x)) \in J^{2,+}u(x)$ $(J^{2,-}u(x))$. On the contrary let us suppose $(p, M) \in J^{2,+}u(x)$, then

$$u(y) \leq u(x) + \langle p, y - x\rangle + \frac{1}{2}\langle M(y - x), y - x\rangle + |y - x|^2\varepsilon(y) \quad \text{ as } y \to x$$

where $\varepsilon(y)$ is a continuous function and $\lim_{y\to x} \varepsilon(y) = 0$. The function

$$\rho(r) = \sup_{B_r(x)\cap\Omega} |\varepsilon(y)|$$

is continuous and increasing on $\mathbb{R}^+$, so it is well defined the function

$$\psi(y) = \int_{|y-x|}^{2|y-x|}\left(\int_t^{2t} \rho(r)\, dr\right) dt.$$

Direct computations show that

$$\psi(x) = 0,\ \psi(y) \geq |y-x|^2 \varepsilon(y),\ D\psi(x) = 0,\ D^2\psi(x) = 0.$$

In this way

$$\varphi(y) = u(x) + \langle p, y-x \rangle + \frac{1}{2} \langle M(y-x), y-x \rangle + \psi(y)$$

is a test function touching u at x from above and

$$\big(D\varphi(x), D^2\varphi(x)\big) = (p, M).$$

So we have proved that

$$J^{2,+}u(x) = \big\{ \big(D\varphi(x), D^2\varphi(x)\big) :\ \varphi \text{ is a } C^2\text{-function and } u - \varphi \text{ has a local maximum at } x \big\}. \tag{1.23}$$

Similarly

$$J^{2,+}u(x) = \big\{ \big(D\varphi(x), D^2\varphi(x)\big) :\ \varphi \text{ is a } C^2\text{-function and } u - \varphi \text{ has a local minimum at } x \big\}. \tag{1.24}$$

From (1.23)-(1.24) it follows next Theorem.

Theorem 1.2.3. *The function $u \in USC(\Omega)$ $(LSC(\Omega))$ is a viscosity solution of*

$$F(x, u, Du, D^2u) \geq (\leq) f(x)$$

if, and only if,

$$F(x, u(x), p, M) \geq (\leq) f(x)$$

for all $(p, M) \in J^{2,+}u(x)$ $(J^{2,-}u(x))$.

Now we define a sort of closure of $J^{2,\pm}u(x)$:

$$\overline{J}^{2,\pm}u(x) = \{(p, M) \in \mathbb{R}^n \times \mathcal{S}^n :\ \exists (x_k, p_k, M_k) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n,\ (p_k, M_k) \in J^{2,\pm}u(x_k) \text{ and } (x_k, u(x_k), p_k, M_k) \to (x, u(x), p, M)\big\}.$$

Theorem 1.2.3 continues to hold by continuity if $(p, M) \in \overline{J}^{2,\pm}u(x)$. More generally one can defines the semijets of a function $u : \Sigma \to \mathbb{R}$ ($\Sigma \subseteq \mathbb{R}^n$ arbitrary set, not necessary open) at $x \in \Sigma$ by

$$J^{2,\pm}_{\Sigma}u(x) = \{(p, M) \in \mathbb{R}^n \times \mathcal{S}^n : u(y) \leq (\geq) u(x) + \langle p, y-x \rangle + 1/2 \langle M(y-x), y-x \rangle + o(|y-x|^2) \quad \text{as } \Sigma \ni y \to x\big\} \tag{1.25}$$

and

$$\overline{J}^{2,\pm}_{\Sigma}u(x) = \{(p, M) \in \mathbb{R}^n \times \mathcal{S}^n :\ \exists (x_k, p_k, M_k) \in \Sigma \times \mathbb{R}^n \times \mathcal{S}^n,\ (p_k, M_k) \in J^{2,\pm}_{\Sigma}u(x_k) \text{ and } (x_k, u(x_k), p_k, M_k) \to (x, u(x), p, M)\Big\}. \tag{1.26}$$

Clearly if $\Sigma_1 \subseteq \Sigma_2$ then $J^{2,\pm}_{\Sigma_1}u(x) \subseteq J^{2,\pm}_{\Sigma_2}u(x)$ and $\overline{J}^{2,\pm}_{\Sigma_1}u(x) \in \overline{J}^{2,\pm}_{\Sigma_2}u(x)$. Moreover if x is an interior point of Σ we have $J^{2,\pm}_{\Sigma}u(x) = J^{2,\pm}_{\overline{\Sigma}}u(x)$ and $\overline{J}^{2,\pm}_{\Sigma}u(x) = \overline{J}^{2,\pm}_{\overline{\Sigma}}u(x)$.

Proposition 1.2.5. *If $u : \Sigma \mapsto \mathbb{R}$ and $\varphi \in C^2$ in a neighborhood of Σ, then*

$$J^{2,\pm}_{\Sigma}(u+\varphi)(x) = \big(D\varphi(x), D^2\varphi(x)\big) + J^{2,\pm}_{\Sigma}u(x), \tag{1.27}$$

$$\overline{J}^{2,\pm}_{\Sigma}(u+\varphi)(x) = \big(D\varphi(x), D^2\varphi(x)\big) + \overline{J}^{2,\pm}_{\Sigma}u(x). \tag{1.28}$$

Proof. Let $(p, M) \in J^{2,\pm}_{\Sigma}(u+\varphi)(x)$. By Taylor's expansion we have

$$\varphi(y) = \varphi(x) + \langle D\varphi, y-x\rangle + \frac{1}{2}\left\langle D^2\varphi(x)(y-x), y-x\right\rangle + o(|y-x|^2) \quad \text{as } y \to x, \tag{1.29}$$

so

$$\begin{aligned} u(y) \le(\ge) u(x) &+ \langle p - D\varphi(x), y-x\rangle \\ &+ \frac{1}{2}\left\langle (M - D^2\varphi(x))(y-x), y-x\right\rangle + o(|y-x|^2) \quad \text{as } y \to x \end{aligned} \tag{1.30}$$

and $(p, M) = (D\varphi(x), D^2\varphi(x)) + (p - D\varphi(x), M - D^2\varphi(x))$ with $(p - D\varphi(x), M - D^2\varphi(x)) \in J^{2,\pm}_{\Sigma}u(x)$ because of (1.30).
Conversely let $(p, M) \in J^{2,\pm}_{\Sigma}u(x)$, using (1.29)

$$\begin{aligned} u(y) + \varphi(y) \le(\ge) u(x) + \varphi(x) &+ \langle p + D\varphi(x), y-x\rangle \\ &+ \frac{1}{2}\left\langle (M + D^2\varphi(x))(y-x), y-x\right\rangle + o(|y-x|^2) \quad \text{as } y \to x, \end{aligned}$$

we conclude $(p + D\varphi(x), M + D^2\varphi(x)) \in J^{2,\pm}_{\Sigma}(u+\varphi)(x)$.
By approximation it follows (1.28).

□

1.2.2 Comparison Principle

This section is concerned with *comparison principle* which is the main issue in viscosity theory to prove uniqueness of Dirichlet problem. Roughly speaking comparison means the following: if u, v are respectively sub and supersolution of (1.3) and $u \le v$ on $\partial\Omega$ then $u \le v$ in Ω. In the proof of comparison principle we will need the following Theorem, due to Ishii's [19, Theorem 3.2].

Lemma 1.2.2 (Ishii). *Let $\mathcal{O}_i$ be a locally compact subset of $\mathbb{R}^{n_i}$ for $i = 1, \dots, k$,*

$$\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_k,$$

$u_i \in USC(\mathcal{O}_i)$ and φ be twice continuously differentiable in a neighborhood of $\mathcal{O}$. Set

$$w(x) = u_1(x_1) + \cdots + u_k(x_k) \quad for\ x = (x_1, \dots, x_k) \in \mathcal{O}$$

and suppose that $w - \varphi$ ha a local maximum at $\hat{x} = (\hat{x}_1, \dots, \hat{x}_k) \in \mathcal{O}$. Then for each $\varepsilon > 0$ there exists $X_i \in \mathcal{S}^{n_i}$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \overline{J}^{2,+}_{\mathcal{O}_i}u_i(\hat{x}_i) \quad for\ i = 1, \dots, k$$

and

$$-\left(\frac{1}{\varepsilon}+\|A\|\right)I \le \begin{pmatrix} X_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_k \end{pmatrix} \le A+\varepsilon A^2$$

where $A = D^2\varphi(\hat{x})$.

DEGENERATE CASE

Theorem 1.2.4 (Maximum principle). *Let* $\Omega \subseteq \mathbb{R}^n$ *be bounded,* $u \in USC(\overline{\Omega})$ *and* $v \in LSC(\overline{\Omega})$ *be respectively sub and supersolution of (*1.3*) in* Ω*. Assume the following conditions:*

1. *there exists* $\gamma > 0$ *such that*

$$F(x,r,p,M) - F(x,s,p,M) \le -\gamma(r-s) \tag{1.31}$$

for $r \ge s$ *and* $(x,p,M) \in \overline{\Omega} \times \mathbb{R}^n \times \mathcal{S}^n$.

2. *There exists a modulus of continuity* $\omega : [0,+\infty) \mapsto [0,+\infty)$, $\omega(0) = 0$, *such that*

$$F(x,r,k(x-y),M) - F(y,r,k(x-y),N) \le \omega(k|x-y|^2 + |x-y|)$$

for any $x, y \in \Omega$, $k \in \mathbb{N}$, $r \in \mathbb{R}$, M *and* N *satisfying*

$$-3k\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \le 3k\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{1.32}$$

Then

$$\max_{\overline{\Omega}}(u-v) \le \max_{\partial\Omega}(u-v)^+. \tag{1.33}$$

Proof. By contradiction we suppose

$$\max_{\overline{\Omega}}(u-v-\max_{\partial\Omega}(u-v)^+) =: \vartheta > 0.$$

For $k \in N$ consider the function $\Phi_k : \overline{\Omega} \times \overline{\Omega} \mapsto \mathbb{R}$ defined by

$$\Phi_k(x,y) = u(x) - u(y) - \frac{k}{2}|x-y|^2 - \max_{\partial\Omega}(u-v)^+$$

and let $(x_k, y_k) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of Φ_k:

$$\Phi_k(x_k,y_k) = \max_{\overline{\Omega}\times\overline{\Omega}} \Phi_k(x,y) \ge \max_{\overline{\Omega}} \Phi_k(x,x) = \vartheta. \tag{1.34}$$

Up to a subsequence we can assume $(x_k, y_k) \to (\overline{x}, \overline{y}) \in \overline{\Omega} \times \overline{\Omega}$. From (1.34) we have

$$\begin{aligned}\frac{k}{2}|x_k-y_k|^2 &\le u(x_k) - v(y_k) - \max_{\partial\Omega}(u-v)^+ - \vartheta \\ &\le \max_{\overline{\Omega}} u - \min_{\overline{\Omega}} v\end{aligned}$$

and $\overline{x} = \overline{y}$ by letting $k \to +\infty$. Moreover

$$\begin{aligned}0 &\le \liminf_{k\to+\infty} \frac{k}{2}|x_k - y_k|^2 \le \limsup_{k\to+\infty} \frac{k}{2}|x_k - y_k|^2 \\ &\le \limsup_{k\to+\infty}(u(x_k) - v(y_k)) - \max_{\partial\Omega}(u-v)^+ - \vartheta \\ &\le u(\overline{x}) - v(\overline{x}) - \max_{\partial\Omega}(u-v)^+ - \vartheta \le 0,\end{aligned}$$

then

$$\lim_{k\to+\infty} \frac{k}{2}|x_k - y_k|^2 = 0$$

and

$$u(\overline{x}) - v(\overline{x}) - \max_{\partial\Omega}(u-v)^+ = \vartheta. \tag{1.35}$$

From (1.35) $\overline{x} \in \Omega$ and $(x_k, y_k) \in \Omega \times \Omega$ for k large enough. In view of Ishii's lemma (taking $\mathcal{O}_1 = \mathcal{O}_2 = \Omega$, $w = u - v$, $\varphi = \frac{k}{2}|x-y|^2 + \max_{\partial\Omega}(u-v)^+$, $\varepsilon = \frac{1}{k}$) there exist $M, N \in \mathcal{S}^n$ such that

$$(k(x_k - y_k), M) \in \overline{J}^{2,+}u(x_k)$$

$$(k(x_k - y_k), N) \in \overline{J}^{2,-}v(y_k)$$

and

$$-3k \le \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le -(k + \|A\|)I = \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \le A + \frac{1}{k}A^2 = 3k \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Using the assumptions 1)-2) we have

$$\begin{aligned}0 &< \gamma(\max_{\partial\Omega}(u-v)^+ + \vartheta) \le \gamma(u(x_k) - v(y_k)) \\ &\le F(x_k, v(y_k), k(x_k - y_k), M) - F(x_k, u(y_k), k(x_k - y_k), M) \\ &\le F(x_k, v(y_k), k(x_k - y_k), M) - F(y_k, v(y_k), k(x_k - y_k), N) \\ &\le \omega(k|x_k - y_k|^2 + |x_k - y_k|)\end{aligned}$$

and we get a contradiction passing to the limit as $k \to +\infty$. □

Remark 1.2.2. If u and v are respectively super and subsolutions of (1.3) we have

$$\min_{\overline{\Omega}}(u - v) \ge -\max_{\partial\Omega}(u-v)^-.$$

From Theorem 1.2.4 it follows a *comparison principle.*

Corollary 1.2.1 (**Comparison principle**). *Under the assumptions of Theorem 1.2.4, if $u \le v$ on $\partial\Omega$ then $u \le v$ in Ω.*

Example 1.2.1. Let $G : \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ be *proper* (degenerate + nonincreasing in r) and $f \in C(\Omega)$. The operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ defined by

$$F(x, r, p, M) = G(r, p, M) - \eta|r|^{d-1}r + f(x)$$

where $d \geq 1$ and $\eta > 0$ satisfies (1.33). In fact if $r \geq s$, using the inequality $||a|^{d-1}a - |b|^{d-1}b| \geq \theta|a-b|^d$ with $\theta > 0$, we obtain

$$F(x,r,p,M) - F(x,s,p,M) \leq -\eta\theta(r-s)^d$$

and this inequality plays the same role of (1.31) in the proof of Theorem 1.2.4. In order to prove assumption 2 let us introduce the modulus of continuity of f:

$$\omega(\rho) = \sup_{|x-y|\leq\rho} |f(x) - f(y)|,$$

so if M an N satisfy (1.32) (in particular $M \leq N$, see Remark 1.2.3) we have

$$\begin{aligned} F(x,r,k(x-y),M) - F(y,r,k(x-y),N) &= G(r,p,M) - G(r,p,N) + f(x) - f(y) \\ &\leq \omega(|x-y|) \leq \omega(k|x-y|^2 + |x-y|). \end{aligned}$$

Remark 1.2.3. The latter inequality in (1.32) implies $M \leq N$ since the matrix $\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$ annihilates vectors of the form $\begin{pmatrix} \xi \\ \xi \end{pmatrix}$.

Remark 1.2.4. Assumption 2 of Theorem 1.2.4 implies degenerate ellipticity. Let $M \leq N$, the idea is to prove the claim with $N + \varepsilon I$, $\varepsilon > 0$, in place of N, then pass to the limit as $\varepsilon \to 0$ and use the continuity of F. Fix $\varepsilon > 0$, for any $\xi, \eta \in \mathbb{R}^n$

$$\begin{aligned} \left\langle \begin{pmatrix} M & 0 \\ 0 & -(N+\varepsilon I) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle &= \langle M\xi, \xi\rangle - \langle N\eta, \eta\rangle - \varepsilon|\eta|^2 \\ &\leq \left(1 + \frac{\|N\|}{\varepsilon}\right) \|N\| \, |\xi - \eta|^2 \\ &\leq 6k|\xi-\eta|^2 = 3k \left\langle \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \end{aligned}$$

for $k \geq \frac{1}{6}\left(1 + \frac{\|N\|}{\varepsilon}\right)\|N\|$ and

$$\left\langle \begin{pmatrix} M & 0 \\ 0 & -(N+\varepsilon I) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \geq -3k(|\xi|^2 + |\eta|^2) = -3k \left\langle \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle$$

if $k \geq \frac{1}{3}\max\left(\|M\|, \|N\| + \varepsilon\right)$. Then if $k \geq \frac{1}{3}\max\left(\|M\|, \|N\| + \varepsilon, \frac{1}{2}\left(1 + \frac{\|N\|}{\varepsilon}\right)\|N\|\right)$, by assumption 2 we have

$$F(x + \frac{p}{k}, r, p, M) - F(x, r, p, N + \varepsilon I) \leq \omega\left(\frac{1}{k}(|p|^2 + |p|)\right)$$

for $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and, letting $k \to +\infty$, $F(x,r,p,M) - F(x,r,p,N+\varepsilon I) \leq 0$. We conclude as $\varepsilon \to 0$.

Remark 1.2.5 (**Comparison without coercivity**)**.** Coercivity assumption 1 in Theorem 1.2.4 can be dropped if the operator F is proper and $u(v)$ is a *strict* sub(super)solution of (1.3), that is

$$F(x,u,Du,D^2u) \geq f(x) + c$$

$$(F(x,v,Dv,D^2v) \leq f(x) + c)$$

for some positive constants c, which provides the contradiction at the end of the proof of Theorem 1.2.4 as $k \to +\infty$:

$$\begin{aligned} c &\leq F(x_k, u(x_k), k(x_k - y_k), M) - F(y_k, v(y_k), k(x_k - y_k), N) \\ &\leq F(x_k, v(y_k), k(x_k - y_k), M) - F(y_k, v(y_k), k(x_k - y_k), N) \\ &\leq \omega(k|x_k - y_k|^2 + |x_k - y_k|). \end{aligned}$$

Instead, if $u(v)$ is a sub(super)solution of (1.3) but for all $\varepsilon > 0$ there exist $\psi_\varepsilon \in USC(\overline{\Omega})$ $(LSC(\overline{\Omega}))$ and $\delta_\varepsilon > 0$ such that $|\psi_\varepsilon| < \varepsilon$ and

$$F(x, u + \psi_\varepsilon, D(u + \psi_\varepsilon), D^2(u + \psi_\varepsilon)) \geq f(x) + \delta_\varepsilon$$

$$(F(x, u + \psi_\varepsilon, D(u + \psi_\varepsilon), D^2(u + \psi_\varepsilon)) \leq f(x) + \delta_\varepsilon),$$

reasoning as above we obtain

$$\max_{\overline{\Omega}}(u + \psi_\varepsilon - v) \leq \max_{\partial\Omega}(u + \psi_\varepsilon - v)^+,$$

so

$$\max_{\overline{\Omega}}(u - v) \leq \max_{\partial\Omega}(u - v)^+ + 2\varepsilon$$

and $\max_{\overline{\Omega}}(u - v) \leq \max_{\partial\Omega}(u - v)^+$ as $\varepsilon \to 0$.

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Theorem 1.2.5. *Let $F \in \mathcal{F}_{\lambda,\Lambda,b}$ be proper. Assume that condition 2 of Theorem 1.2.4 holds. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be respectively sub and supersolution of (1.3) in Ω then*

$$\max_{\overline{\Omega}}(u - v) \leq \max_{\partial\Omega}(u - v)^+.$$

Proof. Fix $\varepsilon > 0$, let $R > 0$ such that $\overline{\Omega} \subseteq B_R(0)$ and $0 < \alpha < \varepsilon e^{-\frac{2b}{\lambda}R}$. The function

$$\psi_\varepsilon(x) = \alpha\left(e^{\frac{2b}{\lambda}x_1} - e^{\frac{2b}{\lambda}R}\right)$$

satisfies the inequality $|\psi_\varepsilon| < \varepsilon$ in Ω and a straightforward computation in viscosity sense shows that

$$F(x, u + \psi_\varepsilon, D(u + \psi_\varepsilon), D^2(u + \psi_\varepsilon)) \geq f(x) + \frac{2\alpha b^2}{\lambda}e^{-\frac{2b}{\lambda}R}.$$

From Remark 1.2.5 we conclude. □

1.2.3 Existence results

First we introduce the notion of semicontinuous envelope of a function $u : \Omega \mapsto \mathbb{R}$.

Definition 1.2.3. The upper (lower) semicontinuous envelope u^* (u_*) of u is the function defined by

$$u^*(x) = \lim_{\varepsilon\to 0} \sup_{B_\varepsilon(x)\cap\Omega} u(y) \qquad \left(u_*(x) = \lim_{\varepsilon\to 0} \inf_{B_\varepsilon(x)\cap\Omega} u(y)\right)$$

From definition it follows $u_* \le u \le u^*$, u^* (u_*) is an upper (lower) semicontinuous function if $u^* < +\infty$ ($u_* > -\infty$) and

$$u^* = \inf\{v : v \in USC(\Omega) \text{ and } v \ge u\}, \quad u_* = \sup\{v : v \in LSC(\Omega) \text{ and } v \le u\}.$$

In the following Theorem we show that the upper semicontinuous envelope of the supremum of viscosity subsolutions is in turn a viscosity subsolution.

Theorem 1.2.6. *Let $(u_\alpha)_{\alpha\in\mathcal{A}} \subseteq USC(\Omega)$ be a family of solutions of*

$$F(x, u_\alpha, Du_\alpha, D^2u_\alpha) \ge f(x) \quad \text{in } \Omega$$

where $f \in C(\Omega)$ and $\mathcal{A}$ arbitrary set of parameters. Let $u = \sup_\alpha u_\alpha$ and assume $u^* < +\infty$ in Ω. Then u^* is a viscosity solution of $F \ge f$.

Proof. Let $x_0 \in \Omega$ and $\varphi \in C^2(B_\delta(x_0))$, $\delta > 0$, such that

$$u^*(x_0) = \varphi(x_0) \text{ and } u^*(x) < \varphi(x)\ \forall x \in B_\delta(x_0).$$

Consider a sequence $(x_k)_{k\in\mathbb{N}}$ converging to x_0 and $u_{\alpha(k)}$, $\alpha(k) \in \mathcal{A}$, satisfying

$$u^*(x_0) = \lim_{k\to+\infty} u_{\alpha(k)}(x_k).$$

Fix $\rho < \delta$ and let $\overline{x}_k$ be the maximum point of $u_{\alpha(k)} - \varphi$ over $\overline{B}_\rho(x_0)$ so that for k big enough

$$u_{\alpha(k)}(x_k) - \varphi(x_k) \le u_{\alpha(k)}(\overline{x}_k) - \varphi(\overline{x}_k).$$

Up to a subsequence $\overline{x}_k \to \hat{x} \in \overline{B}_\rho(x_0)$ and taking into account that

$$\begin{aligned} u^*(\hat{x}) - \varphi(\hat{x}) &\ge \limsup_{k\to+\infty} (u^*(\overline{x}_k) - \varphi(\overline{x}_k)) \ge \limsup_{k\to+\infty} (u(\overline{x}_k) - \varphi(\overline{x}_k)) \\ &\ge \limsup_{k\to+\infty} \left(u_{\alpha(k)}(\overline{x}_k) - \varphi(\overline{x}_k)\right) = u^*(x_0) - \varphi(x_0) \end{aligned}$$

we have $\hat{x} = x_0$, $\overline{x}_k \in B_\rho(x_0)$ for k big enough and, from definition of viscosity subsolution,

$$F(\overline{x}_k, u_{\alpha(k)}(\overline{x}_k), D\varphi(\overline{x}_k), D^2\varphi(\overline{x}_k)) \ge f(x_k). \tag{1.36}$$

Moreover

$$\begin{aligned} u^*(x_0) &= \lim_{k\to+\infty} \big(u_{\alpha(k)}(x_k) - \varphi(x_k) + \varphi(\overline{x}_k)\big) \\ &\le \liminf_{k\to+\infty} u_{\alpha(k)}(\overline{x}_k) \le \limsup_{k\to+\infty} u_{\alpha(k)}(\overline{x}_k) \\ &\le \limsup_{k\to+\infty} u(\overline{x}_k) \le \limsup_{k\to+\infty} u^*(\overline{x}_k) \le u^*(x_0) \end{aligned}$$

then $\lim_{k\to+\infty} u_{\alpha(k)}(\overline{x}_k) = u^*(x_0)$ and letting $k \to +\infty$ in (1.36) we conclude

$$F(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge f(x_0).$$

□

Lemma 1.2.3 (Bump). *Let $u \in USC(\Omega)$ be a solution of $F \ge f$ such that u_* fails to be a supersolution at some point $x \in \Omega$. Then for $\delta > 0$ small enough there exists a subsolution U_δ satisfying*

$$U_\delta \ge u \tag{1.37}$$

$$\sup_\Omega (U_\delta - u) > 0 \tag{1.38}$$

$$U_\delta = u \text{ in } \Omega \backslash B_{\frac{\delta}{2}}(x). \tag{1.39}$$

Proof. Let φ be a test function touching u_* from below at x such that

$$F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) > f(x).$$

By continuity if $\delta > 0$ is small enough the function

$$\psi(y) = \varphi(y) - \frac{\delta}{2}|y-x|^2 + \frac{\delta^3}{8}$$

is a classical solution of $F > f$ in $B_\delta(x)$. In view of Proposition 1.2.2, $\max\{u, \psi\}$ is a viscosity subsolution. For $\frac{\delta}{2} \le |y-x| \le \delta$, $\max\{u,\psi\} \equiv u$ being

$$u(y) \ge u_*(y) \ge \varphi(y) \ge \psi(y).$$

In this way the function

$$U_\delta = \begin{cases} \max\{u, \psi\} & \text{in } B_\delta(x) \\ u & \text{otherwise} \end{cases}$$

is a viscosity solution of $F \ge f$ satisfying (1.37)-(1.39). To prove (1.38) it suffices to consider a sequence $(x_k)_{k\in\mathbb{N}}$ converging to x such that $u(x_k) \to u_*(x)$ and note that

$$\lim_{k\to+\infty} U_\delta(x_k) - u(x_k) = \max\{\psi(x), u_*(x)\} - u_*(x) = \frac{\delta^3}{8} > 0.$$

□

Now we prove the existence of solutions of Dirichlet problem

$$\begin{cases} F(x,u,Du,D^2u) = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases} \tag{DP}$$

with $g \in C(\partial\Omega)$ via *Perron's method.* By a solution (subsolution, supersolution) of (DP) we mean a function $u \in C(\overline{\Omega})$ ($USC(\overline{\Omega})$, $LSC(\overline{\Omega})$) satisfying in Ω the equation $F = f$ ($F \geq f$, $F \leq f$) in the viscosity sense and the boundary condition $u = g$ ($u \leq g$, $u \geq g$) on $\partial\Omega$.

Theorem 1.2.7 (Perron's method). *Assume that comparison principle holds true and that there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (DP) locally bounded satisfying*

$$\underline{u}_* = \overline{u}^* = g \quad on \ \ \partial\Omega.$$

Then $u(x) := \sup_{v\in\mathcal{S}} v(x)$ is a solution of (DP), where

$$\mathcal{S} = \left\{ v \in USC(\overline{\Omega}) : \ F(x,v,Dv,D^2v) \geq f \ in \ \Omega \ and \ \underline{u} \leq v \leq \overline{u} \ in \ \overline{\Omega} \right\}.$$

Proof. The subsolution $\underline{u} \in \mathcal{S}$, so $\mathcal{S} \neq \emptyset$ and u is well defined. On the boundary $\partial\Omega$

$$\underline{u}_* \leq u_* \leq u \leq u^* \leq \overline{u}^*$$

and $u = u_* = u^* = g$. By Theorem 1.2.6 u^* is a subsolution of (DP) and, by comparison, $u^* \leq \overline{u}$. Moreover $u^* \geq u \geq \underline{u}$, hence $u^* \in \mathcal{S}$ and $u^* = u$. If u_* fails to be a supersolution of (DP), via bump Lemma 1.2.3 we find a subsolution U_δ such that, for δ small enough, $U_\delta = g$ on $\partial\Omega$, $U_\delta \geq u \geq \underline{u}$ and by comparison $U_\delta \leq \overline{u}$. Then $U_\delta \in \mathcal{S}$, $u \geq U_\delta$ in view of the maximality of u and we obtain a contradiction because of (1.38). Hence u_* is a supersolution of (DP) and by comparison we conclude $u = u^* = u_*$. □

Remark 1.2.6. The uniqueness of solutions of (DP) follows from comparison.

To complete the analysis of Perron's method we show how to construct the functions $\underline{u}$ and $\overline{u}$ of Theorem 1.2.7. We will make further assumptions on F and on the regularity of $\partial\Omega$.
Fix $x_0 \notin \overline{\Omega}$ and consider the radial function ($r = |x - x_0|$)

$$h(r) = \frac{1}{r_0^\sigma} - \frac{1}{r^\sigma},$$

where σ is a positive constant to be fixed and $0 < r_0 \leq r \leq r_1 \ \forall x \in \Omega$, so that $h(r) \geq 0$ and $h \equiv 0$ on $\partial B_{r_0}(x_0)$.

Proposition 1.2.6. *Let $F \in \mathcal{F}_{\lambda,\Lambda,b}$ be a proper operator. For σ, τ big enough then*

$$F(x, \tau h, D(\tau h), D^2(\tau h)) \leq f \quad in \quad \Omega.$$

Proof. A direct computation yields

$$\begin{aligned} F(x,\tau h, D(\tau h), D(\tau^2 h)) &\le F(x,0,D(\tau h), D^2(\tau h)) \\ &\le \mathcal{P}^+_{\lambda,\Lambda}(D^2(\tau h)) + b|D(\tau h)| \\ &\le \frac{\sigma\tau}{r^{\sigma+2}}\left((n-1)\Lambda - \lambda(\sigma+1) + br_1\right) \\ &\le \inf_{\Omega} f \end{aligned}$$

if $\sigma = \sigma(n,\lambda,\Lambda,b,r_1)$ and $\tau = \tau(\sigma, r_0, f)$ are chosen big enough. □

Regarding the smoothness of the domain Ω we assume the following

Definition 1.2.4. We say that Ω satisfies an *uniform exterior sphere condition* if there exists a radius $r_0 > 0$ such that for any $y \in \partial\Omega$ then $\overline{B}_{r_0}(z_y) \cap \overline{\Omega} = \{y\}$ for a suitable $z_y \notin \overline{\Omega}$.

Proposition 1.2.7. *Under the assumptions of Proposition 1.2.6, the function*

$$\overline{u}(x) = \left(\inf_{y\in\partial\Omega,\varepsilon>0} (g(y) + \varepsilon + C_\varepsilon h(|x - z_y|))\right)_*$$

is a supersolution of $F = f$ in Ω satisfying the boundary condition $\overline{u}^* = g$.

Proof. Let us extend g in $\overline{\Omega}$ so that $g \in C(\overline{\Omega})$ and choose the constant C_ε such that

$$g(y) + \varepsilon + C_\varepsilon h(|x - z_y|) \ge g(x) \quad \forall x \in \overline{\Omega}. \tag{1.40}$$

This choose can be done uniformly in $y \in \partial\Omega$. We put

$$G_{y,\varepsilon}(x) := g(y) + \varepsilon + C_\varepsilon h(|x - z_y|)$$

and observe that $G_{y,\varepsilon} \ge C > -\infty$ in $\overline{\Omega}$. As in Proposition 1.2.6 the function $G_{y,\varepsilon}$ is a supersolution of $F = f$ as well as

$$\overline{u}(x) = \left(\inf_{y\in\partial\Omega,\varepsilon>0} G_{y,\varepsilon}(x)\right)_* .$$

Moreover $\overline{u}(x) \le G_{y,\varepsilon}(x) \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$, then $\overline{u}^*(y) \le G_{y,\varepsilon}(y) = g(y) + \varepsilon$ for $y \in \partial\Omega$. As $\varepsilon \to 0$ $\overline{u}^* \le g$ on $\partial\Omega$. In view of (1.40) we conclude $\overline{u}^* \ge g$ on $\partial\Omega$. □

Remark 1.2.7. The function

$$\underline{u}(x) = \left(\sup_{y\in\partial\Omega,\varepsilon>0} (g(y) - \varepsilon - C_\varepsilon h(|x - z_y|))\right)^*$$

is a subsolution of $F = f$ such that $\underline{u}_* = g$ on $\partial\Omega$.

Remark 1.2.8. The smoothness of the boundary $\partial\Omega$ can be weakened by requiring the more general *uniform exterior cone condition* (see [20]).

1.2.4 Stability

By *stability* we intend the following fundamental problem: let u_k be a solution[2] of $F_k = 0$, if u_k and F_k converge as $k \to +\infty$ respectively to u and F in a suitable sense, is it true that u is a solution of the limit equation $F = 0$? We already know, from Theorem 1.2.2, that the local uniform convergence is a sufficient condition. We generalize this result. To this end we consider the "half-relaxed" limits: given a sequence of function $(u_k)_{k\in\mathbb{N}}$ in Ω set

$$\overline{u}(x) = \limsup_{k\to+\infty}{}^* u_k(x) := \lim_{j\to+\infty} \sup \left\{ u_k(y) :\ k \geq j,\ y \in \Omega \text{ and } |x-y| \leq \frac{1}{j} \right\} \tag{1.41}$$

and

$$\underline{u}(x) = \liminf_{k\to+\infty}{}_* u_k(x) := \lim_{j\to+\infty} \inf \left\{ u_k(y) :\ k \geq j,\ y \in \Omega \text{ and } |x-y| \leq \frac{1}{j} \right\} \tag{1.42}$$

Note that $\liminf_{k\to+\infty *} u_k = -\limsup_{k\to+\infty}{}^*(-u_k)$ and if $u_k \equiv u$ then $\overline{u}$ ($\underline{u}$) coincide with the upper (lower) semicontinuous envelope of u. Moreover for any sequence x_k converging to x we have $\overline{u}(x) \geq \limsup u_k(x_k)$ ($\underline{u}(x) \leq \liminf u_k(x_k)$) and there exists a sequence $x_{k_j} \to x$ such that $u_{k_j}(x_{k_j}) \to \overline{u}(x)(\underline{u}(x))$.

Proposition 1.2.8. *The function $\overline{u}(x)$ ($\underline{u}(x)$) is upper (lower) semicontinuous.*

Proof. We prove that $\overline{u} \in USC(\Omega)$ by showing that the set $A_t = \{x \in \Omega : \overline{u}(x) \geq t\}$ is closed for any $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ and let $(x_m)_{m\in\mathbb{N}} \subseteq A_t$ a sequence converging to x. For any $j \in \mathbb{N}$ let x_{m_j} such that $B_{\frac{1}{j}}(x) \supseteq B_{\frac{1}{2j}}(x_{m_j})$. In this way

$$\sup \left\{ u_k(y) :\ k \geq j,\ y \in \Omega \text{ and } |x-y| \leq \frac{1}{j} \right\} \geq \sup \left\{ u_k(y) :\ k \geq 2j,\ y \in \Omega \text{ and } |x_{m_j}-y| \leq \frac{1}{2j} \right\} \geq t$$

and as $j \to +\infty$ we conclude $\overline{u}(x) \geq t$. The proof $\underline{u}(x) \in LSC(\Omega)$ is similar. □

Analogously for given continuous function $F_k : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ we consider

$$\begin{aligned} \overline{F}(x,r,p,M) &= \limsup_{k\to+\infty}{}^* F_k(x,r,p,M) \\ &= \lim_{j\to+\infty} \sup \left\{ F_k(y,s,q,N) :\ k \geq j,\ y \in \Omega \text{ and } |x-y| \leq \frac{1}{j}, \right. \\ &\qquad \left. |s-r| \leq \frac{1}{j},\ |q-p| \leq \frac{1}{j},\ \|N-M\| \leq \frac{1}{j} \right\} \end{aligned} \tag{1.43}$$

and

$$\begin{aligned} \underline{F}(x,r,p,M) &= \liminf_{k\to+\infty}{}_* F_k(x,r,p,M) \\ &= \lim_{j\to+\infty} \inf \left\{ F_k(y,s,q,N) :\ k \geq j,\ y \in \Omega \text{ and } |x-y| \leq \frac{1}{j}, \right. \\ &\qquad \left. |s-r| \leq \frac{1}{j},\ |q-p| \leq \frac{1}{j},\ \|N-M\| \leq \frac{1}{j} \right\}. \end{aligned} \tag{1.44}$$

[2]In this subsection we refer to the equation $F = 0$, in place of $F = f$, because we will not use the fact that $F(x, 0, 0, 0) = 0$.

As above $\overline{F} \in USC(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$ ($\underline{F} \in LSC(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$), $\limsup_{k \to +\infty} F(x_k, r_k, p_k, M_k) \leq \overline{F}(x, r, p, M)$ ($\liminf_{k \to +\infty} F(x_k, r_k, p_k, M_k) \geq \overline{F}(x, r, p, M)$) for any sequence $(x_k, r_k, p_k, M_k) \to (x, r, p, M)$ as $k \to +\infty$ and the equality holds for a suitable sequence. Now we are in position to prove the following stability result.

Theorem 1.2.8 (Stability 2). *Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of viscosity subsolution (supersolution) of $F_k(x, u_k, Du_k, D^2 u_k) = 0$ in Ω, where F_k are continuous functions. Then $\overline{u}$ ($\underline{u}$) is a viscosity subsolution (supersolution) of $\overline{F}(x, \overline{u}, D\overline{u}, D^2\overline{u}) = 0$ ($\underline{F}(x, \underline{u}, D\underline{u}, D^2\underline{u}) = 0$).*

Proof. We prove the subsolution case. Let $x_0 \in \Omega$, $B_R(x_0) \subseteq \Omega$ and $\varphi \in C^2(B_R(x_0))$ such that

$$\overline{u}(x) - \varphi(x) < \overline{u}(x_0) - \varphi(x_0) = 0 \quad \forall x \in B_R(x_0) \setminus \{x_0\} .$$

Consider a sequence $(x_{k_j})_{j \in \mathbb{N}}$ converging to x_0 such that

$$\lim_{j \to +\infty} u_{k_j}(x_{k_j}) = \overline{u}(x_0)$$

and a sequence $(y_{k_j})_{j \in \mathbb{N}}$ of maximum points of $u_{k_j} - \varphi$ in $\overline{B}_r(x_0)$ with $r < R$. Up to a subsequence we may assume $y_{k_j} \to z \in \overline{B}_r(x_0)$. Then

$$\begin{aligned} 0 &= \lim_{j \to +\infty} (u_{k_j} - \varphi)(x_{k_j}) \leq \liminf_{j \to +\infty} (u_{k_j} - \varphi)(y_{k_j}) = \liminf_{j \to +\infty} u_{k_j}(y_{k_j}) - \varphi(z) \\ &\leq \limsup_{j \to +\infty} u_{k_j}(y_{k_j}) - \varphi(z) \leq (\overline{u} - \varphi)(z). \end{aligned}$$

It follows $z = x_0$ and $\lim_{j \to +\infty} u_{k_j}(y_{k_j}) = \overline{u}(x_0)$. For j big enough $y_{k_j} \in B_r(x_0)$, by definition of subsolution

$$F_{k_j}(y_{k_j}, u_{k_j}(y_{k_j}), D\varphi(y_{k_j}), D^2\varphi(y_{k_j})) \geq 0$$

and the thesis as $j \to +\infty$. □

Remark 1.2.9. Theorem 1.2.2 is a particular case of Theorem 1.2.8. If $(u_k)_{k \in \mathbb{N}}$ is a sequence of upper semicontinuous function converging locally uniformly in Ω to u, then $u = \overline{u}$:

$$u(x) = \limsup_{k \to +\infty} u_k(x) = \lim_{k \to +\infty} \sup \{u_k(x) : \ k \geq j\} \leq \overline{u}(x)$$

and, taking into account that $u \in USC(\Omega)$,

$$\begin{aligned} \overline{u}(x) \leq & \lim_{j \to +\infty} \sup \left\{ u_k(y) - u(y) : \ k \geq j, \, y \in \Omega \text{ and } |x - y| \leq \frac{1}{j} \right\} \\ & + \lim_{j \to +\infty} \sup \left\{ u(y) : \ y \in \Omega \text{ and } |x - y| \leq \frac{1}{j} \right\} \leq u(x). \end{aligned}$$

Analogously if $F_k \to F$ locally uniformly in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ then $\overline{F} = F$.

1.2.5 Approximations

We are concerned with regular approximations of viscosity solution, first introduced by Jensen in [40]. Let $u \in USC(\Omega)$ be a locally bounded function, for any open $H \Subset \Omega$ the *upper ε-envelope* of u with respect to H is the function

$$u^\varepsilon(x) = \sup_{y \in \overline{H}} \left\{ u(y) + \varepsilon - \frac{1}{\varepsilon}|x-y|^2 \right\} \quad \text{for } x \in H. \tag{1.45}$$

Similarly

$$u_\varepsilon(x) = \inf_{y \in \overline{H}} \left\{ u(y) - \varepsilon + \frac{1}{\varepsilon}|x-y|^2 \right\} \quad \text{for } x \in H \tag{1.46}$$

is the *lower ε-envelope* of u with respect to H. In the sequel we analyze the main properties of (1.45), noting that the analogous properties for (1.46) follow from the equality $u_\varepsilon = -(-u)^\varepsilon$. First of all we note that $u^\varepsilon(x) \geq u(x) + \varepsilon$ by choosing $y = x$ in (1.45) and the supremum is achieved, that is there exists a point $y^\varepsilon \in \overline{H}$ such that

$$u^\varepsilon(x) = u(y^\varepsilon) + \varepsilon - \frac{1}{\varepsilon}|x - y^\varepsilon|^2. \tag{1.47}$$

From (1.47)

$$|x - y^\varepsilon|^2 = \varepsilon\left(u(y^\varepsilon) - u^\varepsilon(x) + \varepsilon\right) \leq \varepsilon\left(u(y^\varepsilon) - u(x)\right) \leq \varepsilon \operatorname*{osc}_{\overline{H}} u, \tag{1.48}$$

so $y^\varepsilon \to x$ as $\varepsilon \to 0^+$ and $\frac{|x-y^\varepsilon|^2}{\varepsilon} \to 0$ by semicontinuity. Moreover

$$u^\varepsilon(x) = \sup_{y \in \overline{H} \cap \overline{B}_r(x)} \left\{ u(y) + \varepsilon - \frac{1}{\varepsilon}|x-y|^2 \right\} \tag{1.49}$$

with $r = (\varepsilon \operatorname*{osc}_{\overline{H}} u)^{1/2}$ in view of (1.48).
The upper ε-envelope is a semiconvex function (i.e. $u^\varepsilon(x) + \frac{C}{2}|x|^2$ is convex for a suitable $C > 0$) on every convex subset on H. In fact

$$u^\varepsilon(x) + \frac{1}{\varepsilon}|x|^2 = \sup_{y \in \overline{H}} \left\{ u(y) + \varepsilon - \frac{1}{\varepsilon}|y|^2 - \frac{2}{\varepsilon}\langle x, y\rangle \right\}$$

is convex as supremum of affine functions.

Proposition 1.2.9.

1. *u^ε is a Lipschitz function.*

2. *u^ε is punctually second order differentiable a.e. in H.*

3. *For any $x_0 \in H$ there exists a concave paraboloid of opening $\frac{2}{\varepsilon}$ touching u^ε by below at x_0 in H.*[3]

4. *$u^\varepsilon \downarrow u$ as $\varepsilon \to 0^+$; if $u \in C(H)$ then $u^\varepsilon \downarrow u$ locally uniformly.*

[3] By paraboloid P of opening $c > 0$ we mean $P(x) = a + \langle b, x\rangle \pm \frac{c}{2}|x|^2$, $a \in \mathbb{R}$, $b \in \mathbb{R}^n$

Proof. 1. Let x, $z \in H$, for any $y \in \overline{H}$

$$u^\varepsilon(x) \geq u(y) + \varepsilon - \frac{1}{\varepsilon}|x-y|^2 \geq u(y) + \varepsilon - \frac{1}{\varepsilon}|y-z|^2 - \frac{3\text{diam}H}{\varepsilon}|x-z|.$$

Taking the supremum over $y \in \overline{H}$

$$u^\varepsilon(x) \geq u^\varepsilon(z) - \frac{3\text{diam}H}{\varepsilon}|x-z|$$

and changing the role of x and z we are done.
2. u^ε is semiconvex in any ball $B \subset H$ and, according Alexandroff Theorem [28, Theorem 1 - Section 6.4], is punctually second order differentiable a.e in B from which we conclude.
3. Let $x_0 \in H$ and y_0^ε such that $u^\varepsilon(x_0) = u(y_0^\varepsilon) + \varepsilon - \frac{1}{\varepsilon}|x_0 - y_0^\varepsilon|^2$, then $P(x) = u(y_0^\varepsilon) + \varepsilon - \frac{1}{\varepsilon}|x - y_0^\varepsilon|^2$ is the paraboloid required.
4. For any $\varepsilon' < \varepsilon$, $x \in H$ we have $u^{\varepsilon'}(x) \leq u^\varepsilon(x)$ and

$$u(x) + \varepsilon \leq u^\varepsilon(x) \leq u(y^\varepsilon) + \varepsilon.$$

Taking the limsup and remembering that $y^\varepsilon \to x$ as $\varepsilon \to 0^+$ we have

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(x) = \limsup_{\varepsilon \to 0^+} u^\varepsilon(x) = u(x).$$

If $u \in C(H)$ the uniform convergence on compact sets follows from Dini Theorem (Proposition 11 in Chapter 9 of [50]). □

The class of viscosity subsolutions of $F(Du, D^2u) = 0$ is preserved under the operation of ε-envelope. Precisely the following Proposition holds, see also [14].

Proposition 1.2.10. *Let u be a viscosity subsolution of $F(Du, D^2u) = f(x)$ in Ω and let $H_1 \Subset H$. Then for ε small enough (depending on u, H_1 and H) u^ε is a viscosity subsolution in H_1 of*

$$F(Du^\varepsilon, D^2u^\varepsilon) + \operatorname*{osc}_{B_r(x)} f \geq f(x) \tag{1.50}$$

where r as in (1.49).

Proof. Let $\varphi \in C^2(B_\delta(x))$ touching u^ε by above at x in $B_\delta(x) \subset H_1$. If ε is small enough $u^\varepsilon(x) = u(y^\varepsilon) + \varepsilon - \frac{1}{\varepsilon}|x - y^\varepsilon|^2$ with $y^\varepsilon \in H$. The function $w(z) = \varphi(z + x - y^\varepsilon) \in C^2(B_\delta(y^\varepsilon))$ and

$$u(y^\varepsilon) - w(y^\varepsilon) = -\varepsilon + \frac{1}{\varepsilon}|x - y^\varepsilon|^2.$$

Choosing $\delta' < \delta$ such that $B_{\delta'}(y^\varepsilon) \subset \Omega$ we have

$$u(z) - w(z) \leq u^\varepsilon(z + x - y^\varepsilon) - \varepsilon + \frac{1}{\varepsilon}|x - y^\varepsilon|^2 - \varphi(z + x - y^\varepsilon) \leq -\varepsilon + \frac{1}{\varepsilon}|x - y^\varepsilon|^2,$$

then $u - w$ has a maximum at y^ε in $B_{\delta'}(y^\varepsilon)$ and

$$\begin{aligned} 0 &\leq F(Dw(y^\varepsilon), D^2w(y^\varepsilon)) - f(y^\varepsilon) = F(D\varphi(x), D^2\varphi(x)) - f(x) + (f(x) - f(y^\varepsilon)) \\ &\leq F(D\varphi(x), D^2\varphi(x)) - f(x) + \operatorname*{osc}_{B_r(x)} f \end{aligned}$$

as desired. □

Remark 1.2.10. As a consequence of Propositions 1.2.9-1.2.10 the function u^ε satisfies (1.50) a.e. in H_1 for ε small enough.

Remark 1.2.11. If u is a solution of $F(u, Du, D^2u) \geq f$ with $F(u, \cdot, \cdot)$ nonincreasing, it easy to check that $F(u^\varepsilon - \varepsilon, Du^\varepsilon, D^2u^\varepsilon) + \operatorname{osc}_{B_r(x)} f \geq f(x)$ in H_1.

The next Proposition is based on [14]-[40]. We assume $F \in \mathcal{F}_{\lambda,\Lambda,b}$:

$$\mathcal{P}^-_{\lambda,\Lambda}(M-N) - b|p-q| \leq F(x,r,p,M) - F(x,r,q,N) \leq \mathcal{P}^+_{\lambda,\Lambda}(M-N) + b|p-q| \tag{1.51}$$

for any $x \in \Omega$, $p, q \in \mathbb{R}^n$, $M, N \in \mathcal{S}^n$.

Proposition 1.2.11. *Let $u, v \in C(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$ be respectively viscosity solutions of*

$$F(Dv, D^2v) \leq f \leq F(Du, D^2u) \quad \text{in } \Omega$$

and $f \in C(\Omega)$. If $H_1 \Subset H \Subset \Omega$, then $u^\varepsilon - v_\varepsilon$ is a viscosity solution of

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2(u^\varepsilon - v_\varepsilon)) + b|D(u^\varepsilon - v_\varepsilon)| \geq -2 \operatorname*{osc}_{B_{\overline{r}}(x)} f \quad \text{in } H_1 \tag{1.52}$$

for ε small enough and $r = \max\left((\varepsilon \operatorname{osc}_{\overline{H}} u)^{1/2}, (\varepsilon \operatorname{osc}_{\overline{H}} v)^{1/2}\right)$.

Proof. Fix $x \in H_1$ and let P a paraboloid of opening $c > 0$ such that the function $w(y) = (u^\varepsilon - v_\varepsilon - P)(y)$ satisfies

$$0 = w(x) < w(y) \quad \forall y \in B_r(x) \setminus \{x\} \subseteq H_1.$$

In view of Proposition (1.2.9), $w \in W^{1,\infty}(B_r(x))$ is punctually second order differentiable and

$$D^2 w \geq -\left(\frac{4}{\varepsilon} + c\right) I.$$

Using [40, Lemmas 3.10-3.20] the upper contact set

$$\Gamma^+_\delta(w) = \left\{x \in B_r(x) \,|\, \exists p \in \overline{B}_\delta(0) \text{ and } w(y) \leq w(x) + \langle p, y - x\rangle \ \forall y \in B_r(x)\right\}$$

has positive measure for δ small enough, $D^2 w \leq 0$ a.e. in $\Gamma^+_\delta(w)$ and

$$\int_{\Gamma^+_\delta(w)} \left(\frac{-\mathcal{P}^+_{\lambda,\Lambda}(D^2w)}{|Dw|}\right)^n dx = \lambda^n \int_{\Gamma^+_\delta(w)} \left(\frac{(\operatorname{Trace} D^2 w)^-}{|Dw|}\right)^n dx = +\infty.$$

In this way there exists a subset of $\Gamma^+_\delta(w)$ of positive measure in which

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) \leq -b|Dw|. \tag{1.53}$$

Repeating the above argument for $r_k = \frac{r}{k}$, $k \in \mathbb{N}$, we deduce the existence of a sequence $x_k \to x$ for which (1.50)-(1.53) holds true (remember Remark 1.2.10). Using the structure condition (1.51) we get

$$\begin{aligned}
-2 \operatorname*{osc}_{B_{\overline{r}}(x_k)} f &\leq F(Du^\varepsilon(x_k), D^2u^\varepsilon(x_k)) - F(Dv_\varepsilon(x_k), D^2 v_\varepsilon(x_k)) \\
&\leq \mathcal{P}^+_{\lambda,\Lambda}(D^2(w+P)(x_k)) + b|D(w+P)(x_k)| \\
&\leq \mathcal{P}^+_{\lambda,\Lambda}(D^2P(x_k)) + b|DP(x_k)|
\end{aligned}$$

and the thesis as $k \to +\infty$. ◻

Theorem 1.2.9. *Under the assumptions of Proposition 1.2.11 the difference $u - v$ is a viscosity solution of*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2(u-v)) + b|D(u-v)| \geq 0 \quad in\ \Omega. \tag{1.54}$$

Proof. Let $H_1 \Subset H \Subset \Omega$. The difference $u^\varepsilon - v_\varepsilon$ is a solution of (1.52) and converges uniformly to $u - v$ in H_1. By continuity $\operatorname{osc}_{B_{\overline{r}}(x)} f \to 0$ as $\varepsilon \to 0^+$ and from the stability result (1.2.1)

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2(u-v)) + b|D(u-v)| \geq 0 \quad \text{in } H_1.$$

Since H_1 is arbitrary the proof is complete. □

Remark 1.2.12. Under the structure condition (1.51), viscosity solutions belong to $W^{1,\infty}_{\text{loc}}(\Omega)$ ([39, Theorem VII.2]).

Remark 1.2.13. If $F(u, \cdot, \cdot)$ is nonincreasing we deduce from Remark 1.2.11 that $u - v$ is a solution of (1.54) in $\Omega \cap \{u > v\}$.

1.3 L^p-Viscosity solutions

In this Section we present a suitable viscosity notion for solutions of elliptic equation

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{1.55}$$

when $F(x, \cdot, \cdot, \cdot)$ and $f(x)$ depend measurably (and not continuously as in Section 1.2) on x. For an extensive treatment about L^p-viscosity theory we refer to [9]-[20]-[21]-[42]. In contrast to the definition of C-viscosity solution we consider as test function space the Sobolev space $W^{2,p}_{\text{loc}}$ supposing at least $2p > n$ so that a function $u \in W^{2,p}_{\text{loc}}$ is continuous and pointwise twice differentiable a.e. (in the sense that u has the second-order Taylor expansion) (see Appendix C in[9]). In addition sub(super)solutions are required to be continuous functions.

Definition 1.3.1. Let F be proper and $f \in L^p_{\text{loc}}(\Omega)$. A function $u \in C(\Omega)$ is an *L^p-viscosity subsolution* (*supersolution*) of (1.55), or solution of $F \geq (\leq) f$, if

$$\operatorname*{ess\,lim\,sup}_{y \to x} \big(F(y, u(y), D\varphi(y), D^2\varphi(y)) - f(y)\big) \geq 0$$

$$\left(\operatorname*{ess\,lim\,inf}_{y \to x} \big(F(y, u(y), D\varphi(y), D^2\varphi(y)) - f(y)\big) \leq 0\right)$$

for any $x \in \Omega$ and $\varphi \in W^{2,p}(B_r(x))$ such that $u - \varphi$ has a local maximum (minimum) at x. Equivalently if ε is a positive number such that

$$F(y, u(y), D\varphi(y), D^2\varphi(y)) - f(y) \leq -\varepsilon (\geq \varepsilon)$$

a.e. in some neighborhood of x, then $u - \varphi$ cannot have a local maximum (minimum) at x. Finally $u \in C(\Omega)$ is an *L^p-viscosity solution* of $F = f$ if it is both sub and supersolution.

Remark that if $q \leq p$ and $f \in L^p_{\rm loc}(\Omega)$ then an L^q-viscosity subsolution (supersolution) is automatically L^p-viscosity subsolution (supersolution) because of the inclusion $W^{2,p}_{\rm loc} \subseteq W^{2,q}_{\rm loc}$. It's worth noting that considering C^2 as test function space in the above definition then uniqueness results fails dramatically as shown in [9] (Example 2.4).
In the sequel we will also use the notion of *strong solution*.

Definition 1.3.2. A function $u \in W^{2,p}_{\rm loc}(\Omega)$ satisfying

$$F(x,u(x),Du(x),D^2u(x)) \geq (\leq,=) f(x) \quad \text{a.e. in } \Omega$$

is said to be a *strong subsolution (supersolution,solution)* of (1.55).

To clarify the connection among C-L^p-strong solutions we need a structure condition on F: for each $R>0$ there exists a nondecreasing continuous function ω_R such that $\omega_R(0)=0$ and

$$\begin{aligned}\mathcal{P}^-_{\lambda,\Lambda}(M-N) - b|p-q| - \omega_R((r-s)^+) &\leq F(x,r,p,M) - F(x,s,q,N)\\ &\leq \mathcal{P}^+_{\lambda,\Lambda}(M-N) + b|p-q| - \omega_R((s-r)^+),\end{aligned} \tag{1.56}$$

for $M,N \in \mathcal{S}^n$, $p,q \in \mathbb{R}^n$, $|r|,|s| \leq R$ a.e. $x \in \Omega$. Note that (1.56) for $r=s$ is the uniformly ellipticity plus b-Lipschitz continuity in the gradient variable, while if $M=N$ and $p=q$ the right hand inequality ensures that the mapping $r \mapsto F(\cdot,r,\cdot,\cdot)$ is nonincreasing.

Another ingredient we will use is the generalized maximum principle, GMP for short (see [22],[26], [43], [53]): there exists $p_0 = p_0(n,\Lambda/\lambda) \in (\frac{n}{2},n)$ such that if $f \in L^p(\Omega)$ with $p > p_0$ and $u \in W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ is an L^p-strong solution of the maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + b|Du| \geq f,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C\,\mathrm{diam}(\Omega)^{2-\frac{n}{p}}\|f^-\|_{L^p(\Omega)} \tag{1.57}$$

with C a positive constant depending on $n,\lambda,\Lambda,p,b\,\mathrm{diam}(\Omega)$.

As a consequence of (1.56)-(GMP) L^p-strong solutions are L^p-viscosity solutions [9, Lemma 2.6]. Conversely if $u \in W^{2,p}_{\rm loc}(\Omega)$ is an L^p-viscosity solution then it is an L^p-strong solution [9, Corollary 3.7]. If then F and f are continuous functions the notions of C- and L^p-viscosity solutions coincide [9, Proposition 2.9].

We recall a fundamental tool in the viscosity theory: the Alexandroff-Bakelman-Pucci (ABP for short) maximum principle, see [9, Proposition 3.3], [21, Proposition 2.3], [8, Theorem 3.6] and [9, Proposition 2.12] . The ABP estimates give an upper bound for the supremum of u $(-u)$ in terms of the supremum on the boundary $\partial\Omega$ and the L^n-norm of f, for any subsolution(supersolution) of the equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2u)+\gamma|Du| = f$ $(\mathcal{P}^-_{\lambda,\Lambda}(D^2u)-\gamma|Du|=f)$. In the statement $\Gamma^+(u)$ denotes the upper contact set of the graph of the function u

$$\Gamma^+(u) = \{x \in \Omega \,|\, \exists p \in \mathbb{R}^n \text{ such that } u(y) \leq u(x) + \langle p, y-x\rangle \text{ for } y \in \Omega\}\,.$$

Proposition 1.3.1 (ABP estimates). *Let $f \in L^n(\Omega)$ and $u \in C(\overline{\Omega})$ be an L^n-viscosity solution of the equation*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + b|Du| \geq f \quad \text{in } \{u > 0\} \tag{1.58}$$

$$\left(\mathcal{P}^-_{\lambda,\Lambda}(D^2u) - b|Du| \leq f \quad \text{in } \{u < 0\}\right), \tag{1.59}$$

then there exists a positive constant $C = C(\lambda, b, n, \mathrm{diam}(\Omega))$ such that

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \mathrm{diam}(\Omega) C \left\|f^-\right\|_{L^n(\Gamma^+(u^+))} \tag{1.60}$$

$$\left(\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + \mathrm{diam}(\Omega) C \left\|f^+\right\|_{L^n(\Gamma^+(u^-))}\right) \tag{1.61}$$

Remark 1.3.1. Proposition 1.3.1 applies to subsolution (supersolution) of (1.55) taking into account that, if F satisfies the structure condition (1.56), a subsolution (supersolution) of $F = f$ is in turn a subsolution (supersolution) of $\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + \gamma|Du| = f$ in $\{u > 0\}$ ($\mathcal{P}^+_{\lambda,\Lambda}(D^2u) - \gamma|Du| = f$ in $\{u < 0\}$).

Remark 1.3.2. If the inequalities (1.58)-(1.59) hold in Ω (not only in $\{u > 0\}, \{u < 0\}$), the functions $v = u - \min_{\overline{\Omega}} u + 1$ and $w = u - \max_{\overline{\Omega}} u - 1$ are respectively solutions of

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + \gamma|Dv| \geq f \quad \text{in } \Omega = \{v > 0\},$$

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2w) - \gamma|Dw| \leq f \quad \text{in } \Omega = \{w < 0\}.$$

Since $\Gamma^+(v^+) = \Gamma^+(u)$ and $\Gamma^+(w^-) = \Gamma^+(-u)$ we deduce the estimates

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \mathrm{diam}(\Omega) C \left\|f^-\right\|_{L^n(\Gamma^+(u))},$$

$$\sup_{\Omega}(-u) \leq \sup_{\partial\Omega}(-u) + \mathrm{diam}(\Omega) C \left\|f^+\right\|_{L^n(\Gamma^+(-u))}.$$

For Maximum Principles and ABP estimates in unbounded domains, see [2]-[6]-[12]-[13]-[15]-[54].

Concerning the stability properties of L^p-viscosity solutions there will be useful the following Theorem. To state the result we introduce some notations: for $B_r(x) \subseteq \Omega$ and $\varphi \in W^{2,p}(B_r(x))$ we set for $k \in \mathbb{N}$

$$g_k(y) = F_k(y, u_k(y), D\varphi(y), D^2\varphi(y)) - f_k(y),$$

$$g(y) = F(y, u(y), D\varphi(y), D^2\varphi(y)) - f(y)$$

for $y \in B_r(x)$.

Theorem 1.3.1. *Let F_k, F satisfy (1.56), let f_k, $f \in L^p(\Omega)$ with $p > p_0$ and let $u_k \in C(\Omega)$ be L^p-viscosity subsolution (supersolutions) of*

$$F_k(x, u_k, Du_k, D^2u_k) = f_k \quad \text{in } \Omega$$

for $k \in \mathbb{N}$. Assume that $u_k \to u$ locally uniformly as $k \to +\infty$ and that for $B_r(x) \subseteq \Omega$ and $\varphi \in W^{2,p}(B_r(x))$

$$\left\|(g - g_k)^-\right\|_{L^p(B_r(x))} \to 0 \quad \left(\left\|(g - g_k)^+\right\|_{L^p(B_r(x))} \to 0\right).$$

Then u is an L^p-viscosity subsolution (supersolution) of $F = f$.

Chapter 2

Entire Solutions

2.1 Statement of the problem

We are interested with the well posedness of a class of fully nonlinear second order uniformly elliptic problems

$$F(x, u, Du, D^2u) = f(x) \tag{2.1}$$

in the whole space $\mathbb{R}^n$. Solutions of these problems are said to be *entire solutions*. Starting from Brezis's paper [4], in which the existence and uniqueness of a distributional solution $u \in L^d_{\text{loc}}(\mathbb{R}^n)$ is proved for the semilinear elliptic equation

$$\Delta u - |u|^{d-1}u = f(x) \quad \text{in } \mathbb{R}^n \tag{2.2}$$

with $d > 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and from the related results of Esteban-Felmer-Quaas [27] for the L^n-viscosity solutions of a more general class of uniformly elliptic equations

$$F(D^2u) - |u|^{d-1}u = f(x) \quad \text{in } \mathbb{R}^n \tag{2.3}$$

with $f \in L^n_{\text{loc}}(\mathbb{R}^n)$, we propose to extend these results in various directions: consider a more general class of equations than (2.3), including lower order terms, allowing the dependence on x in the operator F and assuming a minor local summability of f. It is worth to notice that no assumptions on the behaviour of u at infinity and no limitation on the growth at infinity of the datum f are required.

We assume $F \in \mathcal{F}_{\lambda,\Lambda,b}$, namely

$$\mathcal{P}^-_{\lambda,\Lambda}(M - N) - b|p - q| \le F(x, r, p, M) - F(x, r, q, N) \le \mathcal{P}^+_{\lambda,\Lambda}(M - N) + b|p - q| \tag{2.4}$$

for $x \in \mathbb{R}^n$, $r \in \mathbb{R}$, $p, q \in \mathbb{R}^n$, $M, N \in \mathcal{S}^n$ and

$$F(x, 0, 0, 0) = 0. \tag{2.5}$$

The constant $b \ge 0$ plays the role of Lipschitz constant.
As regards the monotonicity in the r-variable we ask something more than usual monotonicity

assumption ($\delta = 0$): there exists $\delta > 0$ such that

$$F(x, r, p, M) - F(x, s, p, M) \leq -\delta(r - s)^d \quad \text{if } r > s \tag{2.6}$$

for $d > 1$. We collect the above assumptions in the structure condition

$$(\mathbf{SC}) := (2.4) - (2.5) - (2.6). \tag{2.7}$$

Making these assumptions we have in mind equations of the form

$$F(x, D^2u) + b|Du| - g(u) = f(x),$$

where F is uniformly elliptic and $g : \mathbb{R} \mapsto \mathbb{R}$ is a function such that $g(r) - g(s) \geq \delta(r - s)^d$ for $r \geq s$ (for example $g(r) = |r|^{d-1}r$ or $g(r) = \sinh r$).

In the sequel $p_0 = p_0(n, \Lambda/\lambda) \in (\frac{n}{2}, n)$ is the exponent such that for $p > p_0$ the GMP (1.57) holds true. It is important to notice that GMP continues to hold for L^p-viscosity subsolutions as $p > p_0$, see [53, Lemma 1.4] and [43, Theorem 3.2].

The key tool to have existence and uniqueness of entire solutions is to prove local uniform estimates for solutions of (2.1)

2.2 Uniform Estimates

Let us introduce the Osserman's barrier function (see [4]-[27]-[49])

$$\Phi(x) = \frac{C_R R^\mu}{(R^2 - |x|^2)^\mu} \quad \text{in } B_R, \tag{2.8}$$

where $\mu = \frac{2}{d-1}$ (recall that $d > 1$) and C_R is a constant depending on R as precised in the next lemma.

Lemma 2.2.1. *Suppose for a.e. $x \in B_R$ that*

$$F(x, s, q, N) \leq \mathcal{P}^+_{\lambda,\Lambda}(N) + b|q| - \delta s^d \tag{2.9}$$

for all $(s, q, N) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^n$, where $\delta > 0$ and $d > 1$.
If we take

$$C_R^{d-1} = 2\mu\delta^{-1}(\Lambda(n + 2(1 + \mu)) + bR),$$

then the function $\Phi \in C^2(B_R)$, defined in (2.8), is a L^p-strong solution of the equation

$$F(x, \Phi, D\Phi, D^2\Phi) \leq 0 \quad \text{in } B_R.$$

Proof. By the assumptions, it is sufficient to show that

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi) + b|D\Phi| - \delta\Phi^d \leq 0.$$

Since $\Phi(x) = \varphi(r) := C_R\, R^\mu (R^2 - r^2)^{-\mu}$, where $r = |x|$, then from Proposition 1.1.3 and Remark 1.1.2

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi) + b|D\Phi| - \delta\Phi^d = \Lambda[\varphi'' + \tfrac{n-1}{r}\,\varphi'] + b\varphi' - \delta\varphi^d$$

and the result follows by choosing $C_R > 0$ as claimed. □

The following Lemma is useful in the sequel.

Lemma 2.2.2. *Let u, v be respectively L^p-viscosity subsolution and strong supersolution of the equations $F(x, u, Du, D^2u) = f$ and $F(x, v, Dv, D^2v) = g$ in Ω. Assuming (2.4)-(2.6) a.e. in Ω, the difference $w = u - v$ is an L^p-viscosity subsolution of the maximal equation*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| - \delta w^d = f - g \tag{2.10}$$

in $\Omega \cap \{w > 0\}$.

Proof. By contradiction let $\varphi \in W^{2,p}(B_r(x))$, with $B_r(x) \subset \Omega \cap \{w > 0\}$, such that $w - \varphi$ has a local maximum at x and $\mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi(y)) + b|D\varphi(y)| - \delta(w(y))^d - (f - g)(y) \leq -\varepsilon$ a.e. in $B_r(x)$ for some $\varepsilon > 0$. Then $v + \varphi \in W^{2,p}(B_r(x))$, $u - (v + \varphi)$ has a local maximum at x and

$$\begin{aligned} F(y, u(y), D(v+\varphi)(y), D^2(v+\varphi)(y)) - f(y) &\leq F(y, u(y), Dv(y), D^2v(y)) \\ &\quad + \mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi(y)) + b|D\varphi(y)| - f(y) \\ &\leq \mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi(y)) + b|D\varphi(y)| - \delta(w(y))^d - (f-g)(y) \\ &\leq -\varepsilon \quad \text{a.e. in } B_r(x) \end{aligned}$$

a contradiction. □

Remark 2.2.1. Lemma 2.2.2 remains valid in the C-viscosity sense: if u and v are respectively C-viscosity subsolution and classical supersolution of $F = f$ and $F = g$, then $w = u - v$ is a C-subsolution of the maximal equation (2.10) in $\Omega \cap \{w > 0\}$.

Lemma 2.2.3. *Let Ω be a domain of $\mathbb{R}^n$ such that $\Omega_R := \Omega \cap B_R \neq \emptyset$. Suppose that F satisfy structure conditions (**SC**) a.e. $x \in \Omega_R$. If $u \in C(\overline{\Omega}_R)$ is a L^p-viscosity solution ($p > p_0$) of the equation*

$$F(x, u, Du, D^2u) \geq f(x)$$

with $f \in L^p(\Omega_R)$, then for each $r \in (0, R)$ we have

$$\sup_{\Omega_r} u \leq u^+_{\partial\Omega} + \frac{C_0(1+R)^{\mu/2}R^\mu}{(R^2 - r^2)^\mu} + C\|f^-\|_{L^p(\Omega_R)} \tag{2.11}$$

with $\mu = 2/(d-1)$, $C_0 = C_0(n, \Lambda, b, d, \delta)$ and $C = C(n, p, \lambda, \Lambda, bR)$ are positive constants. Here

$$u^+_{\partial\Omega} = \begin{cases} \sup\limits_{B_R \cap \partial\Omega} u^+ & \text{if } B_R \cap \partial\Omega \neq \emptyset \\ 0 & \text{if } B_R \subset \Omega\,. \end{cases}$$

Proof. By (**SC**) we have

$$\begin{aligned}F(x,s,q,N) &= F(x,s,q,N) - F(x,s,0,0) + F(x,s,0,0)\\ &\le \mathcal{P}^+_{\lambda,\Lambda}(N) + b|q| - \delta s^d\end{aligned}$$

for all $(s,q,N) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^n$. Thus from Lemma 2.2.1 we deduce that Φ is a L^p-strong supersolution of the equation

$$F(x,\Phi,D\Phi,D^2\Phi) = 0 \quad \text{in } \Omega_R.$$

On the other hand u is a L^p-viscosity subsolution of the equation

$$F(x,u,Du,D^2u) = f(x) \quad \text{in } \Omega_R.$$

Hence by Lemma 2.2.2 the function $w = u - \Phi$ is a L^p-viscosity solution of the equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| \ge \mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| - \delta w^d \ge f(x)$$

in $A = \Omega_R \cap \{u > \Phi\}$.
Let $r \in (0,R)$ be such that $u^+(x) \le \Phi(x)$ as $x \in \Omega$ and $r \le |x| < R$, then $A \subset \Omega_r$ and $w \le 0$ on $\Omega \cap \partial B_r$. Therefore, applying GMP (1.57), we get

$$\sup_A w \le \sup_{B_r\cap\partial\Omega} w + C\|f^-\|_{L^p(\Omega_R)} \le u^+_{\partial\Omega} + C\|f^-\|_{L^p(\Omega_R)}$$

from which

$$u(x) \le u^+_{\partial\Omega} + \frac{C_R R^\mu}{(R^2-|x|^2)^\mu} + C\|f^-\|_{L^p(\Omega_R)}$$

for all $x \in \Omega_R$. □

Proposition 2.2.1. *Let Ω_R, F and f as in Lemma 2.2.3. If $u \in C(\overline{\Omega}_R)$ is a L^p-viscosity solution ($p > p_0$) of the equation*

$$F(x,u,Du,D^2u) = f(x),$$

then for each $r \in (0,R)$ one has

$$\sup_{\Omega_r} |u| \le |u|_{\partial\Omega} + \frac{C_0(1+R)^{\mu/2}R^\mu}{(R^2-r^2)^\mu} + C\|f\|_{L^p(\Omega_R)} \tag{2.12}$$

with C_0, C and $|u|_{\partial\Omega} = \max(u^+_{\partial\Omega}, u^-_{\partial\Omega})$ defined as in Lemma 2.2.3.

Proof. From Lemma 2.2.3 we already know that

$$u(x) \le u^+_{\partial\Omega} + \frac{C_R R^\mu}{(R^2-|x|^2)^\mu} + C\|f^-\|_{L^p(\Omega_R)} \quad \forall x \in \Omega_R.$$

The assert will be proved showing the same inequality for $-u$. To this end firstly observe that the function $v = -u$ satisfies the equation

$$G(x,v,Dv,D^2v) = -f(x) \ \text{ in } \Omega_R,$$

where

$$G(x, s, q, N) = -F(x, -s, -q, -N)$$

that turns out to satisfy (**SC**). Arguing as in Lemma 2.2.3 we conclude

$$-u(x) \leq u^-_{\partial\Omega} + \frac{C_R R^\mu}{(R^2 - |x|^2)^\mu} + C\|f^+\|_{L^p(\Omega_R)}$$

for all $x \in \Omega_R$. □

2.3 Existence results

In this section, using a stronger variant of the structure condition (**SC**) and the uniform estimates of Proposition 2.2.1, we construct an L^p-viscosity solution of the equation

$$F(x, u, Du, D^2u) = f(x)$$

in $\mathbb{R}^n$, assuming $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with $p > p_0$. Remember that $p_0 \in (\frac{n}{2}, n)$ is the exponent for which (GMP) holds true.

We will suppose that for all $R > 0$ there exists a function $\omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega_R(t) \to 0$ as $t \to 0^+$ and

$$|F(x, r, q, N) - F(x, s, q, N)| \leq \omega_R(|r - s|) \tag{2.13}$$

a.e. in x for $|r| + |s| + |q| + \|N\| \leq R$. We put

$$(\mathbf{SC})' := (\mathbf{SC}) \quad \text{and} \quad (2.13). \tag{2.14}$$

It is worth to recall that condition $(\mathbf{SC})'$ is equivalent to (**SC**) in the case that F is continuous.

Theorem 2.3.1. *Let $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ be measurable in x and satisfy the structure condition $(\mathbf{SC})'$ a.e. $x \in \mathbb{R}^n$. If $f \in L^p_{loc}(\mathbb{R}^n)$, then equation*

$$F(x, u, Du, D^2u) = f(x) \tag{2.15}$$

has an L^p-viscosity solution in $\mathbb{R}^n$ for any $p > p_0$.

Proof. In view of [20, Theorem 4.1 - Remark 4.8] we can solve in the L^p-viscosity sense any Dirichlet problem for the equation $F = f$ in the ball B_{2^k} with continuous boundary condition. Choose a solution u_k for each $k \in \mathbb{N}$. Using Proposition 2.2.1, we have for $h > k$

$$\sup_{B_{2^k}} |u_h| \leq C_0 + C\|f\|_{L^p(B_{2^{k+1}})},$$

where $C_0 = C_0(n, \Lambda, b, d, \delta)$ and $C = C(n, p, \lambda, \Lambda, b2^{k+1})$ are positive constants as defined in Lemma 2.2.3. By the structure condition $(\mathbf{SC})'$ we have

$$F(x, s, q, N) \leq \mathcal{P}^+_{\lambda,\Lambda}(N) + b|q| + F(x, -R, 0, 0) \leq \mathcal{P}^+_{\lambda,\Lambda}(N) + b|q| + \omega_R(R)$$

and

$$F(x,s,q,N) \geq \mathcal{P}^-_{\lambda,\Lambda}(N) - b|q| + F(x,R,0,0) \geq \mathcal{P}^-_{\lambda,\Lambda}(N) - b|q| - \omega_R(R)$$

a.e. $x \in \mathbb{R}^n$ for $|s| \leq R$. Therefore for $h > k$ we have

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2u_h) - b|Du_h| - \omega_R(R) \leq f \leq \mathcal{P}^+_{\lambda,\Lambda}(D^2u_h) + b|Du_h| + \omega_R(R)$$

in B_{2^k} with $R = C_0 + C\|f\|_{L^p(B_{2^{k+1}})}$. By C^α-estimates [8, Proposition 4.10] and [52, Theorem 2], we deduce that

$$\|u_h\|_{C^\alpha(B_{2^k})} \leq C_1 \left(1 + \|f\|_{L^p\left(B_{2^{k+1}}\right)}\right),$$

for a positive constant C_1 independent of $h > k$. By a diagonal process, using Ascoli-Arzelà theorem we extract a subsequence $h_k \in \mathbb{N}$ such that $u_{h_k} \to u \in C(\mathbb{R}^n)$ uniformly on every bounded domain. From the stability results for L^p-viscosity solutions (Theorem 1.3.1) u is a solution of the equation (2.15). □

The technique used in the Theorem 2.3.1, combined with a global Hölder continuity result due to Sirakov [52], allows us to solve the Dirichlet problem in any regular domain, even unbounded, of $\mathbb{R}^n$. For other results in unbounded domains see [38].

Theorem 2.3.2. *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain satisfying an uniform exterior cone condition and $F(x,s,q,N)$ be measurable in $x \in \mathbb{R}^n$ for all $(s,q,N) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ such that the structure condition $(\mathbf{SC})'$ holds a.e. $x \in \Omega$. Then for $p > p_0$ the Dirichlet problem*

$$\begin{cases} F(x,u,Du,D^2u) = f & \text{in } \Omega \\ u = \psi & \text{on } \partial\Omega \end{cases} \tag{2.16}$$

has an L^p-viscosity solution $u \in C(\overline{\Omega})$ for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ and every $\psi \in C(\partial\Omega)$.

Proof. For any $k \in \mathbb{N}$ let $u_k \in C(\overline{\Omega}_{2^k})$ be an L^p-viscosity solution in $\Omega_{2^k} = \Omega \cap B_{2^k}$ of the problem

$$\begin{cases} F(x,u,Du,D^2u) = f & \text{in } \Omega_{2^k} \\ u = \psi_k & \text{on } \partial\Omega_{2^k}, \end{cases} \tag{2.17}$$

where ψ_k is a continuous extension to $\mathbb{R}^n$ of $\psi|_{\partial\Omega \cap \overline{B}_{2^k}}$, see [28, Theorem 1-Section 1.2]. Since Ω_{2^k} satisfies the uniform exterior cone condition, we remark that the solvability of (2.17) is a consequence of the already mentioned [20, Theorem 4.1]. Let $R = 2^k$, by Proposition 2.2.1 for $h > k$ we get

$$\sup_{\Omega_R} |u_h| \leq \max_{\partial\Omega \cap \overline{B}_R} |\psi| + C_0 + C\|f\|_{L^p(\Omega_{2R})}.$$

The argument of the proof of Lemma 2.2.3 leads to inequality and therefore the u_h are equibounded in Ω_R. As a consequence, by C^α-estimates they are equi-Hölder continuous in every subset

$$\{x \in \Omega_R \mid \text{dist}\,(x, \partial\Omega) > \varepsilon\}$$

with $\varepsilon > 0$. By [52, Theorem 2]

$$\underset{\Omega_R \cap B_\rho(x)}{\text{osc}} u_h \leq K\,(\rho^{\alpha_k} + \underset{\partial\Omega_R \cap B_{\sqrt{\rho}}(x)}{\text{osc}} \psi_h)$$

for every $x \in \partial\Omega$ and $\rho \leq \rho_k$, and therefore the u_h are also equicontinuous in $\overline{\Omega}_R$. Thus, using a diagonal procedure as in the proof of Theorem 2.3.1 we find a L^p-viscosity solution $u \in C(\overline{\Omega})$ of the Dirichlet problem under consideration. □

2.4 Maximum principle

From the uniform estimates of Section 2.2 we get at once the following Maximum Principle.

Theorem 2.4.1 (**Maximum Principle**). *Let $\delta > 0$, $d > 1$ and Ω be a domain of $\mathbb{R}^n$. Suppose for a.e. $x \in \Omega$ that*

$$F(x, s, q, N) \leq \mathcal{P}^+_{\lambda,\Lambda}(N) + b|q| - \delta|s|^{d-1}s \tag{2.18}$$

for all $(s, q, N) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and $u \in C(\overline{\Omega})$ is a L^p-viscosity solution ($p > p_0$) of the equation

$$F(x, u, Du, D^2u) \geq 0 \quad \text{in } \Omega.$$

M1) *If $\Omega = \mathbb{R}^n$, then $u \leq 0$ in $\mathbb{R}^n$.*
M2) *If $\Omega \subsetneq \mathbb{R}^n$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω.*

Proof. Let $x \in \Omega$ and $r = |x|$. Since

$$F(x, s, q, N) = \mathcal{P}^+_{\lambda,\Lambda}(X) + b|q| - \delta|s|^{d-1}s$$

satisfies (**SC**), we can apply Lemma 2.2.3. Letting $R \to +\infty$ in (2.11) with $f = 0$, we get $u(x) \leq 0$ as asserted. □

Moreover considering the function $u(x) = -v(x)$ and the operator $G(x, s, q, N) = -F(x, -s, -q, -N)$, as in the proof of Proposition 2.2.1, we obtain a Minimum Principle.

Theorem 2.4.2 (**Minimum Principle**). *Let $\delta > 0$, $d > 1$ and Ω be a domain of $\mathbb{R}^n$. Suppose for a.e. $x \in \Omega$ that*

$$F(x, s, q, N) \geq \mathcal{P}^-_{\lambda,\Lambda}(N) - b|q| - \delta|s|^{d-1}s \tag{2.19}$$

for all $(s, q, N) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and $u \in C(\overline{\Omega})$ is a L^p-viscosity solution ($p > p_0$) of the equation

$$F(x, u, Du, D^2u) \leq 0 \quad \text{in } \Omega.$$

M1) *If $\Omega = \mathbb{R}^n$, then $u \geq 0$ in $\mathbb{R}^n$.*
M2) *If $\Omega \subsetneq \mathbb{R}^n$ and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.*

Remark 2.4.1. The above implies that, assuming (**SC**), the function $u = 0$ is the unique viscosity solution of the problem $F = 0$ in $\mathbb{R}^n$.

2.5 Uniqueness

The issue of this Section is to prove uniqueness of entire solutions of

$$F(x, u, Du, D^2u) = f(x). \tag{2.20}$$

The existence results we present follows from Theorem 2.3.1 and the relationship between C-L^p-strong solutions, see Section 1.3. In the sequel we will focus our attention, giving detailed proofs, only on the uniqueness part. As we will see the key point is to use suitable assumptions on F and f to have a homogeneous maximal equation for the difference between two solutions of (2.20) and then get at once the uniqueness using the Maximum Principle of Theorem 2.4.1.
We separate the C-viscosity case from the L^p one.

2.5.1 C-Viscosity case

We establish a first result in the case of F independent of x.

Theorem 2.5.1. *Let $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ be a continuous function satisfying the structure condition* (**SC**) *such that*

$$F(x, s, q, N) = \overline{F}(s, q, N) \tag{2.21}$$

for all $(x, s, q, N) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$. If $f \in C(\mathbb{R}^n)$, there exists a unique C-viscosity solution of equation

$$\overline{F}(u, Du, D^2u) = f(x) \quad in \quad \mathbb{R}^n.$$

Proof. Let u and v be solutions of the equation $F = f$. Set $\Omega = \{x \in \mathbb{R}^n \mid w = u - v > 0\}$. We claim that $\Omega = \emptyset$, so that $u \le v$ in $\mathbb{R}^n$.
Suppose on the contrary $\Omega \neq \emptyset$. Since F is continuous, in view of Theorem 1.2.9 and Remark 1.2.13 (see [14]), and observing that u, v are $C^{1,\alpha}_{\text{loc}}$, we can use the structure condition (**SC**) to have

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| - \delta w^s \ge 0 \tag{2.22}$$

in Ω. Using the Maximum Principle of Theorem 2.4.1 (M2), we should have $w \le 0$ in Ω, a contradiction which proves our claim. Interchanging the role of u and v, we also get $v \le u$ in $\mathbb{R}^n$, and we are done. □

To consider a dependence on x, we need to control the oscillations in the x-variable , and this also requires a uniform bound of the local L^p-norms of f.

Theorem 2.5.2. *Let $F \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$ satisfy the structure condition* (**SC**)*. Suppose also that for all $R > 0$ there exist a constant $K_R > 0$ and a function $\omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to 0^+} \omega_R(t) = 0$ and*

$$|F(y, s, q, N) - F(x, s, q, N)| \le K_R\|N\|\,|y - x| + \omega_R((1 + |q|)|y - x|) \tag{2.23}$$

as $x, y \in \mathbb{R}^n$, $s \in (-R, R)$ *and* $(q, N) \in \mathbb{R}^n \times \mathcal{S}^n$. *If* $p > p_0$, $f \in C(\mathbb{R}^n)$ *and*

$$\|f\|_{M^p} := \sup_{x\in\mathbb{R}^n} \|f\|_{L^p(B_1(x))} < +\infty\,, \tag{2.24}$$

then equation (2.20) *has a unique* C*-viscosity solution.*

Proof. By assumption $f \in L^p_{loc}(\mathbb{R}^n)$ and $\|f\|_{M^p} < +\infty$, then Proposition 2.2.1 implies that u is bounded. In fact, if $x_0 \in \mathbb{R}^n$ and we consider balls centered at x_0, choosing $r \to 0^+$ and $R = 1$ in (2.12), we get

$$|u(x_0)| \le 2^{\mu/2}C_0 + C(n, p, \lambda, \Lambda, b)\|f\|_{M^p}, \tag{2.25}$$

which is finite and independent of x_0, by (2.24). Thus, if u and v are solutions of the equation $F = f$, by (**SC**) and (2.23) we can use Proposition 2.1 of [23] and thus the difference $w = u - v$ satisfies a maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + c|Dw| - \delta w^d \ge 0 \tag{2.26}$$

in $\Omega = \{x \in \mathbb{R}^n \mid u > v\}$ for some positive constant c depending on n, p, λ, Λ, b and d. Therefore we can conclude as in the proof of Theorem 2.5.1 . □

Using the $W^{2,p}$ regularity results of Caffarelli [7]-[8] (see also [26]) the entire solutions of (2.20) belong to $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$, so they are strong solutions. As a consequence the structure conditions can be used "pointwise" to obtain a maximal equation for the difference of two solutions.

We will assume that for every $R > 0$ there exists $c_R > 0$ such that

$$\mathcal{P}^-_{\lambda,\Lambda}(M-N) - b|p-q| - c_R|r-s| \le F(x, r, p, M) - F(x, s, q, N) \le \mathcal{P}^+_{\lambda,\Lambda}(M-N) + b|p-q| + c_R|r-s| \tag{2.27}$$

for $x \in \mathbb{R}^n$, $r, s \in (-R, R)$, $p, q \in \mathbb{R}^n$, $M, N \in \mathcal{S}^n$. We put

$$(\overline{\mathbf{SC}}) := (2.27) - (2.6) - (2.5)$$

Definition 2.5.1. *We say that* F *has* $C^{1,1}$*-estimates at* x_0 *(with constant* C*) if for all* $w_0 \in C^0(\partial B_{r_0}(x_0))$ *there exists a solution* $w \in C^2(B_{r_0}(x_0)) \cap C^0(\overline{B}_{r_0}(x_0))$ *of the Dirichlet problem*

$$\begin{cases} F(x_0, 0, 0, D^2w) = 0 & \text{in} \quad B_{r_0}(x_0) \\ \qquad\qquad\quad w = w_0 & \text{on} \quad \partial B_{r_0}(x_0) \end{cases}$$

such that

$$\|w\|_{C^{1,1}(\overline{B}_{r_0/2}(x_0))} \le Cr_0^{-2}\|w_0\|_{L^\infty(\partial B_{r_0}(x_0))}$$

for some $r_0 > 0$.

Finally, let

$$\beta_F(x, x_0) := \sup_{\substack{X\in\mathcal{S}^n \\ X\neq 0}} \frac{|F(x, 0, 0, X) - F(x_0, 0, 0, X)|}{\|X\|}\,. \tag{2.28}$$

Theorem 2.5.3. *Let $F \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$ satisfying the structure condition $(\overline{\mathbf{SC}})$ such that the $C^{1,1}$-estimates holds true for each $x \in \mathbb{R}^n$ with some $r_0 > 0$. If $f \in C(\mathbb{R}^n)$ and*

$$\sup_{r\in(0,r_0)} \left(\fint_{B_r(x)} |\beta_F(x,y)|^n \, dy \right)^{1/n} \leq \theta, \tag{2.29}$$

for every $x \in \mathbb{R}^n$, with $\theta = \theta(n, p, \lambda, \Lambda, r_0)$, then equation (2.20) *has a unique L^p-strong solution $u \in W^{2,p}_{\rm loc}(\mathbb{R}^n)$, provided that $p > p_0$.*

Proof. In view of [8, Theorem 7.1] and [26, Theorem 1] solutions of $F = f$ are in $W^{2,p}_{\rm loc}(\mathbb{R}^n)$ and hence they are strong solutions. If u, v are such solutions, the difference is a solution a.e. of (2.22) in $\{u > v\}$, from which we conclude again using the Maximum Principle of Theorem 2.4.1 (M2). □

2.5.2 L^p-Viscosity case

By virtue of the results of Winter [56, Theorem 4.2] (see also [53]), the argument of Theorem 2.5.3 can be generalized to the case of F merely measurable in the variable x provided F is convex in the matrix variable. The proof of this result is similar to the previous one, so we omit it.

Theorem 2.5.4. *Let $F(x, \cdot, \cdot, \cdot)$ be a measurable function satisfying a.e. $(\overline{\mathbf{SC}})$. Assume $F(\cdot, \cdot, \cdot, M)$ convex and $f \in L^p_{\rm loc}(\mathbb{R}^n)$. If*

$$\sup_{r\in(0,r_0)} \left(\fint_{B_r(x)} |\beta_F(x,y)|^n \, dy \right)^{1/n} \leq \theta,$$

for every $x \in \mathbb{R}^n$, with $\theta = \theta(n, p, \lambda, \Lambda, r_0)$, then equation (2.20) *has a unique L^p-strong solution $u \in W^{2,p}_{\rm loc}(\mathbb{R}^n)$, provided that $p > p_0$.*

By our assumptions, the L^p-strong solution of Theorem 2.5.4 will be also the unique L^p-viscosity solution. Theorem 2.5.4 can be used for instance in the case of Bellman type equations

$$\sup_\alpha \{L_\alpha u - f_\alpha(x)\} = 0$$

where L_α is a semilinear second order operator

$$L_\alpha u := a^\alpha_{ij}(x) D_{ij} u + b^\alpha_i(x) D_i u + c_1(x) u + c_2(x)|u|^{d-1} u$$

with bounded measurable coefficients such that

$$\lambda \leq \sup_{|\xi|\leq 1} \langle a^\alpha_{ij}(x)\xi, \xi\rangle \leq \Lambda, \quad |b^\alpha_i(x) b^\alpha_i(x)|^{1/2} \leq b, \quad c_1(x) \leq 0, \quad c_2(x) \leq -\tilde{\delta} < 0$$

for a.e. $x \in \mathbb{R}^n$ and every α, provided the a^α_{ij} are uniformly continuous in $\mathbb{R}^n$ with continuity modulus independent of α and $\inf_\alpha f_\alpha \in L^p_{loc}(\mathbb{R}^n)$.

Some cases of Isaacs type equations can be treated with Theorem 2.5.4, as for instance, see [5], the minimum

$$\min\{\inf_\alpha (L_\alpha u - f_\alpha(x)), \sup_\beta (L_\beta u - f_\beta(x))\} = 0$$

between concave and convex operators, which are realized as infimum and supremum, respectively, of two families of semilinear operators, indexed by α and β, with the above conditions.

2.6 Non existence

With arguments similar to those of previous Sections we can prove the non existence of entire solutions for homogeneous equations with exponential decaying in u, namely

$$F(x, Du, D^2u) - e^u = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{2.30}$$

We assume, as usually, $F \in \mathcal{F}_{\lambda,\Lambda,b}$.

Lemma 2.6.1. *The function $\Phi \in C^2(B_R)$*

$$\Phi(x) = \log \frac{A}{(R^2 - |x|^2)^2}, \quad A = 4R^2\left(bR + \Lambda(n+1)\right), \tag{2.31}$$

is a supersolution of the maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi) + b|D\Phi| - e^{\Phi} = 0 \quad \text{in} \quad B_R.$$

Proof. Set $r = |x|$, then

$$\begin{aligned} \mathcal{P}^+_{\lambda,\Lambda}(D^2\Phi(x)) + b|D\Phi(x)| - e^{\Phi(x)} &= \Lambda\left(4\frac{R^2 + r^2}{(R^2 - r^2)^2} + 4\frac{n-1}{R^2 - r^2}\right) + 4b\frac{r}{R^2 - r^2} - \frac{A}{(R^2 - r^2)^2} \\ &\le \frac{1}{(R^2 - r^2)^2}\left(4R^2\left(bR + \Lambda(n+1)\right) - A\right) = 0 \end{aligned}$$

for A as in (2.31) □

Reasoning as in Lemma 2.2.3 we obtain the following

Lemma 2.6.2. *Let $u \in USC(\overline{B}_R)$ be a C-viscosity subsolution of (2.30) in B_R. For every $r < R$*

$$\sup_{B_r} u \le \log \frac{A}{(R^2 - r^2)^2}. \tag{2.32}$$

Proposition 2.6.1. *The equation (2.30) has no entire C-viscosity subsolution.*

Proof. If a such subsolution u exists, for any $x \in \mathbb{R}$, putting $r = |x|$, we obtain

$$u(x) \le \log \frac{A}{(R^2 - r^2)^2} \quad \forall R > 0$$

because of (2.32). Letting $R \to +\infty$ we get the contradiction $u \equiv -\infty$. □

Remark 2.6.1. Following [49] we can bound the radius R of B_R in which a subsolution u of (2.30), bounded from above, exists. In fact, by comparison, $u(0) \le \log \frac{A}{R^4}$ and

$$R \le \left(\frac{A}{e^{u(0)}}\right)^{\frac{1}{4}}.$$

Chapter 3

Blow-up solutions

This chapter is devoted to the existence and the uniqueness of solutions of fully nonlinear second order elliptic equations

$$F(x, u, Du, D^2u) = f(x) \tag{3.1}$$

in a domain Ω, subject to the boundary condition

$$u(x) \to +\infty \quad \text{as} \quad \text{dist}(x, \partial\Omega) \to 0. \tag{3.2}$$

Solutions of (3.1)-(3.2) are called *blow-up* or *large* solutions. These problems have been intensively studied by many authors in connection with several areas of Mathematics and Physics, as conformal and Riemannian geometry, Probability theory, Marked process (superdiffusion). We refer to [24]-[25]-[41]-[47]-[48]-[49] and the bibliography therein for a nice history of the problem.
Using a purely analytical method Marcus and Véron [47] showed the uniqueness of positive large solutions for the equation $\Delta u = u^d$, $d > 1$, in domains with a very general boundary condition, called "local graph property "(see after for the definition). Not having information about the regularity of the boundary $\partial\Omega$, the key point is to prove that for any two solutions u_1, u_2 the ratio $u_1/u_2 \to 1$ as $\text{dist}(x, \partial\Omega) \to 0$.
A different approach, based on an iteration technique due to Safonov, is considered in [25] for classical and strong solutions of semilinear equations $Lu = u^d$ (or $Lu = e^u$) in domains satisfying the "uniform exterior ball condition". There L is a linear second order elliptic operator.
Regarding the existence we present in Section 3.1 a result based on interior estimates of Chapter 2, which provide the local uniform convergence of approximating solutions.
In the following Sections we propose to extend the uniqueness results of Marcus and Véron [47]-[48] for the more general problems (3.1)-(3.2), than the Laplacian, in the larger class of C-viscosity solutions. Our conclusions works for F independent of x.

We will assume the structure condition (**SC**) of previous Chapter, namely $F \in \mathcal{F}_{\lambda,\Lambda,b}$ with the superlinear monotonicity assumption

$$F(x, r, p, M) - F(x, s, p, M) \leq -\delta(r - s)^d \quad \text{if } r > s, \tag{3.3}$$

where $d > 1$ and $\delta > 0$, for all $(x, p, M) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n$.

3.1 Existence result

Theorem 3.1.1. *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain satisfying an uniform exterior cone condition and suppose that at least one of the assumption blocks on F and f of Theorems 2.5.1-2.5.2-2.5.3 holds true in Ω. Then the boundary blow-up problem*

$$\begin{cases} F(x,u,Du,D^2u) = f(x) & \text{in } \Omega \\ \lim\limits_{x\to\partial\Omega} u(x) = +\infty \end{cases} \tag{3.4}$$

has a C-viscosity solution for every $f \in C(\Omega)$.

Proof. For any $k \in \mathbb{N}$, let u_k be a solution of the Dirichlet problem (see Theorem 2.3.2)

$$\begin{cases} F(x,u_k,Du_k,D^2u_k) = f(x) & \text{in } \Omega \\ u_k = k & \text{on } \partial\Omega. \end{cases}$$

The sequence $(u_k)_{k\in\mathbb{N}}$ is nondecreasing, otherwise $w = u_k - u_{k+1}$ would be a solution of

$$\begin{cases} \mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| - \delta w^d \geq 0 & \text{in } \Omega \cap \{w > 0\} \\ w \leq 0 & \text{on } \partial(\Omega \cap \{w > 0\}). \end{cases}$$

and, in view of the Maximum Principle of Theorem 2.4.1, we get the contradiction $w \leq 0$. By using Proposition 2.2.1 and arguing as in Theorem 2.3.1 we deduce that $(u_k)_{k\in\mathbb{N}}$ is bounded and equicontinuous on compact sets of Ω. Thus $(u_k)_{k\in\mathbb{N}}$ converges we to a C-viscosity solution u of

$$F(x,u,Du,D^2u) = f(x) \quad \text{in } \Omega.$$

Being $(u_k)_k$ is nondecreasing, for $y \in \partial\Omega$ we have

$$\liminf_{x\to y} u(x) \geq \lim_{x\to y} u_k(x) = k$$

for any $k \in \mathbb{N}$, whence the assertion follows. □

Remark 3.1.1. The existence of blow-up solutions fails to hold if the domain is not sufficiently regular. In fact assuming $F = \overline{F}(M)$ uniformly elliptic operator such that

$$1 \leq \frac{\Lambda}{\lambda} < n-1 \quad \text{and} \quad d > \frac{(n-1)\lambda + \Lambda}{(n-1)\lambda - \Lambda},$$

Labutin [44] showed that the origin is a removable singularity for the equation

$$\overline{F}(D^2u) - |u|^{d-1}u = 0,$$

that is every viscosity solution in the punctured ball $B_R\backslash\{0\}$ can be continued to a viscosity solution in B_R.

Remark 3.1.2. Theorem 3.1.1 remains valid in the L^p-viscosity framework, see [29, Theorem 1.6].

3.2 Uniqueness results

As mentioned above, the results concerning the uniqueness of large solutions we present, work for operators x-independent. So we refer in the sequel to the equations

$$F(u, Du, D^2u) = f(x) \quad \text{in}\, \Omega. \tag{3.5}$$

We start proving two lemmas.

Lemma 3.2.1. *Let F be an operator satisfying **(SC)** and $\tau \in [0,1]$. If u is a supersolution of (3.5) and $w \geq 0$ is a solution of $\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| - \tau\delta w^d = 0$ then the function $u + w$ is in turn a supersolution of (3.5).*

Proof. By regularity results for convex operators (see [5, Corollary 1.3] and [8, Sections 6.2, 8.1]) we have $w \in C^{2,\gamma}$, with $0 < \gamma < 1$, so w is a classical solution.
Let φ be a test function touching $u + w$ at x from below. Then $\varphi - w$ touches u at x from below and

$$\begin{aligned} F(\varphi(x), D\varphi(x), D^2\varphi(x)) &\leq \mathcal{P}^+_{\lambda,\Lambda}(D^2w(x)) + b|Dw(x)| \\ &\quad + F(\varphi(x), D(\varphi - w)(x), D^2(\varphi - w)(x)) \\ &\leq \mathcal{P}^+_{\lambda,\Lambda}(D^2w(x)) + b|Dw(x)| - \delta(w(x))^d \\ &\quad + F(\varphi(x) - w(x), D(\varphi - w)(x), D^2(\varphi - w)(x)) \\ &\leq f(x). \end{aligned}$$

□

As in the previous Chapters $p_0 \in \left(\frac{n}{2}, n\right)$ is the exponent for which (GMP) holds true (see (1.57)). It will be useful later the following Generalized Comparison Principle (GCP), which is deduced by the uniform estimate of Lemma 2.2.3.

Lemma 3.2.2. *Let Ω be a domain of $\mathbb{R}^n$ and F be an operator satisfying **(SC)**. Suppose that u and v are continuous solutions, respectively, of $F \geq f$ and $F \leq g$ in viscosity sense, where $f, g \in C(\Omega) \cap L^p_{loc}(\mathbb{R}^n)$ for some $p > p_0$. Then for any $y \in \Omega$ and any ball B_R centered at y we have*

$$(u - v)^+(y) \leq \limsup_{x \to \partial\Omega \cap B_R} (u - v)^+(x) + C_0 \left(\frac{1 + bR}{R^2}\right)^{\frac{1}{d-1}} + C_1 \|(f - g)^-\|_{L^p(\Omega \cap B_R)}, \tag{3.6}$$

where $C_0 = C_0(n, \Lambda, d, \delta)$ and $C = C_1(n, p, \lambda, \Lambda, bR)$ are positive constants.
Here, if $\partial\Omega \cap B_R = \emptyset$, one reads $\limsup_{x \to \partial\Omega}(u - v)^+(x) = 0$.

Proof. Since F is independent of x, by means of the Jensen's approximations, we may use the structure conditions (SC) just as for smooth functions (Theorem 1.2.9 and Remark 1.2.13, see also [14]), to deduce that $w = (u - v)^+$ is a viscosity subsolution of

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| - \delta w^s = -(f(x) - g(x))^- \quad \text{in} \quad \Omega.$$

From this, reasoning as in Lemma 2.2.3 and using GMP (1.57), for any ball B_r centered at y of radius $r < R$ we get

$$\sup_{\Omega\cap B_r} w \le \limsup_{x\to\partial\Omega\cap B_R} w(x) + C_0 \left(\frac{R(1+bR)^{\frac{1}{2}}}{R^2-r^2}\right)^{\frac{2}{d-1}} + C_1\|(f-g)^-\|_{L^p(\Omega\cap B_R)}, \tag{3.7}$$

from which (3.6) follows, letting $r \to 0^+$. □

Remark 3.2.1. If $f = g$, letting $R \to \infty$, from Lemma 3.2.2 we obtain the Comparison Principle (CP):

$$(u-v)^+(y) \le \limsup_{x\to\partial\Omega}(u-v)^+(x). \tag{3.8}$$

Note also that Ω is possibly unbounded in Lemma 3.2.2. Nonetheless no assumption is made on the growth of u and v at infinity.

Definition 3.2.1. (Marcus-Véron [47]-[48]) A domain Ω satisfies the *local graph property* at $P \in \partial\Omega$ if there exist a neighborhood Q_P and a function $\psi = \psi \in C(\mathbb{R}^{n-1})$ such that

$$Q_P \cap \Omega = \{x \in Q_P : y_n < \psi(y')\}$$

in a coordinate system $y \equiv (y', y_n)$ obtained by rotation from $x \equiv (x', x_n)$.

Remark 3.2.2. We may assume that Q_P is a spherical cylinder

$$Q_P = \{x \in \mathbb{R}^n\ ,\ |y'| < \rho,\ |y_n| < \sigma\} \tag{3.9}$$

centered at P, of radius $\rho > 0$ and finite height $2\sigma > 0$, as well as $|\psi(y')| < \sigma$ in $\overline{Q}_P$ so that

$$\overline{Q}_P \cap \Omega = \{x \in \mathbb{R}^n\ ,\ |y'| \le \rho,\ -\sigma \le y_n < \psi(y')\}. \tag{3.10}$$

Here $x = Ry + x(P)$ for an orthogonal matrix R (i.e. $R^{-1} = R^T$). As in [47], the class of domains satisfying the local graph property at every $P \in \partial\Omega$ will be denoted by C_{gr}.

Let us consider a C-viscosity solution $w_P \equiv w \in C(Q_P)$ of the boundary blow-up problem

$$\begin{cases} \mathcal{P}^+_{\lambda,\Lambda}(D^2w) + \gamma|Dw| - \delta w^s = 0 & \text{in } Q_P \\ \qquad w(x) \to +\infty & \text{as } \operatorname{dist}(x, \partial Q_P) \to 0 \end{cases} \tag{3.11}$$

provided by Theorem 3.1.1 ([29, Theorem 1.6]). Actually, by [5, Corollary 1.3], $w \in C^{2,\gamma}(Q_P)$ for $\gamma \in (0,1)$.
The main tool to show the uniqueness will be the comparison principle (3.6).

Proposition 3.2.1. *Let Ω be a domain of $\mathbb{R}^n$ satisfying the local graph property at $x_P \in \partial\Omega$, and Q_P the cylinder of Remark 3.2.2. Assume that $F : \mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n \mapsto \mathbb{R}$ satisfies the structure conditions (**SC**). If there exists a viscosity subsolution $u \in C(\overline{Q}_P \cap \Omega)$ of (3.5) such that*

$$u(x) \to +\infty \ \ \textit{locally uniformly} \ \ \textit{as } x \to \Gamma_1 \equiv Q_P \cap \partial\Omega, \tag{3.12}$$

then problem

$$F(v, Dv, D^2v) = f(x) \quad \text{in } Q_P \cap \Omega \tag{3.13}$$

$$v(x) \to +\infty \text{ locally uniformly as } x \to \Gamma_1 \tag{3.14}$$

$$v = 0 \quad \text{on } \Gamma_2 \equiv \partial Q_P \cap \Omega, \tag{3.15}$$

has a viscosity solution $v \in C(\overline{Q}_P \cap \Omega)$ *for every* $f \in C(\Omega) \cap L^p_{loc}(\mathbb{R}^n)$, $p > p_0$.

Remark 3.2.3. Following [47], by condition (3.14) we mean $v(x) \to +\infty$ as $\text{dist}(x, A) \to 0$ for every $A \Subset \Gamma_1$ in the relative topology.

Proof. Following [47], with the notations of (3.9) consider an approximation from below of

$$\Theta \equiv Q_P \cap \Omega = \left\{x \in \mathbb{R}^n : |y'| < \rho, -\sigma < y_n < \psi(y')\right\},$$

where $x = Ry + x(P)$ and $R^{-1} = R^T$, assuming $\psi > 0$ as we may, using a monotone increasing sequence of smooth positive functions $\psi_j \to \psi$ as $j \to +\infty$.
Correspondingly, let

$$\begin{aligned} \Theta_j &= \left\{x \in \mathbb{R}^n : |y'| < \rho,\ -\sigma < y_n < \psi_j(y')\right\} \\ \Gamma_{1j} &= \left\{x \in \mathbb{R}^n : |y'| < \rho,\ y_n = \psi_j(y')\right\} \\ \Gamma_{2j} &= \left\{x \in \mathbb{R}^n : |y'| = \rho,\ -\sigma \le y_n < \psi_j(y')\right\} \\ &\quad \cup \left\{x \in \mathbb{R}^n : |y'| < \rho,\ y_n = -\sigma\right\} \end{aligned}$$

Let also $\Gamma_{2j} = \Gamma'_{2j} \cup \Gamma''_{2j}$ where

$$\begin{aligned} \Gamma'_{2j} &= \{x \in \Gamma_{2j} :\ |y'| = \rho,\ \psi_j(y') - \tfrac{1}{j} \le y_n < \psi_j(y')\} \\ \Gamma''_{2j} &= \{x \in \Gamma_{2j} :\ |y'| = \rho,\ -\sigma \le y_n \le \psi_j(y') - \tfrac{1}{j}\} \\ &\quad \cup \left\{x \in \Gamma_{2j} : |y'| < \rho,\ y_n = -\sigma\right\} \end{aligned}$$

By [19, Theorem 4.1] we can find a continuous viscosity solution of the Dirichlet problem

$$\begin{cases} F(v_{j,k}(x), Dv_{j,k}(x), D^2v_{j,k}(x)) = f(x) & \text{in } \Theta_j \\ v_{j,k}(x) = k & \text{in } \Gamma_{1j} \\ v_{j,k}(x) = j\left(y_n - \psi_j(y') + \frac{1}{j}\right)k & \text{on } \Gamma'_{2j} \\ v_{j,k}(x) = 0 & \text{on } \Gamma''_{2j}. \end{cases}$$

Here we are using the same boundary conditions of [47, Theorem 2.2]. Then by construction for any fixed $j \in \mathbb{N}$ the sequence $(v_{j,k})_{k\in\mathbb{N}}$ is increasing, with respect to $k \in \mathbb{N}$, on $\partial\Theta_j$ and so, by the comparison principle, is also increasing in Θ_j. On the other side, from Proposition 2.2.1 we have uniform boundedness in compact sets K of Θ_j, say

$$\sup_K |v_{j,k}| \le C$$

for a positive constant $C = C(n, \lambda, \Lambda, p, \delta, K)$. Moreover by (**SC**)

$$\begin{aligned} F(v_{j,k}, q, N) &\le \mathcal{P}^+_{\lambda,\Lambda}(N) + b|q| + F(v_{j,k}, 0, 0) \\ &\le \mathcal{P}^+_{\lambda,\Lambda}(N) + b|q| + \max_{|t|\le C} |F(t, 0, 0)|, \\ F(v_{j,k}, q, N) &\ge \mathcal{P}^-_{\lambda,\Lambda}(N) - b|q| + F(v_{j,k}, 0, 0) \\ &\ge \mathcal{P}^-_{\lambda,\Lambda}(N) - b|q| - \max_{|t|\le C} |F(t, 0, 0)|. \end{aligned}$$

By using Hölder estimates (see [8]-[52]), Ascoli-Arzelá theorem and stability results for viscosity solutions, we deduce that

$$v_{j,\infty} = \lim_{k\to+\infty} v_{j,k}$$

is a solution of (3.13) in Θ_j.
Next consider the sequence $(v_{j,\infty})_{j\in\mathbb{N}}$. Since $v_{j+1,k} \le v_{j,k}$ on $\partial\Theta_j$, we have $v_{j+1,\infty} \le v_{j,\infty}$ on $\partial\Theta_j$ so that, again by the comparison principle, the sequence $(v_{j,\infty})_{j\in\mathbb{N}}$ is monotone decreasing in Θ_j and, by reasoning as before to show that $v_{j,\infty}$ are solutions, in turn converges locally uniformly to a solution v of (3.13) in Θ.
It is easy to check that $v = 0$ on Γ_2, which is regular enough in order that the boundary condition is satisfied with continuity, see [20]. In order to prove (3.14), let us observe that for all $k \in \mathbb{N}$

$$v_{j,\infty} \ge v_{j,k} = k \quad \text{on} \quad \Gamma_{1j} \, .$$

Since u is bounded on $\partial\Theta_j$, then $u \le v_{j,\infty}$ on Γ_{1j}, as well as $u \le w$ on Γ_{2j}, by (3.11). Moreover from Lemma (1.2.3) the function $v_{j,\infty} + w$ is a supersolution of (3.13) in Θ_j and hence by the comparison principle

$$u \le v_{j,\infty} + w \quad \text{in } \Theta_j.$$

Passing to the limit as $j \to \infty$ we obtain

$$u \le v + w \tag{3.16}$$

in Θ, from which condition (3.14) follows. □

Using the above Lemma, we can bound the difference between solutions which blow up on the upper boundary Γ_1.

Corollary 3.2.1. *Suppose that the assumptions of Proposition 3.2.1 are satisfied for positive functions $u = u_i \in C(\overline{Q}_P \cap \Omega)$, $i = 1, 2$. Let $Q^*_P \Subset Q_P$ be a spherical cylinder centered at P. Then there exists a positive constant C such that*

$$|u_2 - u_1| \le C \quad \textit{in} \;\; Q^*_P \cap \Omega. \tag{3.17}$$

Proof. In view of (3.16) we have $u_i \le v + w$ in $Q_P \cap \Omega$, for $i = 1, 2$, where, up to a rotation, we may suppose the axis of the cylinder Q_P parallel to x_n, see (3.10). Since w is bounded in Q^*_P, we get then

$$u_i \le v + C \quad \text{in} \;\; Q^*_P \cap \Omega \tag{3.18}$$

with $C = \sup_{Q_P^*} w$. On the other side, consider $v_h(x', x_n) = v(x', x_n - h)$ for sufficiently small $h > 0$: v_h is continuous in $\overline{Q_P^h \cap \Omega}$ and $v_h = 0$ on Γ_2^h. Here Q_P^h and Γ_i^h, $i = 1, 2$, result from the corresponding sets Q_P and Γ_i moved up by h along the axis of Q_P. Then

$$v_h \leq u_i \quad \text{on} \quad \partial(Q_P^h \cap \Omega) \tag{3.19}$$

Since F is independent of x, the function v_h satisfies the equation

$$F(v_h(x), Dv_h(x), D^2 v_h(x)) = f_h(x) \quad \text{in} \quad Q_P^h \cap \Omega. \tag{3.20}$$

where $f_h(x', x_n) = f(x', x_n - h)$.
Therefore, fixing $y \in Q_P \cap \Omega$, choosing $h > 0$ small enough in order that $y \in Q_P^h \cap \Omega$ and applying Lemma 3.2.2 in $Q_P^h \cap \Omega$, for any $R > 0$ we get

$$\begin{aligned}(v_h - u_i)^+(y) &\leq \limsup_{x \to \partial(Q_P^h \cap \Omega) \cap B_R} (v_h - u_i)^+ + C_0 \left(\frac{1 + bR}{R^2}\right)^{\frac{1}{d-1}} + C_1 \|(f - f_h)^-\|_{L^p(Q_P^h \cap \Omega \cap B_R)} \\ &\leq C_0 \left(\frac{1 + bR}{R^2}\right)^{\frac{1}{d-1}} + C_1 \|(f - f_h)^-\|_{L^p(\Omega \cap B_R)}.\end{aligned}$$

Letting $h \to 0^+$, we have

$$(v - u_i)^+(y) \leq C_0 \left(\frac{1 + bR}{R^2}\right)^{\frac{1}{d-1}}$$

and as $R \to \infty$, since $y \in Q_P \cap \Omega$ is arbitrary,

$$v \leq u_i \quad \text{in} \quad Q_P \cap \Omega. \tag{3.21}$$

From (3.18) and (3.21) the result follows. □

Now we are in position to prove the following uniqueness Theorem.

Theorem 3.2.1. *Let Ω be a domain of class C_{gr}, F be an operator satisfying the structure conditions (**SC**) and $f \in C(\Omega) \cap L^p_{loc}(\mathbb{R}^n)$. In addition, we suppose that*

$$\liminf_{k \to 1^+} \left(F(s, q, N) - \varphi(k) F\left(\frac{s}{k}, \frac{q}{k}, \frac{N}{k}\right) \right) \leq 0 \tag{3.22}$$

uniformly with respect to $(s, q, N) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^n$, where $1 \leq \varphi(k) \to 1$ as $k \to 1^+$. Then problem (3.1)-(3.2) has at most one non-negative solution.

Proof. Let u_1, u_2 be non-negative blow-up solutions of $F(u, Du, D^2u) = f(x)$ in Ω. Let $\varepsilon > 0$ and setting $k_\varepsilon = 1 + \varepsilon$, $u_{1\varepsilon} = (1 + \varepsilon) u_1$. Using (3.22), we have

$$F(u_{1\varepsilon}, Du_{1\varepsilon}, D^2 u_{1\varepsilon}) \leq \varphi(k_\varepsilon) F(u_1, Du_1, D^2 u_1) + o(1) = \varphi(k_\varepsilon) f(x) + o(1) \tag{3.23}$$

in the viscosity sense, where $o(1) \to 0$ as $\varepsilon \to 0^+$.
Take, for every $P \in \partial\Omega$, a spherical cylinder Q_P^* centered at P of radius ρ^* and height $2\sigma^*$, as in Corollary 3.2.1. By (3.17) we have constructed an open covering $\{Q_P^*\}_{P \in \partial\Omega}$ of $\partial\Omega$ such that

$$1 - \frac{C_P}{u_1} \leq \frac{u_2}{u_1} \leq 1 + \frac{C_P}{u_1} \quad \text{in} \quad Q_P^* \cap \Omega. \tag{3.24}$$

Since $u_1 \to +\infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, then

$$u_2(x) \le (1+\varepsilon)u_1(x) = u_{1\varepsilon}(x), \tag{3.25}$$

in N_P^*, an open neighborhood of $Q_P^* \cap \partial\Omega$.
Collecting all N_P^* we obtain a neighborhood N_ε of $\partial\Omega$ where (3.25) holds true.
We claim that (3.25) holds in Ω for ε small enough.
By contradiction, suppose $\Omega_\varepsilon := \{u_2 > u_{1\varepsilon}\} \neq \emptyset$ for infinitely many $\varepsilon \to 0^+$.Using (3.23)-(3.25) and recalling that $F(u_2, Du_2, Du_2) \ge f$, we have by (3.6)

$$(u_2 - u_{1\varepsilon})^+(y) \le C_0 \left(\frac{1+bR}{R^2}\right)^{\frac{1}{d-1}} + C_1 \left((\varphi(k_\varepsilon) - 1)\|f^+\|_{L^p(\Omega\cap B_R(y))} + o(1)\right) \tag{3.26}$$

for all $y \in \Omega$ and $R > 0$. Thus, letting $\varepsilon \to 0^+$ and then $R \to \infty$, we get $u_2 \le u_1$ in Ω, which contradicts $\Omega_\varepsilon \neq \emptyset$ and proves the claim.
Hence $u_2 \le (1+\varepsilon)u_1$ in Ω definitively as $\varepsilon \to 0^+$ and taking the limit we have $u_2 \le u_1$ in Ω. Interchanging u_1 and u_2 we finish the proof. □

Remark 3.2.4. Condition (3.22) on F is satisfied with $\varphi(k) = k^\alpha$ in the case of operators

$$F(s, q, N) = F_1(q, N) - |s|^{d-1}s$$

such that F_1 is positively homogeneous of degree $\alpha \in (0, d]$. In fact, when $s \ge 0$ and $k \ge 1$

$$F(s,q,N) = F_1(q,N) - s^d = k^\alpha F_1\left(\frac{q}{k}, \frac{N}{k}\right) - s^d \le k^\alpha F_1\left(\frac{q}{k}, \frac{N}{k}\right) - k^\alpha \frac{s^d}{k^d} = k^\alpha F\left(\frac{s}{k}, \frac{q}{k}, \frac{N}{k}\right).$$

Remark 3.2.5. As one can see in Remark 3.2.1, $f \le 0$ is a sufficient condition to have non-negative solutions.

Remark 3.2.6. By Theorem 3.2.1 and Remark 3.2.4 we have uniqueness of non-negative blow-up solutions for the maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + b|Du| - |u|^{d-1}u = f(x)$$

and more generally for Bellman and Isaacs type equations like

$$\sup_\alpha \inf_\beta \{\operatorname{Trace}(A^{\alpha\beta}(x)D^2u) + \langle b^{\alpha\beta}(x), Du\rangle\} - |u|^{d-1}u = f(x),$$

with $A^{\alpha\beta}(x) \in \mathcal{S}^n$ such that $\lambda I \le A^{\alpha\beta}(x) \le \Lambda I$ and $|b^{\alpha\beta}(x)| \le b \in \mathbb{R}_+$.

3.3 A generalization

In this Section, following [48], a more general class of uniformly elliptic operators $F \in \mathcal{F}_{\lambda,\Lambda,b}$, obtained adding a "positive semilinearity "

$$F(s, q, N) := F_1(q, N) + c|s|^{\alpha-1}s - |s|^{d-1}s, \tag{3.27}$$

is considered. Here $F_1 \in \mathcal{F}_{\lambda,\Lambda,b}$ is positively homogeneous of degree $\beta \in [\alpha, d]$, $\alpha \in (0, d)$ and $c \ge 0$. In the next Lemma we state a comparison principle for (3.27).

Lemma 3.3.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ and let F be an uniform elliptic operator as in (3.27). Suppose that u and v are continuous subsolutions and supersolutions, respectively, of $F = f$ in viscosity sense, where $f \in C(\Omega)$ and $f \leq 0$. If $u, v \in C^1(\Omega)$ and $v > 0$ in Ω, then*

$$\limsup_{x \to \partial\Omega} (u - v) \leq 0 \quad \Rightarrow \quad u \leq v \quad \text{in } \Omega. \tag{3.28}$$

Proof. By contradiction, suppose $\Omega^+ \equiv \{x \in \Omega \,|\, u(x) > v(x)\} \neq \emptyset$. Setting $u = e^U$ and $v = e^V$, by straightforward computation, we obtain, in viscosity sense,

$$F_1(DU, D^2U + DU \otimes DU) + ce^{(\alpha-\beta)U} - e^{(d-\beta)U} \geq e^{-\beta U} f(x) \tag{3.29}$$

and

$$F_1(DV, D^2V + DV \otimes DV) + ce^{(\alpha-\beta)V} - e^{(d-\beta)V} \leq e^{-\beta V} f(x), \tag{3.30}$$

where we have used the positive homogeneity of F_1.
Let $w = U - V$. Subtracting (3.30) from (3.29), as we may in viscosity setting since F_1 is independent of x (see [14] and Theorem 1.2.9), we have

$$\begin{aligned} &\mathcal{P}^+_{\lambda,\Lambda}(D^2w + (DU \otimes Dw + Dw \otimes DV)) + b|Dw| \\ &\qquad \geq -c(e^{(\alpha-\beta)U} - e^{(\alpha-\beta)V}) + (e^{(d-\beta)U} - e^{(d-\beta)V}) + (e^{-\beta U} - e^{-\beta V})f(x). \end{aligned}$$

Using the fact that $c \geq 0$, $\alpha \leq \beta$ and $f \leq 0$ we obtain

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b(x)|Dw| \geq 0 \quad \text{in} \quad \Omega^+,$$

where $b(x) = \Lambda(|DU(x)| + |DV(x)|) + b$.
Now, w is positive in Ω^+ and $\limsup_{x \to \partial\Omega^+} w \leq 0$, so its positive maximum M would be achieved in Ω^+. Then by the strong maximum principle $w \equiv M$ in Ω^+, which contradicts the boundary condition. Therefore $U \leq V$ and consequently $u \leq v$ in Ω. □

Theorem 3.3.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ of class C_{gr}. Let F be an uniformly elliptic operator of the form (3.27). If $f \in C(\Omega)$, $f \leq 0$, then problem*

$$\begin{cases} F(u, Du, D^2u) = f(x) & \text{in } \Omega \\ u(x) \to +\infty & \text{as } \operatorname{dist}(x, \partial\Omega) \to 0 \end{cases} \tag{3.31}$$

has at most one positive solution.

Proof. Consider two positive solutions u_1, u_2 of problem (3.31). By standard results on viscosity solutions, see [8] and [53], u_1, u_2 have Hölder first derivatives.
By the local graph property and the boundary blow-up condition, for every $P \in \partial\Omega$ we can find a spherical cylinder Q_P as (3.9) such that (3.10) holds true and $u_i \geq (2c)^{\frac{1}{d-\alpha}}$ in $Q_P \cap \Omega$, $i = 1, 2$. Then

$$F_1(Du_i, D^2u_i) - \frac{1}{2}u_i^d \geq f(x) \quad \text{in} \quad Q_P \cap \Omega. \tag{3.32}$$

Since the operator $F_1(q,N) - \frac{1}{2}|s|^{d-1}s$ associated to (3.32) satisfies the structure conditions (**SC**), we conclude as in Corollary 3.2.1 that for $Q_P^* \Subset Q_P$ there exists C such that (3.17) holds true. Following the proof of Theorem 3.2.1, for any $\varepsilon > 0$, we find a neighborhood N_ε of $\partial\Omega$ where (3.25) holds true. As there, we set $\Omega_\varepsilon = \{x \in \Omega\,,\, u_2 > (1+\varepsilon)u_1 = u_{1\varepsilon}\}$ and infer that $\Omega_\varepsilon = \emptyset$. By contradiction, suppose $\Omega_\varepsilon \neq \emptyset$. Setting $k_\varepsilon = 1+\varepsilon$, using the positive homogeneity of F_1 and the fact that $f \leq 0$, from $F_1(Du_1, D^2u_1) + cu_1^\alpha - u_1^d \leq f$ we have

$$F_1(Du_{1\varepsilon}, D^2u_{1\varepsilon}) + cu_{1\varepsilon}^\alpha - u_{1\varepsilon}^d \leq k_\varepsilon^\beta f(x) \leq f(x)$$

so that, being $F_1(Du_2, D^2u_2) + cu_2^\alpha - u_2^d \geq f$, Lemma 3.3.1 yields $u_2 \leq u_{1\varepsilon}$ in Ω_ε, against $\Omega_\varepsilon \neq \emptyset$. From this $u_2 \leq (1+\varepsilon)u_1$ and, letting $\varepsilon \to 0^+$, $u_2 \leq u_1$ in Ω. Interchanging u_1 and u_2, we also have $u_1 \leq u_2$, as claimed. □

Remark 3.3.1. Note that if $c \leq 0$ the operator F satisfies the superlinear monotonicity assumption (3.3). Moreover if $\beta = \alpha$ the uniqueness of blow-up solutions is provided by Theorem 3.2.1.

Chapter 4

Extended maximum principle and removable singularities

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. In view of the Maximum Principle we know that a subsolution of the uniformly elliptic equation $F(D^2u) = 0$ is non positive in Ω assuming $u \leq 0$ on the boundary $\partial\Omega$. More generally

$$\limsup_{x \to \partial\Omega} u(x) \leq 0 \Rightarrow u \leq 0 \quad \text{in } \ \Omega. \tag{4.1}$$

A natural question arises from (4.1): can we weaken the boundary condition, by requiring $u \leq 0$ on a "relevant"part of $\partial\Omega$, in order that the Maximum Principle continues to holds? Setting $\partial\Omega = P \cup E$, we propose to give conditions on E which ensure the validity of the following *Extended Maximum Principle*, EMP for short:

$$u \in USC(\Omega), \ \ F(D^2u) \geq 0 \ \text{ in } \ \Omega, \ \ u \text{ bounded above}, \ \limsup_{x \to P} u(x) \leq 0 \Rightarrow u \leq 0 \ \text{ in } \ \Omega. \tag{4.2}$$

If E is a Borel set for which EMP holds we say that E is a F-exceptional set on $\partial\Omega$, while if EMP holds true for any $F \in \mathcal{F}_{\lambda,\Lambda}$ then E is an $\mathcal{F}_{\lambda,\Lambda}$-exceptional set.
In the sequel we will consider the pairs (λ, Λ) such that

$$\alpha^* = \alpha^*\left(n, \frac{\lambda}{\Lambda}\right) := \frac{\lambda}{\Lambda}(n-1) - 1 \geq 0. \tag{4.3}$$

Finite sets are simple examples of $\mathcal{F}_{\lambda,\Lambda}$-exceptional sets. Indeed let $E = \{x_1, \dots, x_m\} \subseteq \partial\Omega$, $\Omega \subseteq B_{\frac{R}{2}}$ and consider the so called fundamental solution of the maximal Pucci operator $\mathcal{P}^+_{\lambda,\Lambda}$ ([44]-[3]), i.e.

$$\mathcal{U}(x) = \begin{cases} \log\left(R|x|^{-1}\right) & \text{if } \alpha^* = 0 \\ |x|^{-\alpha^*} & \text{if } \alpha^* > 0. \end{cases} \tag{4.4}$$

By a standard calculation we have

$$\mathcal{P}^+_{\lambda,\Lambda}\left(D^2\left(\sum_{i=1}^m \mathcal{U}(x - x_i)\right)\right) \leq \sum_{i=1}^m \mathcal{P}^+_{\lambda,\Lambda}\left(D^2\mathcal{U}(x - x_i)\right) = 0 \quad \text{in} \quad \mathbb{R}^n \backslash E. \tag{4.5}$$

If $u \in USC(\Omega)$ is a subsolution, bounded above, of $F(D^2u) = 0$ satisfying the boundary condition

$$\limsup_{x \to P} u(x) \leq 0, \tag{4.6}$$

for any $\varepsilon > 0$ we put $v(x) = u(x) - \varepsilon \left(\sum_{i=1}^{m} \mathcal{U}(x - x_i)\right)$. Then

$$\mathcal{P}^{+}_{\lambda,\Lambda}\left(D^2 v(x)\right) \geq 0 \quad \text{in} \quad \Omega, \quad \limsup_{x \to \partial\Omega} v(x) \leq 0. \tag{4.7}$$

By the Maximum Principle (4.1) $v(x) \leq 0$ in Ω, that is $u(x) \leq \varepsilon \sum_{i=1}^{m} \mathcal{U}(x - x_i)$ and letting $\varepsilon \to 0^+$ we conclude $u(x) \leq 0$ in Ω.
We notice that the above argument continues to work assuming

$$F(M) \leq \mathcal{P}^{+}_{\lambda,\Lambda}(M) \quad \forall M \in \mathcal{S}^n, \tag{4.8}$$

instead of $F \in \mathcal{F}_{\lambda,\Lambda}$, and

$$\begin{cases} u^+(x) = o\left(\log|x - x_i|^{-1}\right) & \text{if } \alpha^* = 0 \\ u^+(x) = o\left(|x - x_i|^{-\alpha^*}\right) & \text{if } \alpha^* > 0 \end{cases} \tag{4.9}$$

as $x \to x_i$ for $i = 1, \ldots, m$ instead of the boundedness of u.
Looking at the Laplace operator, it is well know that a set E is Δ-exceptional if, and only if, its capacity $C_{n-2}(E)$, in dimension $n \geq 3$, and logarithmic capacity $C_{\log}(E)$, in dimension 2, vanish (see [37]-[46]). We report the definitions of capacity in Section 4.1.
The sets of null capacity play the role of null sets in various questions of Potential Theory, a role somewhat analogous to the sets of zero measure in the Integration Theory, even if it is worth to remark that capacity is not a measure because it fails to be additive on disjoint sets.
In the Section 4.2 we propose to show sufficient conditions about EMP in the nonlinear viscosity setting, first dealing with the pure second order case $F(D^2u) = 0$ (Subsection 4.2.1) and then for the more general case including the gradient variable (Subsection 4.2.2).

An interesting application of EMP concerns the problem of *removable singularities*. Let E be a Borel subset of the domain Ω of $\mathbb{R}^n$, we say that E is a F-*removable* set in Ω if every bounded viscosity solution $u \in C(\Omega')$ of the equation $F(Du, D^2u) = f(x)$ in $\Omega' = \Omega \backslash E$ can be extended to a viscosity solution $\overline{u} \in C(\Omega)$ of the same equation in the whole Ω. If the same holds for every $F \in \mathcal{F}_{\lambda,\Lambda,b}$ then E if $\mathcal{F}_{\lambda,\Lambda,b}$-removable.
It is plain that a set E having non empty interior cannot be removable: otherwise considering a solution of $F = 0$ in $B_2 \backslash B_1$ (where B_1 and B_2 are balls satisfying $B_2 \supset E \supseteq \overset{\circ}{E} \supset B_1$) with $u = 0$ on ∂B_2 and $u \neq 0$ in $B_2 \backslash B_1$ it should be $\overline{u} = 0$ in B_2.
As before, using EMP, it easy to check that points are $\mathcal{F}_{\lambda,\Lambda,b}$-removable sets. More generally we will show in Section 4.3 that $\mathcal{F}_{\lambda,\Lambda,b}$-exceptional sets are removable ones.
In order to characterize the $\mathcal{F}_{\lambda,\Lambda}$-removable sets, D.Labutin [45] propose a different definition of capacity, establishing the equivalence of the removability with the vanishing of his capacity and so generalizing the results concerning with the Laplace operator. On the contrary we will use the

classical Riesz capacity showing that it provide a sufficient condition for the removability problem. Although this approach does not provide a complete characterization of the removable sets, nonetheless it looks advantageous if we ask to what extent the lower order terms influence the removable sets with respect to the main term of a fully nonlinear second order operator.

In Section 4.4 we deal with a class of pure second order degenerate elliptic operators considered by R.Harvey and B.Lawson [31]-[32]-[33]-[34]-[35]-[36] and by L.Caffarelli, Y.Y.Li, L.Nirenberg [10]-[11]:

$$\mathcal{P}_p^-(D^2u) = \lambda_1(D^2u) + \cdots + \lambda_p(D^2u) \tag{4.10}$$

where $\lambda_1(M) \leq \cdots \leq \lambda_n(M)$ are the eigenvalues of the $n \times n$ real symmetric matrix M. R.Harvey and B.Lawson formulate a geometric approach to nonlinear ellipticity, which is realized as a monotonicity condition on the set $\mathbf{F} = \{M \in \mathcal{S}^n \mid F(M) \geq 0\}$ with respect to a convex cone $\mathbf{P}$:

$$\mathbf{F} - \mathbf{P} \subset \mathbf{F}. \tag{4.11}$$

In this perspective degenerate and uniform ellipticity correspond to $\mathbf{P} = \{M \leq 0\}$ and $\mathbf{P} = \{M \in \mathcal{S}^n \mid \mathcal{P}^+_{\lambda,\Lambda}(M) \leq 0\}$, respectively. We discuss how the results of [32] are more general for removable singularities of pure second order elliptic operators and at the same time check our method indicating an alternative proof. Moreover we extend the results about Maximum Principle and exceptional sets to this class of degenerate elliptic equations.

4.1 Riesz potential and capacity

We recall some definitions and results from Potential Theory, referring to [37]-[46] as main source. The function

$$K_\alpha(x) = |x|^{-\alpha}, \quad 0 < \alpha < n, \tag{4.12}$$

is called *Riesz kernel*, it is continuous for $x \neq 0$ and lower semicontinuous in $\mathbb{R}^n$.
For any compact set $E \subset \mathbb{R}^n$ we denote by $\mathcal{M}^+(E)$ the set of nonnegative Borel measure on E and by $\mathcal{M}^+_1(E)$ the subset of $\mathcal{M}^+(E)$ consisting of unit measure (i.e. $\mu(E) = 1$). The α-*Riesz potential* of $\mu \in \mathcal{M}^+(E)$ is defined as

$$V^\mu_\alpha(x) = \int_E K_\alpha(x-y)\, d\mu(y). \tag{4.13}$$

Since $K_\alpha(x)$ is superharmonic in $\mathbb{R}^n$ for $\alpha \leq n-2$, it follows that $V^\mu_\alpha(x)$ is superharmonic too. Moreover in $\mathbb{R}^n \backslash E$ it is harmonic if $\alpha = n-2$ and subharmonic when $\alpha > n-2$.
Next we consider the nonnegative *energy integral*

$$I_\alpha(\mu) = \iint_{E \times E} K_\alpha(x-y)\, d\mu(x) d\mu(y) \tag{4.14}$$

and the α-*equilibrium value* $V_\alpha(E)$ of E

$$V_\alpha(E) = \inf_{\mu \in \mathcal{M}^+_1(E)} I_\alpha(\mu). \tag{4.15}$$

Since $K_\alpha(x-y) \geq (\mathrm{diam}(E))^{-\alpha}$, then $0 < V_\alpha(E) \leq +\infty$.

Definition 4.1.1. *The α-capacity of the compact E is the nonnegative number*

$$C_\alpha(E) = (V_\alpha(E))^{-1}. \tag{4.16}$$

We deduce that $C_\alpha(E) = 0$ if, and only if, $I_\alpha(\mu) = +\infty$ for any $\mu \in \mathcal{M}_1^+(E)$ and $C_\alpha(E) > 0$ when there exists a unit measure for which the energy integral is finite.
The infimum in (4.15) is attained for a suitable minimizing measure $\mu \in \mathcal{M}_1^+(E)$, called *$\alpha$-equilibrium measure*. The corresponding α-Riesz potential is the *α-conductor potential* of E.

Definition 4.1.2. *For $E \subseteq \mathbb{R}^n$ (not necessarily compact) the α-inner capacity is defined as*

$$\underline{C}_\alpha(E) = \sup_{\substack{K \subset E \\ K \text{ compact}}} C_\alpha(K) \tag{4.17}$$

and the α-outer capacity as

$$\overline{C}_\alpha(E) = \inf_{\substack{A \supset E \\ A \text{ open}}} C_\alpha(A). \tag{4.18}$$

Inner and outer capacity are monotone increasing set functions and $\underline{C}_\alpha(E) \leq \overline{C}_\alpha(E)$. When the equality holds we say that E is *capacitable*, its common value is denoted by $C_\alpha(E)$. Borel sets are capacitable [46, Theorem 2.8].

Definition 4.1.3. *A set E is of type F_σ if it is representable in the form $E = \bigcup_{k \in \mathbb{N}} E_k$, where E_k are closed sets.*

An F_σ set is capacitable and

$$C_\alpha(E) = 0 \Leftrightarrow C_\alpha(E_k) = 0 \quad \forall k \in \mathbb{N}. \tag{4.19}$$

To consider the case $\alpha = 0$ we can repeat the above construction starting from the logarithmic kernel $K_0(x) = \ln(d/|x|)$ to define the energy integral $I_0(\mu)$, for a compact subset $E \subset B_d \equiv B_d(0)$. In this case we set

$$V_0(E) = \inf_{\mu \in \mathcal{M}_1^+(E)} I_0(\mu) \quad \text{and} \quad C_0(E) = e^{-V_0} \tag{4.20}$$

as the 0-equilibrium (log-equilibrium) value and the 0-capacity (logarithmic capacity) of E, respectively.
As before we also define the 0-equilibrium (log-equilibrium) measure and the corresponding 0-conductor (logarithmic) potential for E.
Furthermore, again as above, we also define by approximation the 0-capacity and the 0-capacitability for any $E \Subset B_d$. In doing this, we notice that for our purposes all d are equivalent, since it is easy to check that a different choice of $d > 0$ such that $E \Subset B_d$ changes the value of $C_0(E)$ only by a positive multiplicative constant.

It is worth to notice that for $\alpha = n - 2$ the above definition agrees with the classical definition of capacity. In this case we omit $\alpha = n-2$ when referring to potential, minimizing measure, equilibrium value and capacity.

4.2 Extended Maximum Principle

4.2.1 Case $F(D^2u) = 0$

It is well known that, if E is a compact set in $\mathbb{R}^n$ with $C(E) = 0$, there exists a superharmonic function v in $\mathbb{R}^n$ such that $v(x) = +\infty$ for $x \in E$ and $0 < v(x) < +\infty$ outside E ([37, Theorems 5.11-5.32]). Our aim is to generalize this result to fully nonlinear uniformly elliptic operators.
We start showing that the α-potential of a measure $\mu \in \mathcal{M}_1^+(E)$ is a supersolution of the maximal Pucci equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2u) = 0$ in $\mathcal{C}E = \mathbb{R}^n \backslash E$.

Lemma 4.2.1. *Let $0 \le \alpha \le \alpha^*$, E be a compact subset of $\mathbb{R}^n$, $\mu \in \mathcal{M}_1^+(E)$ and*

$$V^\mu_\alpha(x) = \int_E K_\alpha(x-y)\, d\mu(y)$$

the α-potential of μ. Then $\mathcal{P}^+_{\lambda,\Lambda}(D^2V^\mu_\alpha(x)) \le 0$ for $x \in \mathcal{C}E = \mathbb{R}^n \backslash E$.

Proof. Indeed $V^\mu_\alpha \in C^\infty(\mathcal{C}E)$ and for $x \in \mathcal{C}E$ it turns out that

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}(D^2V^\mu_\alpha(x)) &= \sup_{\lambda I \le A \le \Lambda I} \text{Trace}\left(A\, D^2 \int_E K_\alpha(x-y)\, d\mu(y)\right) \\ &= \sup_{\lambda I \le A \le \Lambda I} \int_E \text{Trace}\left(A\, D^2K_\alpha(x-y)\right)\, d\mu(y) \\ &\le \int_E \sup_{\lambda I \le A \le \Lambda I} \text{Trace}\left((A\, D^2K_\alpha(x-y)\right)\, d\mu(y) = \int_E \mathcal{P}^+_{\lambda,\Lambda}(D^2K_\alpha(x-y))\, d\mu(y)\end{aligned}$$

and

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2K_\alpha(x-y)) = \begin{cases} -\dfrac{\alpha\Lambda(\alpha^*-\alpha)}{|x-y|^{\alpha+2}} & \text{if } \alpha > 0 \\ -\dfrac{\Lambda\alpha^*}{|x-y|^2} & \text{if } \alpha = 0 \end{cases}$$

from which the result follows. □

Theorem 4.2.1. *Suppose that $\alpha^* = \frac{\lambda}{\Lambda}(n-1)-1 \ge 0$. Let E be a Borel set of $\mathbb{R}^n$ such that $C_{\alpha^*}(E) = 0$. If E is compact, there exists a supersolution v of the maximal Pucci equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) = 0$ such that*

$$v(x) = +\infty \ \text{ in } \ E, \quad v(x) < +\infty \ \text{ outside } \ E.$$

If E is F_σ, for any $x_0 \notin E$ there exists a supersolution v of the maximal Pucci equation such that

$$v(x) = +\infty \ \text{ in } \ E, \ \ v(x_0) < +\infty.$$

Such supersolutions can be chosen non-negative in $\mathbb{R}^n$ in $\alpha^* > 0$ and non-negative on an arbitrary bounded set of $\mathbb{R}^n$ if $\alpha^* = 0$ (logarithmic capacity).

Proof. Firstly suppose E to be a compact set of α^*-capacity zero. From [46, Theorem 3.1] and the argument thereafter we find a positive measure $\mu \in \mathcal{M}_1^+(E)$ such that for the α^*-potential of μ, which is lower semicontinuous, we have $V^\mu_{\alpha^*}(x) = +\infty$ on E and $V^\mu_{\alpha^*}(x) < +\infty$ outside E. From

Lemma 4.2.1 above we have $\mathcal{P}^+_{\lambda,\Lambda}(D^2 V^\mu_{\alpha^*}(x)) \leq 0$, and we can choose $v(x) = V^\mu_{\alpha^*}(x)$ in this case. Now suppose $E = \bigcup_{k\in\mathbb{N}} E_k$, where E_k are compact sets. For each $k \in \mathbb{N}$ the above construction provides a supersolution $v_k(x)$ of the maximal Pucci equation such that $v_k(x) = +\infty$ on E_k and $v_k(x) < +\infty$ outside E_k. Let $v_k(x_0) = c_k^{-1}$ in order that

$$v(x) = \sum_{k=1}^{\infty} \frac{c_k}{k^2} v_k(x)$$

is finite at $x = x_0$. Then $v(x)$ is the limit of an increasing sequence of supersolutions of the maximal Pucci equation, from Theorem 1.2.8 (see also [1, Lemma 2.1]) we deduce the result. □

Remark 4.2.1. Let E be a F_σ-set of α^*-capacity zero. In view of the maximality of $\mathcal{P}^+_{\lambda,\Lambda}$, by Theorem 4.2.1 for all $x_0 \notin E$ we find a non-negative function on bounded sets $v \in LSC(\mathbb{R}^n)$ such that

$$F(D^2 v) \leq 0 \ \text{ in } \ \mathbb{R}^n, \ \ v(x) = +\infty \ \text{ in } \ E, \ \ v(x_0) < +\infty$$

for all $F \in \mathcal{F}_{\lambda,\Lambda}$.

Now we are in position to answer to the question posed at the beginning of this Chapter.

Theorem 4.2.2. [**Extended Maximum Principle**] *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\partial\Omega = P \cup E$ with E an F_σ-set. If $C_{\alpha^*}(E) = 0$ then E is an $\mathcal{F}_{\lambda,\Lambda}$-exceptional set on $\partial\Omega$.*

Proof. Let $u \in USC(\Omega)$ be a subsolution, bounded above, of the equation $F(D^2u) = 0$ in Ω satisfying the boundary condition $\limsup_{x\to P} u(x) \leq 0$. Here $F \in \mathcal{F}_{\lambda,\Lambda}$. In view of the maximality of Pucci operators in the class $\mathcal{F}_{\lambda,\Lambda}$, u is a subsolution of $\mathcal{P}^+_{\lambda,\Lambda}(D^2u) = 0$ in Ω. Let ε be a positive number and fix $x_0 \in \Omega$. The function $w(x) = u(x) - \varepsilon v(x)$, where $v(x)$ is the supersolution of $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) = 0$ provided by Theorem 4.2.1, is upper semicontinuous in Ω and

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) \geq \mathcal{P}^+_{\lambda,\Lambda}(D^2u) - \varepsilon\mathcal{P}^+_{\lambda,\Lambda}(D^2v) \geq 0$$

in the viscosity sense. From the boundedness of u and the non-negativity of v we deduce that

$$\limsup_{x\to\partial\Omega} w(x) \leq 0$$

and so, by Maximum Principle,

$$u(x_0) \leq \varepsilon v(x_0).$$

Letting $\varepsilon \to 0^+$ we conclude that $u(x_0) \leq 0$ and since $x_0 \in \Omega$ is arbitrary the proof is complete. □

4.2.2 Case $F(Du, D^2u) = 0$

The key point in the proof of Theorem 4.2.2 is the possibility to construct a supersolution of a maximal equation which is finite in $\Omega\backslash E$ and blows up on E. This was done using Riesz Potential. Now we want to show in which way this results can be extended to the class of second order operators with non zero first order terms. We start with the following

Lemma 4.2.2. *Suppose that* $\alpha^* = \frac{\lambda}{\Lambda}(n-1) - 1 > 0$. *For every* $0 \le \alpha < \alpha^*$ *and* $b > 0$ *there exists* $\delta > 0$ *such that, if* E *is a compact subset of* $B_\delta(\overline{x})$ *for* $\overline{x} \in E$, *then*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 V^\mu_\alpha(x)) + b|DV^\mu_\alpha(x)| \le 0$$

for all $x \in B_\delta(\overline{x}) \cap \mathcal{C}E$.

Proof. Suppose that $E \subset B_\delta(\overline{x})$ for $\overline{x} \in E$ and $\delta > 0$ to be chosen. Following the calculations of Lemma 4.2.1, for $x \in \mathcal{C}E$ we have

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}(D^2 V^\mu_\alpha(x)) + b|DV^\mu_\alpha(x)| &\le \int_E \mathcal{P}^+_{\lambda,\Lambda}(D^2 K_\alpha(x-y)) + b|DK_\alpha(x-y)|\, d\mu(y) \\ &= \begin{cases} \displaystyle\int_E \alpha b \left(\frac{\Lambda}{b}(\alpha - \alpha^*) + |x-y|\right) \frac{d\mu(y)}{|x-y|^{\alpha+2}} & \text{if } \alpha > 0 \\ \displaystyle\int_E b \left(-\frac{\Lambda}{b}\alpha^* + |x-y|\right) \frac{d\mu(y)}{|x-y|^2} & \text{if } \alpha = 0 \end{cases}\end{aligned}$$

so that, choosing

$$\delta = \frac{\Lambda}{2b}(\alpha^* - \alpha) > 0 \tag{4.21}$$

we get

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 V^\mu_\alpha(x)) + b|DV^\mu_\alpha(x)| \le 0$$

for $x \in B_\delta(\overline{x}) \backslash E$, as we wanted to show. ☐

In order to prove a generalization of Theorem 4.2.1 to the class $\mathcal{F}_{\lambda,\Lambda,b}$ we need the following result concerning the extension of supersolution.

Lemma 4.2.3. *Let* V *be a viscosity supersolution of the equation* $F(DV, D^2V) = f(x)$ *in the annulus* $A_{s\rho,t\rho} \equiv B_{t\rho}(y) \backslash \overline{B}_{s\rho}(y)$, $s < 1 < t$, *for* $F \in \mathcal{F}_{\lambda,\Lambda,b}$, *where* $f(x)$ *is a continuous function in* $\overline{A}_{s\rho,R}$, *with* $R > t\rho$. *We suppose that* V *is continuous in* $\overline{A}_{s\rho,t\rho}$. *There exists a continuous function* V_R *such that*

$$\begin{cases} F(DV_R, D^2V_R) \le f(x) & \text{in } A_{s\rho,R} \\ V_R = V & \text{in } A_{s\rho,\rho} \end{cases} \tag{4.22}$$

in the viscosity sense.

Proof. We may suppose, up to a translation, that $y = 0$ and , eventually changing V with $V + c$ for a suitable constant c, that $V \ge 0$. Let $u \in C(\overline{A}_{\rho,t\rho})$ be the solution of the Dirichlet problem

$$\begin{cases} F(Du, D^2u) = f(x) & \text{in } A_{\rho,t\rho} \\ u = V & \text{on } |x| = \rho \\ u = 0 & \text{on } |x| = t\rho \end{cases}$$

By the comparison principle $u \le V$ in $A_{\rho,t\rho}$. Hence from [8, Proposition 2.8] we deduce that the function $V(x)$ is an extension of the continuous supersolution $u(x)$ from $A_{\rho,t\rho}$ on $A_{s\rho,t\rho}$.
Next, we set $|x| = r$ and

$$w(x) = \gamma(e^{\beta t\rho} - e^{\beta r}).$$

Setting $M = \max\{u(x) = V(x),\ |x| = \rho\}$ and $N = \max\{f^-(x),\ |x| \le R\}$, we choose

$$\gamma = \max\left(1, \frac{M}{e^{\beta t\rho} - e^{\beta\rho}}\right)$$

and β a positive number such that

$$-\lambda\beta^2 + b\beta + N \le 0\,.$$

By direct calculation, since $F \in \mathcal{F}_{\lambda,\Lambda,b}$ we have

$$\begin{aligned} F(Dw, D^2w) &\le \mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| \le e^{\beta r}\gamma(-\lambda\beta^2 + b\beta) \\ &\le -\lambda\beta^2 + b\beta \le -N \le f(x), \end{aligned}$$

so that $w(x)$ is a supersolution in $B_R\backslash\{0\}$. Moreover, $w(x) \ge M \ge u(x)$ on $|x| = \rho$ and $w(x) = 0 = u(x)$ on $|x| = t\rho$, so that by the comparison principle we also have $w \ge u$ in $A_{\rho,t\rho}$. From this it follows that $w(x)$ is an extension of the continuous supersolution $u(x)$ from $A_{\rho,t\rho}$ on $A_{\rho,R}$. We conclude that

$$V_R(x) = \begin{cases} V(x) & \text{in } \overline{A}_{s\rho,\rho} \\ u(x) & \text{in } A_{\rho,t\rho} \\ w(x) & \text{on } \overline{A}_{t\rho,R} \end{cases} \tag{4.23}$$

is a possible choice for the required extension. □

Although Lemma 4.2.3 is sufficient to our aim, for the sake of completeness we show that the supersolution V can be extended to a supersolution in $\mathbb{R}^n\backslash\overline{B}_{s\rho}(y)$.

Lemma 4.2.4. *Let V as in the Lemma 4.2.3. Assuming $f(x)$ to be continuous in $\mathbb{R}^n\backslash\overline{B}_{s\rho}(y)$, then there exists a continuous viscosity supersolution of $F(Dv, D^2v) = f(x)$ in $\mathbb{R}^n\backslash\overline{B}_{s\rho}(y)$ such that $v = V$ in $A_{s\rho,\rho}$*

Proof. We may assume $y = 0$. Let $R > 2t\rho$ so that $B_{\frac{R}{2}} \supseteq B_{t\rho}$. Set

$$v \equiv V_R \quad \text{in} \quad A_{s\rho,\frac{R}{2}}\,, \tag{4.24}$$

where V_R is the function (4.23) satisfying (4.22). The function V_R, restricted to the set $A_{s\rho,\frac{R}{2}}$, has an extension V_{2R}, in view of Lemma 4.2.3, from $A_{s\rho,\frac{R}{2}}$ to $A_{s\rho,2R}$. We put

$$v \equiv V_{2R} \quad \text{in} \quad A_{s\rho,\frac{3}{2}R}\,, \tag{4.25}$$

noting that (4.24)-(4.25) are well defined being $V_R = V_{2R}$ in $A_{s\rho,\frac{R}{2}}$. Iterating the argument above, for any $n \in \mathbb{N}$ we define

$$v \equiv V_{(n+1)R} \quad \text{in} \quad A_{s\rho,\left(n+\frac{1}{2}\right)R}, \tag{4.26}$$

where $V_{(n+1)R}$ is the extension from $A_{s\rho,(n-\frac{1}{2})R}$ to $A_{s\rho,(n+1)R}$ of the function V_{nR} restricted to $A_{s\rho,(n-\frac{1}{2})R}$. □

Theorem 4.2.3. *Let $E \subseteq \mathbb{R}^n$ be a Borel subset of the ball B_R such that $C_\alpha(E) = 0$ for some non-negative $\alpha < \alpha^* = \frac{\lambda}{\Lambda}(n-1) - 1$. If E is compact, for all $b > 0$ there exists a non negative viscosity supersolution v of the maximal equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b|Dv| = 0$ in B_R such that*

$$v(x) = +\infty \ \ in \ \ E, \quad v(x) < +\infty \ \ outside \ \ E\,.$$

If E is a F_σ-set, for every $x_0 \in B_R \backslash E$ there exists a non negative viscosity supersolution v of the maximal equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b|Dv| = 0$ in B_R such that

$$v(x) = +\infty \ \ in \ \ E, \ \ v(x_0) < +\infty.$$

Proof. Firstly, suppose E to be a compact subset of $B_{\delta/2}(\overline{x})$ for some $\overline{x} \in E$. Here δ is the positive number of Lemma 4.2.2. Consider $\overline{R} > 0$ such that $B_{\overline{R}}(\overline{x}) \supseteq B_R$ and $\delta < \overline{R}$. Reasoning as in Theorem 4.2.1, since $C_\alpha(E) = 0$, there exists $\mu \in \mathcal{M}^+_1(E)$ such that $V^\mu_\alpha(x) = +\infty$ on E and $V^\mu_\alpha(x) < +\infty$ outside E. Next, using Lemma 4.2.3 with $\rho = \frac{3}{4}\,\delta$, $s = \frac{2}{3}$ and $t = \frac{4}{3}$, we construct an extension of the supersolution $V^\mu_\alpha(x)$ in $B_{\delta/2}(\overline{x})$ to a supersolution $v(x)$ of equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b|Dv| = 0$ in $B_{\overline{R}}(\overline{x})$ and therefore in B_R. We complete the proof in this case adding a suitable constant to get non negativity in B_R.
Suppose now that E is the union of a finite number N compact subsets $E_i \subset B_{\delta/2}(\overline{x}_i)$ for some $\overline{x}_i \in E_i$, from the previous case we can construct a non negative supersolution v_i in B_R such that $v_i(x) = +\infty$ on E_i and $v_i(x) < +\infty$ outside E_i. Thus by additivity the function $v = \sum_{i=1}^N v_i$ is a non negative supersolution of the equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b|Dv| = 0$ in B_R and yields the result for an arbitrary compact set E.
Finally, in the case that E is a countable union of closed sets, we conclude proceeding along the same lines of Theorem 4.2.1 . □

Remark 4.2.2. If E is compact, using Lemma 4.2.4 it is possible to construct a supersolution of $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b|Dv| = 0$ in $\mathbb{R}^n$ such that

$$v(x) = +\infty \ \text{ in } \ E, \quad v(x) < +\infty \ \text{ outside } \ E, \quad v(x) \geq 0 \ \text{ in } \ B_R.$$

Theorem 4.2.4. [**Extended Maximum Principle 2**] *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\partial\Omega = P \cup E$ with E an F_σ-set. If there exists $\alpha \in [0, \alpha^*)$ such that $C_\alpha(E) = 0$ then E is an $\mathcal{F}_{\lambda,\Lambda,b}$-exceptional set on $\partial\Omega$.*

Proof. Let $F \in \mathcal{F}_{\lambda,\Lambda,b}$. Suppose that u is a subsolution of the equation $F(Du, D^2u) = 0$ in Ω, bounded above, such that $\limsup_{x\to P} u(x) \leq 0$. Then u is a solution of the maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + b|Du| \geq 0 \quad \text{in} \quad \Omega\,.$$

Let ε be a positive number and $x_0 \in \Omega$. Consider $w(x) = u(x) - \varepsilon v(x)$, where $v(x)$ is the supersolution of the maximal equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b|Dv| = 0$ provided by Theorem 4.2.3 . Then $w(x)$ is upper semicontinuous and by viscosity tools we have

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| \geq \mathcal{P}^+_{\lambda,\Lambda}(D^2u) + b|Du| - \varepsilon\,(\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b|Dv|) \geq 0.$$

As in Theorem 4.2.2 it follows $\limsup_{x\to\partial\Omega} w(x) \leq 0$ and $u(x_0) \leq \varepsilon v(x_0)$ form which we conclude letting $\varepsilon \to 0^+$. □

Arguing as in the proofs of Theorems 4.2.2-4.2.4 and considering the function $w(x) = u(x)+\varepsilon v(x)$ in place of $w(x) = u(x) - \varepsilon v(x)$ we also deduce the following *extended minimum principle.*

Theorem 4.2.5. **[Extended Minimum Principle]** *Let Ω, P, E and F be as in Theorems 4.2.2-4.2.4. Let also $u \in LSC(\Omega)$ be a bounded below function satisfying in the viscosity sense*
A) $F(D^2u) \leq 0$ *in* Ω *and* $C_{\alpha^*}(E) = 0$
or
B) $F(Du, D^2u) \leq 0$ *in* Ω *and* $C_\alpha(E) = 0$ *for some* $\alpha \in [0, \alpha^*)$.
Then

$$\liminf_{x\to P} u(x) \geq 0 \Rightarrow u \geq 0 \ \ in \ \ \Omega.$$

The above EMPs allow us to extend the comparison principle in bounded domains.

Theorem 4.2.6. *Let $\Omega \in \mathbb{R}^n$ be a bounded domain and $\partial\Omega = P \cup E$ with E an F_σ-set. Suppose $f \in C(\Omega)$, let $u \in USC(\Omega)$ and $v \in LSC(\Omega)$ be respectively bounded above and below viscosity solution of*
A) $F(D^2v) \leq f(x) \leq F(D^2u)$ *in* Ω *and* $C_{\alpha^*}(E) = 0$
or
B) $F(Dv, D^2v) \leq f(x) \leq F(Du, D^2u)$ *in* Ω *and* $C_\alpha(E) = 0$ *for some* $\alpha \in [0, \alpha^*)$.
Then

$$\limsup_{x\to P} (u(x) - v(x)) \leq 0 \Rightarrow u \leq v \ \ in \ \ \Omega.$$

Proof. In view of Theorem 1.2.9, the difference $w = u - v \in USC(\Omega)$ is a viscosity subsolution, bounded above, of

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| = 0 \ \text{ in } \ \Omega$$

($b = 0$ in the case A)) satisfying the boundary condition

$$\limsup_{x\to P} w(x) \leq 0.$$

The conclusion follows from Theorems 4.2.2-4.2.4. □

4.3 Removable singularities

We discuss the application of the previous results to the problem of removable singularities. Remember that, as stated in the introduction of the Chapter, a set $E \Subset \Omega$ is said to be $\mathcal{F}_{\lambda,\Lambda,b}$-removable if for every $F \in \mathcal{F}_{\lambda,\Lambda,b}$ and for any bounded viscosity solution u of $F(Du, D^2u) = f(x)$ in $\Omega \backslash E$ there exists a solution $\widetilde{u}$ of the equation in Ω such that $\tilde{u} = u$ in $\Omega \backslash E$.
The main tool we use to construct extensions of solutions are EMPs and the existence results for the Dirichlet problem in bounded domains with smooth boundaries.

Proposition 4.3.1. *Let $\Omega \in \mathbb{R}^n$ be a bounded domain. Suppose that $E \Subset \Omega$ is a F_σ-set and $\Omega' = \Omega \backslash E$. If E is a $\mathcal{F}_{\lambda,\Lambda,b}$-exceptional set on $\partial\Omega'$, then each bounded solution $u \in C(\Omega')$ of the equation $F(Du, D^2u) = f(x)$ in Ω', where $F \in \mathcal{F}_{\lambda,\Lambda,b}$ and $f \in C(\Omega)$, can be extended to a solution $\widetilde{u} \in C(\Omega)$ of the same equation in Ω.*

Proof. Let Ω_0 be a bounded domain with sufficiently regular boundary (the uniform exterior cone condition is enough) such that $E \Subset \Omega_0 \Subset \Omega$, then solve the Dirichlet problem

$$\begin{cases} F(Dv, D^2v) = f(x) & \text{in } \Omega_0 \\ v = u & \text{on } \partial\Omega_0. \end{cases}$$

Setting $w = u - v$, by Theorem 1.2.9 then we get

$$\begin{cases} \mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b|Dw| \geq 0 & \text{in } \Omega_0 \backslash E \\ w = 0 & \text{on } \partial\Omega_0. \end{cases}$$

By assumptions, w is bounded above. Therefore EMP implies $u \leq v$ in $\Omega_0 \backslash E$. Reversing the role of u and v we also get $v \leq u$, so that $u = v$ and the function

$$\widetilde{u}(x) = \begin{cases} v(x) & \text{if } x \in E \\ u(x) & \text{if } x \in \Omega \backslash E \end{cases}$$

yields the desired extension of u to Ω. □

From Theorems 4.2.2-4.2.4 and Proposition 4.3.1 we deduce at once the following

Theorem 4.3.1. *Let Ω, Ω', E and f as in Proposition 4.3.1.*
A) *If u is a bounded viscosity solution of $F(D^2u) = f(x)$ in Ω' and $C_{\alpha^*}(E) = 0$ then E is $\mathcal{F}_{\lambda,\Lambda}$-removable in Ω.*
B) *If u is a bounded viscosity solution of $F(Du, D^2u) = f(x)$ in Ω' and $C_\alpha(E) = 0$, for some $\alpha \in [0, \alpha^*)$, then E is $\mathcal{F}_{\lambda,\Lambda,b}$-removable in Ω.*

4.4 A class of degenerate elliptic equations

Let $p \leq n$ be a positive integer, and $G(p,n)$ be the Grassmanian of p-dimensional subspaces of $\mathbb{R}^n$. Following Harvey-Lawson [32] and Caffarelli-Li-Nirenberg [10]-[11] we define

$$\mathcal{P}^-_p(M) = \inf_{W \in G(p,n)} \text{Trace}_W(Z) \equiv \lambda_1(M) + \cdots + \lambda_p(M),$$

where $\text{Trace}_W(M)$ is the trace of the quadratic form M restricted to W and $\lambda_i(M)$, $i = 1, \ldots, n$, are the eigenvalues of M in nondecreasing order. By duality we can define

$$\mathcal{P}^+_p(M) = \sup_{W \in G(p,n)} \text{Trace}_W(M) \equiv \lambda_{n-p+1}(M) + \cdots + \lambda_n(M).$$

Next, we denote by $\mathcal{F}_p$ the set of degenerate elliptic operators $F : \mathcal{S}^n \to \mathbb{R}$ such that $F(0) = 0$ and

$$F(N) \geq 0,\ \mathcal{P}_p^+(M) \leq 0 \quad \Rightarrow \quad \mathcal{P}_p^+(N - M) \geq 0. \tag{4.27}$$

Suppose now $0 \leq \alpha \leq p - 2$. The same calculations of Lemma 4.2.1 show that

$$\mathcal{P}_p^+(D^2 K_\alpha(x-y)) = \begin{cases} -\dfrac{\alpha(p-2-\alpha)}{|x-y|^{\alpha+2}} & \text{if } \alpha > 0 \\ -\dfrac{p-2}{|x-y|^2} & \text{if } \alpha = 0 \end{cases}$$

and therefore, if E is a compact set of $\mathbb{R}^n$ and $\mu \in \mathcal{M}_1^+(E)$, the α-potentials of μ are smooth solutions of

$$\mathcal{P}_p^+(D^2 V_\alpha^\mu(x)) \leq 0 \quad \text{in} \quad \mathbb{R}^n \backslash E. \tag{4.28}$$

As before, we may assume $V_\alpha^\mu(x)$ positive in any ball of $\mathbb{R}^n$, large as we want, and we may also take μ such that

$$V_\alpha^\mu(x) = +\infty \quad \text{on} \quad E, \quad V_\alpha^\mu(x) < +\infty \quad \text{elsewhere.} \tag{4.29}$$

Theorem 4.4.1. [**Maximum Principle**] Let Ω be a bounded domain of $\mathbb{R}^n$ and $u \in USC(\Omega)$ such that $\mathcal{P}_p^+(D^2 u) \geq 0$ in Ω in the viscosity sense. If $u \leq 0$ on $\partial\Omega$ then $u \leq 0$ in Ω.

Proof. We assume that $\Omega \subset (-R, R)^n$ for some $R > 0$. Suppose by contradiction that $u(0) = \max_{\overline{\Omega}} u = k > 0$ and consider the function

$$h(x) = k\left[1 + e^{-R}\left(1 - \frac{1}{n}\sum_{k=1}^{n} e^{x_i}\right)\right],$$

such that $h(0) = k$ and $h(x) \geq ke^{-R}$ for $x \in \overline{\Omega}$. Eventually moving up the graph of $h(x)$, we find a point $\overline{x} \in \overline{\Omega}$ where the graph of $h(x)$ touches above the graph of $u(x)$, i.e.

$$u(\overline{x}) = h(\overline{x}), \quad u(x) \leq h(x) \ in \ \overline{\Omega}.$$

Since $u \leq 0 < h$ on $\partial\Omega$, then $\overline{x} \in \Omega$ and $h(x)$ is a test function for the viscosity subsolution $u(x)$ at $\overline{x}$. This leads to the following contradiction:

$$0 \leq \lambda_{n-p+1}(D^2 h(\overline{x})) + \cdots + \lambda_n(D^2 h(\overline{x})) \leq -ke^{-2R}\,\frac{p}{n},$$

which concludes the proof. □

Now we can prove the following EMP for a pure second order elliptic operators satisfying the above elliptic condition (4.27).

Theorem 4.4.2. *Suppose $p \geq 2$. Let Ω be a bounded domain of $\mathbb{R}^n$ and $E \subset \partial\Omega$ be a F_σ-set. If $C_{p-2}(E) = 0$, then E is $\mathcal{F}_p$-exceptional on $\partial\Omega$.*

Proof. The proof follows the same lines of Theorem 4.2.2, using the structure condition $\mathcal{F}_p$ instead of $\mathcal{F}_{\lambda,\Lambda}$ with the potential $v = V_\alpha^\mu(x)$ constructed above (4.29) with $\alpha = p-2$. Since $\mathcal{P}_p^+(D^2v) \le 0$, assuming $F(D^2u) \ge 0$, we have by (4.27)

$$\mathcal{P}_p^+(D^2w) \ge 0$$

for $w = u - \varepsilon v$, and the maximum principle of the previous lemma completes the proof. □

Using existence results, see for instance [31, Theorem 6.2], and arguing as in Proposition 4.3.1-Theorem 4.3.1 we obtain an alternative proof of the removability of the sets with $(p-2)$-capacity zero for the class $\mathcal{F}_p$, as already established in [32, Theorem 10.5]. This result is more general with respect to Theorem 4.3.1 A) because of the inclusion $\mathcal{F}_{\lambda,\Lambda} \subset \mathcal{F}_p$. In other words the sets of null $(p-2)$-capacity are removable for a class larger than the uniformly elliptic one. To prove this we need the following

Lemma 4.4.1. *If $\alpha^* = p-2$, or equivalently $p = \frac{\lambda}{\Lambda}(n-1)+1$, then*

$$\mathcal{P}_{\lambda,\Lambda}^+(M) \le \max\left\{\lambda\mathcal{P}_p^+(M), \frac{n}{p}\Lambda\mathcal{P}_p^+(M)\right\}$$

for any $M \in \mathcal{S}^n$.

Proof. If

$$\lambda_1(M) \le \cdots \le \lambda_n(M) \le 0,$$

then

$$\mathcal{P}_{\lambda,\Lambda}^+(M) = \lambda(\lambda_1(M) + \cdots + \lambda_n(M)) \le \lambda\mathcal{P}_p^+(M)\,.$$

Suppose that

$$\lambda_1(M) \le \cdots \le \lambda_k(M) \le 0 \le \lambda_{k+1}(M) \le \cdots \le \lambda_n(M).$$

If $n-p+1 \le k \le n-1$ then

$$\begin{aligned}
\mathcal{P}_{\lambda,\Lambda}^+(M) &= \lambda(\lambda_1(M) + \cdots + \lambda_{n-p}(M)) + \lambda(\lambda_{n-p+1}(M) + \cdots + \lambda_k(M)) \\
&\quad + \Lambda(\lambda_{k+1}(M) + \cdots + \lambda_n(M)) \\
&\le \lambda\left(\frac{n-p}{k-n+p}+1\right)(\lambda_{n-p+1}(M) + \cdots + \lambda_k(M)) + \Lambda(\lambda_{k+1}(M) + \cdots + \lambda_n(M)) \\
&= \lambda\,\frac{k}{k-n+p}\,(\lambda_{n-p+1}(M) + \cdots + \lambda_k(M)) + \Lambda(\lambda_{k+1}(M) + \cdots + \lambda_n(M)) \\
&\le \lambda\,\frac{n-1}{p-1}(\lambda_{n-p+1}(M) + \cdots + \lambda_k(M)) + \Lambda(\lambda_{k+1}(M) + \cdots + \lambda_n(M)) = \Lambda\mathcal{P}_p^+(M)
\end{aligned}$$

If $1 \le k = n-p$ then

$$\mathcal{P}_{\lambda,\Lambda}^+(M) = \lambda(\lambda_1(M) + \cdots + \lambda_{n-p}(M)) + \Lambda(\lambda_{n-p+1}(M) + \cdots + \lambda_n(M)) \le \Lambda\mathcal{P}_p^+(M)$$

If $1 \leq k \leq n-p-1$ then

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}(M) &= \lambda(\lambda_1(M)+\cdots+\lambda_k(M)) + \Lambda(\lambda_{k+1}(M)+\cdots+\lambda_n(M))\\ &\leq \Lambda\left(\lambda_{k+1}(M)+\cdots+\lambda_{n-p}(M)\right) + \Lambda\left(\lambda_{n-p+1}(M)+\cdots+\lambda_n(M)\right)\\ &\leq \Lambda\left(\frac{n-p-k}{p}+1\right)\left(\lambda_{n-p+1}(M)+\cdots+\lambda_n(M)\right)\\ &= \Lambda\,\frac{n-k}{p}\,\mathcal{P}^+_p(M) \leq \Lambda\,\frac{n}{p}\,\mathcal{P}^+_p(Y)\,.\end{aligned}$$

If

$$0 \leq \lambda_1(M) \leq \cdots \leq \lambda_n(M)$$

then

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}(M) &= \Lambda(\lambda_1(M)+\cdots+\lambda_n(M))\\ &\leq \Lambda\left(\lambda_1(M)+\cdots+\lambda_{n-p}(M)\right) + \Lambda\left(\lambda_{n-p+1}(M)+\cdots+\lambda_n(M)\right)\\ &= \Lambda\left(\frac{n-p}{p}+1\right)\left(\lambda_{n-p+1}(M)+\cdots+\lambda_n(M)\right) = \ \Lambda\,\frac{n}{p}\,\mathcal{P}^+_p(M)\,.\end{aligned}$$

□

Now we are in position to prove the inclusion $\mathcal{F}_{\lambda,\Lambda} \subset \mathcal{F}_p$. Let $F \in \mathcal{F}_{\lambda,\Lambda}$. If $F(N) \geq 0$ and $\mathcal{P}^+_p(M) \leq 0$, by previous lemma $\mathcal{P}^+_{\lambda,\Lambda}(M) \leq 0$ and

$$\begin{aligned}\max(\lambda\mathcal{P}^+_p(N-M), \frac{n}{p}\Lambda\mathcal{P}^+_p(N-M)) &\geq \mathcal{P}^+_{\lambda,\Lambda}(N-M)\\ &\geq F(N-M) \geq F(N) - \mathcal{P}^+_{\lambda,\Lambda}(M) \geq 0,\end{aligned}$$

as we wanted to show.

Bibliography

[1] M.E.Amendola, G.Galise and A.Vitolo, Riesz capacity, maximum principle and removable sets of fully nonlinear second order elliptic operators, *accepted for publications on Differential and Integral Equations*

[2] M.E. Amendola, L.Rossi and A. Vitolo, Harnack Inequalities and ABP Estimates for Nonlinear Second Order Elliptic Equations in Unbounded Domains, *Abstract and Applied Analysis*, (ISSN:1085-3375), Vol. 2008, pp.1-19

[3] S.N.Armstrong, B.Sirakov and C.K.Smart, Fundamental solutions of homogeneous fully nonlinear elliptic equations, *Comm. Pure Appl. Math.*, Vol. 64, pp. 737-777 (2011)

[4] H.Brezis, Semilinear equations in $\mathbb{R}^n$ without conditions at infinity, *Appl. Math. Optim.* 12 (1984), 271–282

[5] X.Cabré and L.A.Caffarelli, Interior $C^{2,\alpha}$-regularity theory for a class of nonconvex fully nonlinear elliptic equations, *J. Math. Pures Appl.* 82 (2003), n.5, 573–612

[6] V.Cafagna and A.Vitolo, On the maximum principle for second- order elliptic operators in unbounded domains, *Comptes Rendus Mathematique* (ISSN:1631-073X), (2002), Vol. 334, pp.359-363

[7] L.A.Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, *Annals Math.* 130 (1989), n.1, 189–213

[8] L.A.Caffarelli and X.Cabré, *Fully Nonlinear Elliptic Equations*, Colloquium Publications 43, American Mathematical Society, Providence, RI (1995)

[9] L.A.Caffarelli, M.G.Crandall, M.Kocan and A.Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients, *Comm. Pure Appl. Math.* 9 (1996), n.4, 365-398

[10] L.A.Caffarelli,Y.Y.Li and L.Nirenberg, Some remarks on singular solutions of nonlinear elliptic equations. I, *J. Fixed Point Theory Appl.* 5 (2009), 353-395

[11] L.A.Caffarelli,Y.Y.Li and L.Nirenberg , Some remarks on singular solutions of nonlinear elliptic equations. III: viscosity solutions, including parabolic operators, *Comm. Pure Appl. Math.*, doi: 10.1002/cpa.21412 (2012)

[12] I.Capuzzo Dolcetta and A.Vitolo, On the Maximum Principle for Viscosity Solutions of Fully Nonlinear Elliptic Equations in General Domains, *Le Matematiche* (ISSN:0373-3505), (2007), Vol. 62 pp.69-91

[13] I.Capuzzo Dolcetta and A.Vitolo, Local and Global Estimates for Viscosity Solutions of Fully Nonlinear Elliptic Equations, *Dynamics of Continuous, Discrete and Impulsive System, Series A: Mathematical Analysis*, (ISSN:1201-3390), (2007), Vol. 14 (S2), pp.11-16

[14] I.Capuzzo Dolcetta and A.Vitolo, Glaeser's type gradient estimates for non-negative solutions of fully nonlinear elliptic equations, *Discrete Contin. Dyn. Syst.* 28 n.2, Dedicated to Louis Nirenberg on the Occasion of his 85th Birthday, Part I, Caffarelli et al. eds. (2010), 539–557

[15] I.Capuzzo Dolcetta, F.Leoni and A.Vitolo, The Alexandrov- Bakelman-Pucci weak Maximum Principle for fully nonlinear equations in unbounded do- mains, *Comm. Partial Differential Equations*, (ISSN:0360- 5302), (2005), Vol. 30, pp.1863-1881

[16] M.G.Crandall and P.L.Lions, Condition d'unicité pour les solutions generalisées des équations de Hamilton - Jacobi du premier order, *C.R. Acad. Sci.* 292 (1981), 183–186

[17] M.G.Crandall and P.L.Lions, Viscosity solutions of Hamilton - Jacobi equations, *Trans. Amer. MAth. Soc.* 277 (1983), 1–42

[18] M.G.Crandall, L.C. Evans and P.L.Lions, Some properties of viscosity solutions of Hamilton - Jacobi equations, *Trans. Amer. MAth. Soc.* 282 (1984), 487–502

[19] M.G.Crandall, H.Ishii and P.L.Lions, User's guide to viscosity solutions of second-order partial differential equations, *Bull. Amer. Math. Soc.* 27 (1992), n.1, 1–67

[20] M.G.Crandall, M.Kocan, P.L.Lions and A.Świech, Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations, *EJDE* 24 (1999), pp. 1–20

[21] M. G. Crandall, M. Kocan, P. Soravia, A. Świech On the equivalence of various weak notions of solutions of elliptic PDEs with measurable ingredients. in "Progress in Elliptic and Parabolic P.D.E.'s" (A. Alvino et al. eds.), Pitman Research Notes in Math., n. 50, (1996),pp. 136–162.

[22] M.G.Crandall and A.Świech, A note on generalized maximum principles for elliptic and parabolic PDE, *Lecture Notes in Pure and Appl. Math.* 235, Proc. in honour of J.A. Goldstein's 60th birthday, Goldstein et al. eds. (2003), 121–127

[23] F.Da Lio and B.Sirakov, Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations, *J. Eur. Math. Soc. (JEMS)* 9 (2007), n.2, 317–330

[24] G.Dìaz and R.Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, *Nonlinear Analysis, Theory, Methods and Applications*, Vol. 20, No 2, 97-125, 1993

[25] H.Dong, S.Kim and M.Safonov, On uniqueness boundary blow-up solutions of a class of nonlinear elliptic equations, *Commun. Partial Differ. Equations* 33 (2008), 177–188

[26] L.Escauriaza, $W^{2,n}$ a priori estimates for solutions to fully nonlinear equations, *Indiana Univ. Math. J.* **42** (1993), 413–423

[27] M.J.Esteban, P.L.Felmer and A.Quaas, Superlinear elliptic equations for fully nonlinear operators without growth restrictions for the data, *Proc. Edinb. Math. Soc.* **53** (2010), 125–141

[28] L.C.Evans and R.F.Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton (1992)

[29] G.Galise and A.Vitolo, Viscosity Solutions of Uniformly Elliptic Equations without Boundary and Growth Conditions at Infinity, *Int. J. Differ. Equations*, vol. 2011, Article ID 453727, 18 pages, 2011. doi:10.1155/2011/453727.

[30] D.Gilbarg and N.S.Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren Math. Wiss. 224, Springer-Verlag, Berlin-New York (1983)

[31] R.Harvey and B.Lawson Jr., Dirichlet duality and the Nonlinear Dirichlet problem, *Comm. Pure Appl. Math.* **62** (2009), 396–443

[32] R.Harvey and B.Lawson Jr., Removable singularities for nonlinear subequations, *Stony Brook Preprint*

[33] R.Harvey and B.Lawson Jr., The foundations of p-convexity and p-plurisubharmonicity in riemannian geometry, *ArXiv: 1111.3895*

[34] R.Harvey and B.Lawson Jr., Existence, uniqueness and removable singularities for nonlinear partial differential equations in geometry, *Stony Brook Preprint*

[35] R.Harvey and B.Lawson Jr., Plurisubharmonicity in a general geometric context , *Geometry and Analysis* **1** (2010), 363-401. ArXiv:0804.1316

[36] R.Harvey and B.Lawson Jr., Dirichlet duality and the nonlinear Dirichlet problem on Riemannian manifolds , J. Diff. Geom. **88** (2011), 395-482

[37] N.K.Hayman and P.B.Kennedy, *Subharmonic functions Vol. I*, London Mathematical Society Monographs No. 9, Academic Press, London (1976)

[38] H.Ishii, On uniqueness and Existence of Viscosity Solutions of Fully Nonlinear Second-Order Ellptic PDE's, *Comm. Pure Appl. Math.* 42 (1989), 15–45

[39] H.Ishii and P.L.Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, *Journal of differential equations*, 83, 26-78 (1990)

[40] R. Jensen, The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations, *Arch. Ration. Mech. Anal*, 101 (1988), 1-27

[41] J.B.Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math*, vol. X, 503-510 (1957)

[42] S.Koike, *A Beginners Guide to the Theory of Viscosity Solutions*, MSJ Memoirs 13, Math. Soc. Japan, Tokyo (2004)

[43] S.Koike and A.Święch, Maximum principle and existence of L^p viscosity solutions for fully nonlinear uniformly elliptic equations with measurable and quadratic terms, *Nonlinear differ. equ. appl.* 11 (2004), 491-509

[44] D.A.Labutin, Removable Singularities for Fully Nonlinear Elliptic Equations, *Arch. Ration. Mech. Anal.* 155 (2000), 201–214

[45] D.Labutin, Singularities of viscosity solutions of fully nonlinear elliptic equations, *Viscosity Solutions of Differential Equations and Related Topics*, Ishii ed., RIMS Kôkyûroku No. 1287, Kyoto University, Kyoto (2002), 45–57

[46] N.S.Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag (1972)

[47] M.Marcus and L.Véron, Uniqueness of solutions with blowup at the boundary for a class of nonlinear elliptic equations, *C.R. Acad. Sci. Paris*, 317 séries I (1993), pp. 559-563

[48] M.Marcus and L.Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. Henri Poincaré,Analyse non linéaire* Vol. 14, no 2, 1997, p. 231-274

[49] R.Osserman, On the inequlity $\Delta u \geq f(u)$, *Pacific J. Math*, Vol. 7, n.4 (1957), 1641-1647

[50] H.L.Royden, *Real Analysis*, 2^{nd} ed., Macmillan (1988)

[51] S.Salsa, *Lezioni su equazioni ellittiche*, PhD course, Milano (2009)

[52] B.Sirakov, Solvability of uniformly elliptic fully nonlinear PDE, *Arch. Ration. Mech. Anal.* 195 n.2 (2010), 579–607

[53] A.Święch, $W^{1,p}$-interior estimates for solutions of fully nonlinear, uniformly elliptic equations, *Adv. Differential Equations* 2 n.6 (1997), 1005–1027

[54] A.Vitolo, On the Maximum Principle for Complete Second-Order Elliptic Operators in General Domains, *Journal of Differential Equations*, (ISSN:0022-0396), (2003), Vol. 194, pp.166-184

[55] A.Vitolo, M.E.Amendola and G.Galise, On the uniqueness of blow-up solutions of fully nonlinear elliptic equations, *accepted for publications on DCDS Series S*

[56] N.Winter, $W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear uniformly elliptic equations, *Z. Anal. Anwend.*, 28 (2009), 129-164